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Conformal invariance and Hojman conserved quantities of canonical Hamilton systems*

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This paper discusses the conformal invariance by infinitesimal transformations of canonical Hamilton systems. The necessary and sufficient conditions of conformal invariance being Lie symmetrical simultaneously by the action of infinitesimal transformations are given. The determining equations of the conformal invariance are gained. Then the Hojman conserved quantities of conformal invariance by special infinitesimal transformations are obtained. Finally an illustrative example is given to verify the results.

Keywords: canonical Hamilton systems, infinitesimal transformations, conformal invariance, Hojman conserved quantities

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1. Introduction

Lagrange equations can be transformed into Hamilton canonical equations by Legendre transformations. The dynamical systems are called Hamilton systems which are described by Hamilton equations. And the dynamical character can be described only by utilizing the Hamiltonian. In recent years the generalized Hamilton systems, KAM theorems in the approximately integral systems and the symplectic arithmetic in computational mathematics are presented. The research fruits of Hamilton systems have produced important influence on non-linear analysis, algebraic topology, mathematical physics and differential geometry.[1−3] The subject of symmetries and conserved quantities are very important in mathematics and physics, and plenty of research fruits are obtained.[1,4−26] Recently conformal Hamiltonian systems was discussed by geometric methods, the authors researched into the geometric structure of conformal systems with symmetry and pointed out the difference of conformal systems from arbitrary symmetric systems.[27] In 1997, Galiullin *et al* studied the conformal invariance of Birkhoff equations and their relation with the Lie symmetry.[28] In this paper the conformal invariance of canonical Hamilton equations is discussed in detail. The conformal invariance is first defined and the conformal factors of canonical Hamilton equations are obtained by infinitesimal transformations. Finally we obtain the Hojman conserved quantities of conformal invariance by special infinitesimal transformations.

2. The differential equations of dynamical systems

Lagrange equations can be transformed into Hamilton canonical equations by Legendre transformations. The canonical equations are

$$\dot{q}_s = \frac{\partial H}{\partial p_s}, \quad \dot{p}_s = -\frac{\partial H}{\partial q_s} \quad (s = 1, \ldots, n). \tag{1}$$

Hamilton equations are equivalent to Lagrange equations when the Lagrangian L is contacted to the Hamiltonian H by the Legendre transformations. Equations (1) are equivalent to

$$\dot{q}_s = g_s\left(t, \boldsymbol{q}, \boldsymbol{p}\right), \quad \dot{p}_s = h_s\left(t, \boldsymbol{q}, \boldsymbol{p}\right). \tag{2}$$

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3. The conformal invariance of Hamilton canonical equations

The differential equations of Hamilton systems can be written down as

$$\begin{aligned} F'_s &= \dot{q}_s - g_s\left(t, \boldsymbol{q}, \boldsymbol{p}\right), \\ F''_s &= \dot{p}_s - h_s\left(t, \boldsymbol{q}, \boldsymbol{p}\right). \end{aligned} \tag{3}$$

Discuss the conformal invariance of Hamilton systems, that is to seek the one-parameter group of infinitesimal transformations of expressions (3). Considering the conformal invariance of expressions (3), we adopt the infinitesimal transformations of t and q_s as

$$\begin{aligned} &t^* = t + \Delta t, \\ &q_s^*\left(t^*\right) = q_s + \Delta q_s, \\ &p_s^*\left(t^*\right) = p_s\left(t\right) + \Delta p_s. \end{aligned}$$

Or expand it as

$$\begin{aligned} t^* &= t + \varepsilon\xi_0\left(t, \boldsymbol{q}, \boldsymbol{p}\right), \\ q_s^*\left(t^*\right) &= q_s + \varepsilon\xi_s\left(t, \boldsymbol{q}, \boldsymbol{p}\right), \\ p_s^*\left(t^*\right) &= p_s + \varepsilon\eta_s\left(t, \boldsymbol{q}, \boldsymbol{p}\right), \end{aligned} \tag{4}$$

where ε is the infinitesimal parameter, $\xi_0,\ \xi_s,\ \eta_s$ are the generators of the one-parameter group of infinitesimal transformations.

Taking the infinitesimal generator vectors as

$$X^{(0)} = \xi_0\frac{\partial}{\partial t} + \xi_k\frac{\partial}{\partial q_k} + \eta_k\frac{\partial}{\partial p_k},$$

we obtain its first prolonged vectors

$$X^{(1)} = X^{(0)} + \left(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0\right)\frac{\partial}{\partial \dot{q}_k} + \left(\dot{\eta}_k - \dot{p}_k\dot{\xi}_0\right)\frac{\partial}{\partial \dot{p}_k}.$$

Definition 1 In the infinitesimal transformations of infinitesimal generators $\xi_0\left(t, \boldsymbol{q}, \boldsymbol{p}\right)$, $\xi_s\left(t, \boldsymbol{q}, \boldsymbol{p}\right)$, $\eta_s\left(t, \boldsymbol{q}, \boldsymbol{p}\right)$, if F_i satisfy the following determining expressions:

$$X^{(1)}F_i = \ell_i^j F_j, \tag{5}$$

then the infinitesimal transformations have conformal invariance for the corresponding first order differential equations.

4. The conformal factors of conformal invariance

In order to get the conformal factors, we need to suppose that the Hamilton equations have conformal invariance and Lie symmetry simultaneously under the infinitesimal transformations.

When Eq.(3) has Lie symmetry, namely

$$X^{(1)}F_i\,|_{F_i=0} = 0 \qquad (i = 1, \ldots, 2n), \tag{6}$$

to gain the general expression of conformal factors, we calculate the difference

$$X^{(1)}F_i - X^{(\mathbf{1})}F_i\,|_{F_i=0}\,. \tag{7}$$

There exist

$$\begin{aligned} X^{(1)}F' &= \left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right) - X^{(\mathbf{0})}\left(g_s\right) \\ &= \left(\frac{\partial \xi_s}{\partial t} + \frac{\partial \xi_s}{\partial q_l}\dot{q}_l + \frac{\partial \xi_s}{\partial p_l}\dot{p}_l - \dot{q}_s\dot{\xi}_0\right) \\ &\quad - X^{(0)}\left(g_s\right), \end{aligned} \tag{8}$$

and

$$\begin{aligned} X^{(1)}F'' &= \left(\dot{\eta}_k - \dot{p}_k\dot{\xi}_0\right) - X^{(0)}\left(h_s\right) \\ &= \left(\frac{\partial \eta_k}{\partial t} + \frac{\partial \eta_k}{\partial q_l}\dot{q}_l + \frac{\partial \eta_k}{\partial p_l}\dot{p}_l - \dot{p}_k\dot{\xi}_0\right) \\ &\quad - X^{(0)}\left(h_s\right). \end{aligned} \tag{9}$$

Equivalently, we have

$$\begin{aligned} X^{(1)}F'_s|_{F'_s=0} &= \left(\dot{\xi}_s - g_s\dot{\xi}_0\right) - X^{(0)}\left(g_s\right) \\ &= \frac{\partial \xi_s}{\partial t} + \frac{\partial \xi_s}{\partial q_l}g_l + \frac{\partial \xi_s}{\partial p_l}h_l - g_s(\dot{\xi}_0) \\ &\quad - X^{(0)}\left(g_s\right), \end{aligned} \tag{10}$$

and

$$\begin{aligned} X^{(1)}F''_s|_{F''_s=0} &= \left(\dot{\eta}_s - h_s\dot{\xi}_0\right) - X^{(0)}\left(h_s\right) \\ &= \frac{\partial \eta_s}{\partial t} + \frac{\partial \eta_s}{\partial q_l}g_l + \frac{\partial \eta_s}{\partial p_l}h_l - h_s(\dot{\xi}_0) \\ &\quad - X^{(0)}\left(h_s\right). \end{aligned} \tag{11}$$

Substituting Eqs.(8) and (10), (9) and (11) into Eq.(7) separately, we obtain

$$\begin{aligned} &X^{(1)}F'_s - X^{(2)}F'_s\,|_{F'_s=0} \\ =&\frac{\partial \xi_s}{\partial q_l}\left(\dot{q}_l - g_l\right) + \frac{\partial \xi_s}{\partial p_l}\left(\dot{p}_l - h_l\right) \\ &- \left(\dot{q}_l - g_l\right)\delta_l^s\dot{\xi}_0, \end{aligned} \tag{12}$$

and

$$\begin{aligned} &X^{(1)}F''_s - X^{(2)}F''_s\,|_{F''_s=0} \\ =&\frac{\partial \eta_s}{\partial q_l}\left(\dot{q}_l - g_l\right) + \frac{\partial \eta_s}{\partial p_l}\left(\dot{p}_l - h_l\right) \\ &- \left(\dot{p}_l - h_l\right)\delta_l^s\dot{\xi}_0. \end{aligned} \tag{13}$$

Due to

$$\dot{q}_l - g_l = F'_l, \qquad \dot{p}_l - h_l = F''_l,$$

therefore, Eqs.(12) and(13) are transformed into

$$\begin{aligned} & X^{(1)}F'_s - X^{(2)}F'_s\,|_{F'_s=0} \\ &= \left(\frac{\partial \xi_s}{\partial q_l} - \delta^s_l \dot{\xi}_0\right) F'_l + \frac{\partial \xi_s}{\partial p_l} F''_l, \end{aligned} \tag{14}$$

and

$$\begin{aligned} & X^{(1)}F''_s - X^{(2)}F''_s\,|_{F''_s=0} \\ &= \left(\frac{\partial \eta_s}{\partial p_l} - \delta^s_l \dot{\xi}_0\right) F''_l + \frac{\partial \eta_s}{\partial q_l} F'_l. \end{aligned} \tag{15}$$

Let

$$\begin{aligned} \beta'^s_l &= \frac{\partial \xi_s}{\partial q_l} - \delta^s_l \dot{\xi}_0, \\ \beta''^s_l &= \frac{\partial \xi_s}{\partial p_l}, \\ \beta^s_l &= \beta'^s_l \oplus \beta''^s_l, \\ \gamma'^s_l &= \frac{\partial \eta_s}{\partial p_l} - \delta^s_l \dot{\xi}_0, \\ \gamma''^s_l &= \frac{\partial \eta_s}{\partial q_l}, \\ \gamma^s_l &= \gamma'^s_l \oplus \gamma''^s_l, \end{aligned} \tag{16}$$

then we have

$$\begin{aligned} & X^{(1)}F_i - X^{(\mathbf{1})}F_i\,|_{F_i=0} = \alpha^j_i\, F_j, \\ & \alpha^j_i = \beta^s_l \oplus \gamma^s_l. \end{aligned} \tag{17}$$

When Eq.(3) has conformal invariance and Lie symmetry simultaneously by the action of the infinitesimal transformations of one-parameter group (4), we can get

$$\ell^j_i F_j - \alpha^j_i F_j = X^{(1)}F_i\,|_{F_i=0} = 0\ . \tag{18}$$

That is

$$\ell^j_i = \alpha^j_i. \tag{19}$$

Thus, from the above discussion there exists a criterion.

Proposition 1 For the canonical equations (3), the necessary and sufficient conditions of its conformal invariance which has Lie symmetry simultaneously by the action of the one-parameter group of infinitesimal transformations are there exist the following conformal factors:

$$\ell^j_i = \begin{pmatrix} \left(\dfrac{\partial \xi_s}{\partial q_l} - \delta^s_l \dot{\xi}_0\right) & \left(\dfrac{\partial \xi_s}{\partial p_l}\right) \\ \left(\dfrac{\partial \eta_s}{\partial q_l}\right) & \left(\dfrac{\partial \eta_s}{\partial p_l} - \delta^s_l \dot{\xi}_0\right) \end{pmatrix}, \qquad (i,j = 1,\ldots,2n). \tag{20}$$

5. The conformal invariance and Hojman conserved quantities

By the action of special infinitesimal transformations, there exist the conformal factors which make the canonical equations conformally invariant and Lie symmetrical simultaneously. The Hojman conserved quantities can be educed. Assume the special infinitesimal transformations with time invariance

$$\begin{aligned} & t^* = t, \\ & q^*_s(t^*) = q_s + \varepsilon \xi_s(t, \boldsymbol{q}, \boldsymbol{p}), \\ & p^*_s(t^*) = p_s + \varepsilon \eta_s(t, \boldsymbol{q}, \boldsymbol{p}). \end{aligned} \tag{21}$$

Proposition 2 For the Hamiltonian canonical equations (1), by the action of special infinitesimal transformations which are conformally invariant and Lie symmetrical simultaneously, besides the infinitesimal generators fulfil the following expressions:

$$\begin{aligned} \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\xi_s &= \frac{\partial^2 H}{\partial p_s \partial q_k}\xi_k + \frac{\partial^2 H}{\partial p_s \partial p_k}\eta_k, \\ \frac{\bar{\mathrm{d}}}{\mathrm{d}t}\eta_s &= -\frac{\partial^{\mathbf{2}} H}{\partial q_s \partial q_k}\xi_k - \frac{\partial^2 H}{\partial q_s \partial p_k}\eta_k, \end{aligned} \tag{22}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \frac{\partial H}{\partial p_s}\frac{\partial}{\partial q_s} - \frac{\partial H}{\partial q_s}\frac{\partial}{\partial p_s}.$$

And there exist the Hojman conserved quantities

$$I_{\mathrm{H}} = \frac{\partial \xi_s}{\partial q_s} + \frac{\partial \eta_s}{\partial p_s} = \text{const.} \tag{23}$$

The proof can consult Ref.[1].

6. Illustrative example

Consider the following Hamilton system with two degrees of freedom:

$$H = \frac{1}{2}\left(p_1^2 + p_2^2\right) + q_2. \tag{24}$$

The differential equations are

$$F = \begin{pmatrix} \dot{q}_1 - p_1 = 0 \\ \dot{q}_2 - p_2 = 0 \\ \dot{p}_1 = 0 \\ \dot{p}_2 + 1 = 0 \end{pmatrix}. \tag{25}$$

When the infinitesimal generators are

$$\begin{aligned} &\xi_1 = 0, \\ &\xi_2 = (p_2 + t)\left(q_2 + \frac{1}{2}t^2\right), \\ &\eta_1 = 0, \\ &\eta_2 = (p_2 + t)^2, \end{aligned} \tag{26}$$

then

$$\begin{aligned} X^{(1)} =& \xi_s \frac{\partial}{\partial q_s} + \eta_s \frac{\partial}{\partial p_s} + \left(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0\right)\frac{\partial}{\partial \dot{q}_s} \\ &+ \left(\dot{\eta}_s - \dot{p}_s\dot{\xi}_0\right)\frac{\partial}{\partial \dot{p}_s} \\ =& \xi_2 \frac{\partial}{\partial q_2} + \eta_2 \frac{\partial}{\partial p_2} + \dot{\xi}_2 \frac{\partial}{\partial \dot{q}_2} + \dot{\eta}_2 \frac{\partial}{\partial \dot{p}_2}, \end{aligned}$$

accordingly, we can obtain

$$\begin{aligned} X^{(1)}F =& \left(\xi_2 \frac{\partial}{\partial q_2} + \eta_2 \frac{\partial}{\partial p_2} + \dot{\xi}_2 \frac{\partial}{\partial \dot{q}_2} + \dot{\eta}_2 \frac{\partial}{\partial \dot{p}_2}\right) \begin{pmatrix} F_1 = \dot{q}_1 - p_1 \\ F_2 = \dot{q}_2 - p_2 \\ F_3 = \dot{p}_1 \\ F_4 = \dot{p}_2 + 1 \end{pmatrix} \\ =& \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (p_2 + t) & 0 & \left(q_2 + \frac{1}{2}t^2\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(p_2 + t) \end{pmatrix} \begin{pmatrix} F_1 = \dot{q}_1 - p_1 \\ F_2 = \dot{q}_2 - p_2 \\ F_3 = \dot{p}_1 \\ F_4 = \dot{p}_2 + 1 \end{pmatrix}. \end{aligned}$$

So we obtain the conformal factors

$$\ell_i^j = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (p_2 + t) & 0 & \left(q_2 + \frac{1}{2}t^2\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(p_2 + t) \end{pmatrix}. \tag{27}$$

Utilizing Proposition 1, we can also obtain

$$\ell_i^j = \begin{pmatrix} \left(\frac{\partial \xi_s}{\partial q_l} - \delta_l^s \dot{\xi}_0\right) & \left(\frac{\partial \xi_s}{\partial p_l}\right) \\ \left(\frac{\partial \eta_s}{\partial q_l}\right) & \left(\frac{\partial \eta_s}{\partial p_l} - \delta_l^s \dot{\xi}_0\right) \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (p_2 + t) & 0 & \left(q_2 + \frac{1}{2}t^2\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2(p_2 + t) \end{pmatrix}. \tag{28}$$

To all appearances, the result is equivalent to Eq.(27). Therefore, the determining equations of conformal invariance of Eq.(25) by the action of special infinitesimal transformations are

$$X^{(1)}F = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & (p_2+t) & 0 & \left(q_2+\frac{1}{2}t^2\right) \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 2\,(p_2+t) \end{pmatrix} \begin{pmatrix} F_1=\dot{q}_1-p_1 \\ F_2=\dot{q}_2-p_2 \\ F_3=\dot{p}_1 \\ F_4=\dot{p}_2+1 \end{pmatrix}. \tag{29}$$

So Eq.(25) by the action of Eq.(26) are conformal invariant and Lie symmetrical simultaneously. When we utilize Proposition 2, the Hojman conserved quantities are gained

$$I_{\mathrm{H}} = \frac{\partial \xi_2}{\partial q_2} + \frac{\partial \eta_2}{\partial p_2} = 3\,(p_2+t) = \text{const.} \tag{30}$$

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