Vol 18 No 3, March 2009
1674-1056/2009/18(03)/0850-06

Chinese Physics B

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Discrete integrable system and its integrable coupling

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(Received 5 June 2008; revised manuscript received 5 August 2008)

This paper derives new discrete integrable system based on discrete isospectral problem. It shows that the hierarchy is completely integrable in the Liouville sense and possesses bi-Hamiltonian structure. Finally, integrable couplings of the obtained system is given by means of semi-direct sums of Lie algebras.

Keywords: isospectral problem, Hamiltonian structure, integrable coupling, semi-direct sums
PACC: 0220, 0365G, 0304K

1. Introduction

It is well known that many continuous integrable systems with physical significance and their corresponding integrable coupling system have been obtained by using the Tu model.[1−17] With the development of the soliton theory, it becomes an important topic to find discrete integrable systems and its related properties.

Integrable coupling is a new and very important topic in the study of integrable system. The concept of integrable coupling and the related theories were brought forward in Refs.[18, 19]. Recently, the idea of semi-direct sums of Lie algebras was also used for studying integrable couplings in Ref.[20]. A systematic theory between semi-direct sums of Lie algebras and integrable couplings was developed very recently. Let G be a Lie algebra, take a semi-direct sum of G with another Lie algebra G_c as follows:

$$\bar{G} = G \oplus G_c. \tag{1}$$

The notion of semi-direct sums means that G and G_c satisfy

$$[G, G_c] \subseteq G_c, \tag{2}$$

where $[G, G_c] = \{[A, B]|A \in G, B \in G_c\}$. Hence, G_c is an ideal Lie subalgebra of the enlarged Lie algebra $\bar{G}$. According to Eqs.(1) and (2), if we set

$$\bar{U} = U + U_c, \bar{V} = V + V_c,\ U, V \in G,\ U_c, V_c \in G_c, \tag{3}$$

and make a pair of enlarged matrix spectral problems

$$\begin{aligned} \bar{\phi}_x &= \bar{U}\bar{\phi}, \\ \bar{\phi}_t &= \bar{V}\bar{\phi}, \end{aligned} \tag{4}$$

then the resulting compatibility condition presents the following enlarged zero curvature equation

$$\bar{U}_t - \bar{V}_x + [\bar{U}, \bar{V}] = 0, \tag{5}$$

which is equivalent to

$$\begin{aligned} &U_t - V_x + [U, V] = 0, \\ &U_{c,t} - V_{c,x} + [U, V_c] + [U_c, V] + [U_c, V_c] = 0. \end{aligned} \tag{6}$$

The first equation in Eq.(6) exactly presents the known equation $u_t = K(u)$, the second one gives the new equation $v_t = S(u, v)$. Thus, the enlarged integrable system

$$\begin{aligned} u_t &= K(u), \\ v_t &= S(u, v) \end{aligned} \tag{7}$$

is just a coupling system.

In this paper, first, we construct a discrete isospectral problem based on loop algebra $\widetilde{G}$, then by use of Tu scheme we obtain a hierarchy of Liouville integrable equations. Then, we work out the integrable couplings of the obtained system by the method of semi-direct sums of Lie algebras.

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2. Discrete integrable system

First, we set

$$G = \mathrm{span}\{\boldsymbol{e}_1, \boldsymbol{e}_2, \boldsymbol{e}_3, \boldsymbol{e}_4\}, \tag{8}$$

where

$$\boldsymbol{e}_1 = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{e}_2 = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix},$$
$$\boldsymbol{e}_3 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad \boldsymbol{e}_4 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}. \tag{9}$$

It is easy to see that G is closed under the following matrix multiplication:

$$\begin{aligned} &\boldsymbol{e}_1\boldsymbol{e}_1 = \boldsymbol{e}_3\boldsymbol{e}_4 = \boldsymbol{e}_1, \\ &\boldsymbol{e}_2\boldsymbol{e}_2 = \boldsymbol{e}_4\boldsymbol{e}_3 = \boldsymbol{e}_2, \\ &\boldsymbol{e}_1\boldsymbol{e}_3 = \boldsymbol{e}_3\boldsymbol{e}_2 = \boldsymbol{e}_3, \\ &\boldsymbol{e}_2\boldsymbol{e}_4 = \boldsymbol{e}_4\boldsymbol{e}_1 = \boldsymbol{e}_4, \\ &\boldsymbol{e}_1\boldsymbol{e}_2 = \boldsymbol{e}_2\boldsymbol{e}_1 = \boldsymbol{e}_3\boldsymbol{e}_3 = \boldsymbol{e}_4\boldsymbol{e}_4 = \boldsymbol{e}_2\boldsymbol{e}_3 \\ =&\boldsymbol{e}_3\boldsymbol{e}_1 = \boldsymbol{e}_1\boldsymbol{e}_4 = \boldsymbol{e}_4\boldsymbol{e}_2 = 0. \end{aligned} \tag{10}$$

Set $\boldsymbol{e}_i(n) = \boldsymbol{e}_i\lambda^n, i = 1, 2, 3, 4$, then $\mathrm{span}\{\boldsymbol{e}_i(n)\}_{i=1}^4 \equiv \widetilde{G}$ is an obvious loop algebra, by use of loop algebra $\widetilde{G}$ the discrete isospectral problem is established by

$$\begin{aligned} E\psi =&\boldsymbol{U}_n\psi, \quad \lambda_t = 0, \quad \boldsymbol{u} = \begin{pmatrix} p \\ q \end{pmatrix}, \\ \boldsymbol{U}_n =& \begin{pmatrix} 1 & \dfrac{q}{p} \\ \lambda p & \lambda(1+q) \end{pmatrix} \\ =&\boldsymbol{e}_1(0) + (1+q)\boldsymbol{e}_2(1) + \frac{q}{p}\boldsymbol{e}_3(0) + p\boldsymbol{e}_4(1). \end{aligned} \tag{11}$$

Let

$$\begin{aligned} \boldsymbol{\Gamma} =& \begin{pmatrix} a & \dfrac{b}{\lambda} \\ c & -a \end{pmatrix} \\ =&\sum_{m\geq 0}(a_m\boldsymbol{e}_1(-m) - a_m\boldsymbol{e}_2(-m) \\ &+ b_m\boldsymbol{e}_3(-m-1) + c_m\boldsymbol{e}_4(-m)). \end{aligned} \tag{12}$$

Solving the stationary zero curvature equation

$$(E\boldsymbol{\Gamma})\boldsymbol{U}_n - \boldsymbol{U}_n\boldsymbol{\Gamma} = 0, \tag{13}$$

we obtain the recursion relation

$$\begin{aligned} &a_m^{(1)} - a_m + pb_m^{(1)} - \frac{q}{p}c_m = 0, \\ &\frac{q}{p}(a_m^{(1)} + a_m) - b_{m-1} + (1+q)b_m^{(1)} = 0, \\ &p(a_m^{(1)} + a_m) - c_{m-1}^{(1)} + (1+q)c_m = 0, \\ &\frac{q}{p}c_{m-1}^{(1)} - pb_{m-1} + (1+q)(a_m - a_m^{(1)}) = 0. \end{aligned} \tag{14}$$

We choose $a_0 = -\dfrac{1}{2}, b_0 = \dfrac{q^{(-1)}}{p^{(-1)}(1+q^{(-1)})}, c_0 = \dfrac{p}{1+q}$ and require $a_j\big|_{[u]=0} = 0$, $b_j\big|_{[u]=0} = 0$, $c_j\big|_{[u]=0} = 0$ $(j \geq 1)$, where $[u] = (u, Eu, E^{-1}u, \ldots)$. Thus, the recursion relation (14) uniquely determines $a_j, b_j, c_j, j \geq 1$, and the first few quantities are given as follows:

$$a_1 = -\frac{pq^{(-1)}}{p^{(-1)}(1+q^{(-1)})(1+q)},$$

$$\begin{aligned} b_1 =&\frac{q^{(-2)}}{p^{(-2)}(1+q^{(-2)})(1+q^{(-1)})} \\ &+ \frac{q^{(-1)}q^{(-2)}}{p^{(-2)}(1+q^{(-2)})\left(1+q^{(-1)}\right)^2} \\ &+ \frac{p\left(q^{(-1)}\right)^2}{\left(q^{(-1)}\right)^2(1+q)\left(1+q^{(-1)}\right)^2}, \end{aligned}$$

$$\begin{aligned} c_1 =&\frac{p}{(1+q)^2} + \frac{p^2q^{(-1)}}{p^{(-1)}(1+q)^2(1+q^{(-1)})} \\ &+ \frac{qp^{(1)}}{(1+q)^2(1+q^{(1)})}, \ldots. \end{aligned}$$

For any integer $m \geq 0$, we choose

$$\begin{aligned} \boldsymbol{\Gamma}_+ =& \sum_{m=0}^{n}(a_m\boldsymbol{e}_1(n-m) - a_m\boldsymbol{e}_2(n-m) \\ &+ b_m\boldsymbol{e}_3(n-m-1) + c_m\boldsymbol{e}_4(n-m)), \end{aligned} \tag{15}$$

then direct computation gives

$$\begin{aligned} (E\boldsymbol{\Gamma}_+)\boldsymbol{U}_n - \boldsymbol{U}_n\boldsymbol{\Gamma}_+ =& \left(\frac{q}{p}c_n^{(1)} - pb_n\right)\boldsymbol{e}_2(1) - b_n\boldsymbol{e}_3(0) \\ &+ c_n^{(1)}\boldsymbol{e}_4(1). \end{aligned} \tag{16}$$

Take $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_+$, then the discrete zero curvature equation admits the following Lax integrable system:

$$\boldsymbol{u}_{t_n}=\begin{pmatrix}p\\q\end{pmatrix}_{t_n}=\begin{pmatrix}c_n^{(1)}\\ \frac{q}{p}c_n^{(1)}-pb_n\end{pmatrix}=\begin{pmatrix}0 & Ep\\ -pE^{-1} & \frac{q}{p}Ep-pE^{-1}\frac{q}{p}\end{pmatrix}\begin{pmatrix}b_n^{(1)}-\frac{q}{p^2}c_n\\ \frac{c_n}{p}\end{pmatrix}. \tag{17}$$

From Eq.(14) the recurrence operator $\boldsymbol{L}$ satisfies

$$\begin{pmatrix}b_{n+1}^{(1)}-\frac{q}{p^2}c_{n+1}\\ \frac{c_{n+1}}{p}\end{pmatrix}=\boldsymbol{L}\begin{pmatrix}b_n^{(1)}-\frac{q}{p^2}c_n\\ \frac{c_n}{p}\end{pmatrix},\quad \boldsymbol{L}=\begin{pmatrix}L_{11} & L_{12}\\ L_{21} & L_{22}\end{pmatrix}, \tag{18}$$

where

$$L_{11}=\frac{1}{1+q}E^{-1},\quad L_{12}=\frac{1}{1+q}E^{-1}\frac{q}{p}-\frac{q}{p^2(1+q)}Ep,\quad L_{21}=\frac{1}{1+q}(E+1)(E-1)^{-1}\frac{p}{1+q}E^{-1},$$
$$L_{22}=\frac{1}{p(1+q)}Ep-\frac{1}{1+q}(E+1)(E-1)^{-1}\frac{q}{p(1+q)}Ep+\frac{1}{1+q}(E+1)(E-1)^{-1}\frac{p}{1+q}E^{-1}\frac{q}{p}.$$

Then the system (17) can be rewritten as

$$\boldsymbol{u}_{t_n}=\begin{pmatrix}p\\q\end{pmatrix}_{t_n}=\boldsymbol{J}\boldsymbol{L}^n\begin{pmatrix}b_0^{(1)}-\frac{q}{p^2}c_0\\ \frac{c_0}{p}\end{pmatrix}=\boldsymbol{J}\boldsymbol{L}^n\begin{pmatrix}0\\ \frac{1}{1+q}\end{pmatrix}. \tag{19}$$

Taking $n=0,q=0,p=u$, we find that the system (19) reduces to the discrete equation $u_t=u_{n+1}$, it is an important equation in mathematical physics.

3. The Hamiltonian structure of the system (19)

Define

$$\begin{aligned}\boldsymbol{V}=&\boldsymbol{\Gamma}\boldsymbol{U}^{-1}=\left((1+q)a-\frac{pb}{\lambda}\right)\boldsymbol{e}_1\\ &-\frac{1}{\lambda}\left(a+\frac{qc}{p}\right)\boldsymbol{e}_2+\left(\frac{b}{\lambda^2}-\frac{qa}{\lambda p}\right)\boldsymbol{e}_3\\ &+(c+qc+pa)\,\boldsymbol{e}_4,\end{aligned} \tag{20}$$

and $<\boldsymbol{A},\boldsymbol{B}>=\mathrm{Tr}(\boldsymbol{A}\boldsymbol{B})$, where $\boldsymbol{A},\boldsymbol{B}$ are the same order square matrices.

It is easy to calculate that $\mathrm{Tr}(\boldsymbol{e}_1\boldsymbol{e}_1)=\mathrm{Tr}(\boldsymbol{e}_2\boldsymbol{e}_2)=\mathrm{Tr}(\boldsymbol{e}_3\boldsymbol{e}_4)=\mathrm{Tr}(\boldsymbol{e}_4\boldsymbol{e}_3)=1$, the rest are zero, and

$$\frac{\partial U}{\partial\lambda}=(1+q)\boldsymbol{e}_2+p\boldsymbol{e}_4,$$
$$\frac{\partial U}{\partial p}=\lambda\boldsymbol{e}_4-\frac{q}{p^2}\boldsymbol{e}_3,$$
$$\frac{\partial U}{\partial q}=\lambda\boldsymbol{e}_2+\frac{1}{p}\boldsymbol{e}_3.$$

So, we obtain

$$\left\langle\boldsymbol{V},\frac{\partial\boldsymbol{U}}{\partial\lambda}\right\rangle=-\frac{a^{(1)}}{\lambda},$$
$$\left\langle\boldsymbol{V},\frac{\partial\boldsymbol{U}}{\partial p}\right\rangle=b^{(1)}-\frac{qc}{p^2},$$
$$\left\langle\boldsymbol{V},\frac{\partial\boldsymbol{U}}{\partial q}\right\rangle=\frac{c}{p}.$$

From the trace identity

$$\frac{\delta}{\delta u}\sum_{n\in Z}\left\langle\boldsymbol{V},\frac{\partial\boldsymbol{U}}{\partial\lambda}\right\rangle(n)=\lambda^{-\epsilon}\frac{\partial}{\partial\lambda}\lambda^{\epsilon}\left\langle\boldsymbol{V},\frac{\partial\boldsymbol{U}}{\partial u}\right\rangle, \tag{21}$$

we have

$$\frac{\delta}{\delta u}\sum_{n\in Z}\left(-\frac{a^{(1)}}{\lambda}\right)(n)=\lambda^{-\epsilon}\frac{\partial}{\partial\lambda}\lambda^{\epsilon}\begin{pmatrix}b^{(1)}-\frac{qc}{p^2}\\ \frac{c}{p}\end{pmatrix}. \tag{22}$$

Comparing the coefficients of λ^{-n-1} we obtain

$$\frac{\delta}{\delta u}\sum_{n\in Z}(-a_n^{(1)})(n)=(\epsilon-n)\begin{pmatrix}b_n^{(1)}-\frac{q}{p^2}c_n\\ \frac{c_n}{p}\end{pmatrix}, \tag{23}$$

set $n=0$, then $\epsilon=0$. Therefore, we obtain

$$\begin{pmatrix}b_n^{(1)}-\frac{q}{p^2}c_n\\ \frac{c_n}{p}\end{pmatrix}=\frac{\delta H_n}{\delta u},$$

$$H_n=\sum_{n\in Z}\left(\frac{a_n^{(1)}}{n}\right)(n),\ n\geq 1,\ H_0=\ln(1+q). \tag{24}$$

So, the system (19) possesses the bi-Hamiltonian structure

$$\boldsymbol{u}_{t_n} = \boldsymbol{J}\frac{\delta H_n}{\delta u} = \boldsymbol{K}\frac{\delta H_{n-1}}{\delta u}, \tag{25}$$

where the Hamiltonian operator $\boldsymbol{K}$ is given by

$$\boldsymbol{K} = \boldsymbol{J}\boldsymbol{L} = \begin{pmatrix} K_{11} & K_{12} \\ K_{21} & K_{22} \end{pmatrix}. \tag{26}$$

In Eq.(26)

$$\begin{aligned}
K_{11} =& E\frac{p}{1+q}(E+1)(E-1)^{-1}\frac{p}{1+q}E^{-1},\\
K_{12} =& E\frac{1}{1+q}Ep - E\frac{p}{1+q}(E+1)(E-1)^{-1}\frac{q}{p(1+q)}Ep + E\frac{p}{1+q}(E+1)(E-1)^{-1}\frac{p}{1+q}E^{-1}\frac{q}{p},\\
K_{21} =& \frac{q}{p}E\frac{p}{1+q}(E+1)(E-1)^{-1}\frac{p}{1+q}E^{-1} - pE^{-1}\frac{1}{1+q}E^{-1} - pE^{-1}\frac{q}{p(1+q)}(E+1)(E-1)^{-1}\frac{p}{1+q}E^{-1},\\
K_{22} =& \frac{q}{p}E\frac{1}{1+q}Ep + pE^{-1}\frac{q}{p(1+q)}(E+1)(E-1)^{-1}\frac{q}{p(1+q)}Ep - pE^{-1}\frac{q}{p(1+q)}(E+1)(E-1)^{-1}\frac{p}{1+q}E^{-1}\frac{q}{p}\\
& - pE^{-1}\frac{1}{1+q}E^{-1}\frac{q}{p} + \frac{q}{p}E\frac{p}{1+q}(E+1)(E-1)^{-1}\frac{p}{1+q}E^{-1}\frac{q}{p} - \frac{q}{p}E\frac{p}{1+q}(E+1)(E-1)^{-1}\frac{q}{p(1+q)}Ep.
\end{aligned}$$

It is easy to verify that the Hamiltonian operator $\boldsymbol{K}$ is skew-symmetric, so the system (19) is integrable in the Liouville sense.

4. Integrable coupling of the system (19)

We construct the extending Lie algebra G_1 of G as

$$G_1 = \operatorname{span}\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, \boldsymbol{f}_4, \boldsymbol{f}_5, \boldsymbol{f}_6, \boldsymbol{f}_7, \boldsymbol{f}_8\}, \tag{27}$$

where

$$\begin{aligned}
&\boldsymbol{f}_1 = \begin{pmatrix} 1&0&0&0\\0&0&0&0\\0&0&1&0\\0&0&0&0 \end{pmatrix}, \quad \boldsymbol{f}_2 = \begin{pmatrix} 0&0&0&0\\0&1&0&0\\0&0&0&0\\0&0&0&1 \end{pmatrix}, \quad \boldsymbol{f}_3 = \begin{pmatrix} 0&1&0&0\\0&0&0&0\\0&0&0&1\\0&0&0&0 \end{pmatrix},\\
&\boldsymbol{f}_4 = \begin{pmatrix} 0&0&0&0\\1&0&0&0\\0&0&0&0\\0&0&1&0 \end{pmatrix}, \quad \boldsymbol{f}_5 = \begin{pmatrix} 0&0&1&0\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \boldsymbol{f}_6 = \begin{pmatrix} 0&0&0&0\\0&0&0&1\\0&0&0&0\\0&0&0&0 \end{pmatrix},\\
&\boldsymbol{f}_7 = \begin{pmatrix} 0&0&0&1\\0&0&0&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}, \quad \boldsymbol{f}_8 = \begin{pmatrix} 0&0&0&0\\0&0&1&0\\0&0&0&0\\0&0&0&0 \end{pmatrix}.
\end{aligned} \tag{28}$$

It is easy to see that G_1 is closed under the following matrix multiplication:

$$\begin{aligned}
&\boldsymbol{f}_1\boldsymbol{f}_1 = \boldsymbol{f}_3\boldsymbol{f}_4 = \boldsymbol{f}_1, \boldsymbol{f}_2\boldsymbol{f}_2 = \boldsymbol{f}_4\boldsymbol{f}_3 = \boldsymbol{f}_2, \boldsymbol{f}_1\boldsymbol{f}_3 = \boldsymbol{f}_3\boldsymbol{f}_2 = \boldsymbol{f}_3, \boldsymbol{f}_2\boldsymbol{f}_4 = \boldsymbol{f}_4\boldsymbol{f}_1 = \boldsymbol{f}_4,\\
&\boldsymbol{f}_1\boldsymbol{f}_5 = \boldsymbol{f}_3\boldsymbol{f}_8 = \boldsymbol{f}_5\boldsymbol{f}_1 = \boldsymbol{f}_7\boldsymbol{f}_4 = \boldsymbol{f}_5, \boldsymbol{f}_2\boldsymbol{f}_6 = \boldsymbol{f}_4\boldsymbol{f}_7 = \boldsymbol{f}_6\boldsymbol{f}_2 = \boldsymbol{f}_8\boldsymbol{f}_3 = \boldsymbol{f}_6,\\
&\boldsymbol{f}_1\boldsymbol{f}_7 = \boldsymbol{f}_3\boldsymbol{f}_6 = \boldsymbol{f}_5\boldsymbol{f}_3 = \boldsymbol{f}_7\boldsymbol{f}_2 = \boldsymbol{f}_7, \boldsymbol{f}_2\boldsymbol{f}_8 = \boldsymbol{f}_4\boldsymbol{f}_5 = \boldsymbol{f}_6\boldsymbol{f}_4 = \boldsymbol{f}_8\boldsymbol{f}_1 = \boldsymbol{f}_8.
\end{aligned} \tag{29}$$

Denote $G_2 = \operatorname{span}\{\boldsymbol{f}_1, \boldsymbol{f}_2, \boldsymbol{f}_3, \boldsymbol{f}_4\}, G_3 = \operatorname{span}\{\boldsymbol{f}_5, \boldsymbol{f}_6, \boldsymbol{f}_7, \boldsymbol{f}_8\}$, then we find G_2, G_3 are isomorphic to G, and $G_1 = G_2 \oplus G_3$, $[G_2, G_3] \subseteq G_3$, G_2 is an ideal of the Lie algebra G_1. Set $\boldsymbol{f}_i(n) = \boldsymbol{f}_i\lambda^n$, $i = 1, 2, \ldots, 8$, then

$\mathrm{span}\{\boldsymbol{f}_i(n)\}_{i=1}^{8} \equiv \widetilde{G}_1$ is an obvious loop algebra, by use of loop algebra $\widetilde{G}_1$ the discrete isospectral problem is established by

$$\begin{aligned} E\psi_n =&\boldsymbol{U}_n\psi_n, \quad \lambda_t = 0,\\ \boldsymbol{U}_n =&\boldsymbol{f}_1(0) + \boldsymbol{f}_5(0) + (1+q)\boldsymbol{f}_2(1) + \frac{q}{p}\boldsymbol{f}_3(0) + p\boldsymbol{f}_4(1)\\ &+ (1+s)\boldsymbol{f}_6(1) + \frac{s}{r}\boldsymbol{f}_7(0) + r\boldsymbol{f}_8(1). \end{aligned} \tag{30}$$

Let

$$\begin{aligned} \boldsymbol{\Gamma} = &\sum_{m\geq 0}(a_m\boldsymbol{f}_1(-m) - a_m\boldsymbol{f}_2(-m) + b_m\boldsymbol{f}_3(-m-1) + c_m\boldsymbol{f}_4(-m)\\ &+ d_m\boldsymbol{f}_5(-m) - d_m\boldsymbol{f}_6(-m) + g_m\boldsymbol{f}_7(-m-1) + h_m\boldsymbol{f}_8(-m)). \end{aligned} \tag{31}$$

Solving the stationary zero curvature equation

$$(E\boldsymbol{\Gamma})\boldsymbol{U}_n - \boldsymbol{U}_n\boldsymbol{\Gamma} = 0, \tag{32}$$

we obtain the recursion relation

$$\begin{aligned} &a_m^{(1)} - a_m + pb_m^{(1)} - \frac{q}{p}c_m = 0,\\ &\frac{q}{p}(a_m^{(1)} + a_m) - b_{m-1} + (1+q)b_m^{(1)} = 0,\\ &p(a_m^{(1)} + a_m) - c_{m-1}^{(1)} + (1+q)c_m = 0,\\ &\frac{q}{p}c_{m-1}^{(1)} - pb_{m-1} + (1+q)(a_m - a_m^{(1)}) = 0,\\ &a_m^{(1)} - a_m + d_m^{(1)} - d_m + pg_m^{(1)} + rb_m^{(1)} - \frac{q}{p}h_m - \frac{s}{r}c_m = 0,\\ &\frac{q}{p}h_{m-1}^{(1)} + \frac{s}{r}c_{m-1}^{(1)} - rb_{m-1} - pg_{m-1} + (1+s)(a_m - a_m^{(1)}) + (1+q)(d_m - d_m^{(1)}) = 0,\\ &\frac{q}{p}(d_m^{(1)} + d_m) + \frac{s}{r}(a_m^{(1)} + a_m) - b_{m-1} - g_{m-1} + (1+s)b_m^{(1)} + (1+q)g_m^{(1)} = 0,\\ &r(a_m^{(1)} + a_m) + p(d_m^{(1)} + d_m) - c_{m-1}^{(1)} - h_{m-1}^{(1)} + (1+q)h_m + (1+s)c_m = 0. \end{aligned} \tag{33}$$

We choose $a_0 = -\dfrac{1}{2}$, $b_0 = \dfrac{q^{(-1)}}{p^{(-1)}(1+q^{(-1)})}$, $c_0 = \dfrac{p}{1+q}$, $d_0 = 0$, $g_0 = \dfrac{s^{(-1)}}{r^{(-1)}(1+q^{(-1)})} - \dfrac{q^{(-1)}(1+s^{(-1)})}{p^{(-1)}(1+q^{(-1)})^2}$, $h_0 = \dfrac{r}{1+q} - \dfrac{(1+s)p}{(1+q)^2}$ and require $a_j\,|_{[u]=0} = 0$, $b_j\,|_{[u]=0} = 0$, $c_j\,|_{[u]=0} = 0$, $d_j\,|_{[u]=0} = 0$, $g_j\,|_{[u]=0} = 0$, $h_j|_{[u]=0} = 0$, $(j \geq 1)$, where $[u] = (u, Eu, E^{-1}u, \ldots)$. Thus, the recursion relation (33) uniquely determines a_j, b_j, c_j, d_j, g_j, h_j, $j \geq 1$.

For any integer $m \geq 0$, we choose

$$\begin{aligned} \boldsymbol{\Gamma} = &\sum_{m\geq 0}(a_m\boldsymbol{f}_1(n-m) - a_m\boldsymbol{f}_2(n-m) + b_m\boldsymbol{f}_3(n-m-1) + c_m\boldsymbol{f}_4(n-m)\\ &+ d_m\boldsymbol{f}_5(n-m) - d_m\boldsymbol{f}_6(n-m) + g_m\boldsymbol{f}_7(n-m-1) + h_m\boldsymbol{f}_8(n-m)), \end{aligned} \tag{34}$$

then direct computation gives

$$\begin{aligned} (E\boldsymbol{\Gamma}_+)\boldsymbol{U}_n - \boldsymbol{U}_n\boldsymbol{\Gamma}_+ =& \left(\frac{q}{p}c_n^{(1)} - pb_n\right)\boldsymbol{f}_2(1) - b_n\boldsymbol{f}_3(0) + c_n^{(1)}\boldsymbol{f}_4(1) - (b_n + g_n)\boldsymbol{f}_7(0)\\ &+ (c_n^{(1)} + h_n^{(1)})\boldsymbol{f}_8(1) + \left(\frac{q}{p}h_n^{(1)} + \frac{s}{r}c_n^{(1)} - pg_n - rb_n\right)\boldsymbol{f}_6(1). \end{aligned} \tag{35}$$

Take $\boldsymbol{\Gamma} = \boldsymbol{\Gamma}_+$, then the discrete zero curvature equation admits the following equations:

$$
\begin{aligned}
(p)_{t_m} &= c_n^{(1)}, \\
(q)_{t_m} &= \frac{q}{p} c_n^{(1)} - p b_n, \\
(r)_{t_m} &= c_n^{(1)} + h_n^{(1)}, \\
(s)_{t_m} &= \frac{q}{p} h_n^{(1)} + \frac{s}{r} c_n^{(1)} - p g_n - r b_n. \qquad (36)
\end{aligned}
$$

When we take $r = s = 0$, the system (36) reduces to the system (19). Therefore, in term of the definition of integrable coupling, we conclude that the system (36) is an integrable coupling of the system (19).

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