Vol 18 No 3, March 2009
1674-1056/2009/18(03)/0845-05

Chinese Physics B

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Multi-component Harry–Dym hierarchy and its integrable couplings as well as their Hamiltonian structures*

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(received 27 August 2007; revised manuscript received 28 July 2008)

This paper obtains the multi-component Harry–Dym (H–D) hierarchy and its integrable couplings by using two kinds of vector loop algebras $\widetilde{G}_3$ and $\widetilde{G}_6$. The Hamiltonian structures of the above system are given by the quadratic-form identity. The method can be used to produce the Hamiltonian structures of the other integrable couplings or multi-component hierarchies.

Keywords: Hamiltonian structure, H–D hierarchy, quadratic-form identity
PACC: 0220, 0340K, 0365G

1. Introduction

A central and important topic in the study of integrable systems is searching for new Lax or Liouville systems as many as possible and such that they are associated with certain evolution equations with physical meaning. Professor Tu once proposed a simple and efficient method for generating soliton equation hierarchy and Hamiltonian structure,[1] which was called the Tu model by Professor Ma in Ref.[2]. Under the frame of the Tu model, a lot of integrable system associated with physics backgrounds are obtained,[1−6] such as the Ablowitz–Kaup–Newell–Segur hierarchy, Kaup–Newell hierarchy and so on. With the development of soliton theory, integrable couplings become an important and hot topic. By use of semi-direct sum of Lie algebras some multi-component integrable couplings such as the multi-component Burgers hierarchy, Guo hierarchy, etc., are worked out in Refs.[4,7–10]. In this paper, we first constructed a vector loop algebra and its expanding loop algebra, then obtained multi-component Harry–Dym (H–D) hierarchy and its couplings. Their Hamiltonian structures are presented by quadratic-form identity.

2. Multi-component H–D hierarchy

Set two row vectors $\boldsymbol{a} = (a_1, a_2, \ldots, a_m)$, $\boldsymbol{b} = (b_1, b_2, \ldots, b_m)$ and assume that c is constant, define their multiplication and division as

$$\begin{aligned}\boldsymbol{ab} &= (a_1b_1, a_2b_2, \ldots, a_mb_m),\\ \frac{\boldsymbol{a}}{\boldsymbol{b}} &= \left(\frac{a_1}{b_1}, \frac{a_2}{b_2}, \ldots, \frac{a_m}{b_m}\right),\\ \frac{\boldsymbol{c}}{\boldsymbol{a}} &= \left(\frac{c}{a_1}, \frac{c}{a_2}, \ldots, \frac{c}{a_m}\right),\end{aligned}$$

set $G_3 = \{\boldsymbol{a} = (a_{ij})_{3\times M} = (\boldsymbol{a}_1,\ \boldsymbol{a}_2,\ \boldsymbol{a}_3)^{\mathrm{T}},\ \boldsymbol{a}_i = (a_{i1},\ a_{i2}, \ldots, a_{im})\}$.

The commutative operation $[\boldsymbol{a}, \boldsymbol{b}]$ is defined as

$$\begin{aligned}[\boldsymbol{a}, \boldsymbol{b}]_1 =&(\boldsymbol{a}_2\boldsymbol{b}_3 - \boldsymbol{a}_3\boldsymbol{b}_2, 2(\boldsymbol{a}_1\boldsymbol{b}_2 - \boldsymbol{a}_2\boldsymbol{b}_1),\\ &2(\boldsymbol{a}_3\boldsymbol{b}_1 - \boldsymbol{a}_1\boldsymbol{b}_3))^{\mathrm{T}}.\end{aligned} \tag{1}$$

It is easy to test that G_3 is a Lie algebra. The following loop algebra is given as

$$\begin{aligned}\widetilde{G}_3 =&\{\boldsymbol{a}\lambda^m, \boldsymbol{a} \in G_3, [\boldsymbol{a}\lambda^m, \boldsymbol{b}\lambda^n]_1\\ =&[\boldsymbol{a}, \boldsymbol{b}]_1\lambda^{m+n}, \forall \boldsymbol{a}, \boldsymbol{b} \in G_3\}.\end{aligned} \tag{2}$$

Taking the following isospectral problem:

$$\begin{aligned}&\boldsymbol{\psi}_x = [\boldsymbol{U}, \boldsymbol{\psi}]_1,\ \boldsymbol{U}, \boldsymbol{V}, \boldsymbol{\psi} \in \tilde{G}_3,\\ &\boldsymbol{\psi}_t = [\boldsymbol{V}, \boldsymbol{\psi}]_1, \lambda_t = 0,\end{aligned} \tag{3}$$

where

$$\boldsymbol{U} = (-\mathrm{i}\lambda\boldsymbol{I}_M, (\boldsymbol{q} - \boldsymbol{I}_M)\lambda, -\lambda\boldsymbol{I}_M)^{\mathrm{T}},$$

$$\boldsymbol{V} = \sum_{m\geq 0}(\lambda\boldsymbol{a}_m, \boldsymbol{b}_{mx} + \mathrm{i}\lambda(\boldsymbol{q} - \boldsymbol{I}_M)\boldsymbol{a}_m, \boldsymbol{c}_{mx} - \mathrm{i}\lambda\boldsymbol{a}_m)^{\mathrm{T}}\lambda^{-m},$$

$$\boldsymbol{I}_M = (1, 1, \ldots 1)_{1\times M}, \boldsymbol{q} = (q_1, q_2, \ldots, q_M)^{\mathrm{T}},$$

$$\boldsymbol{a}_m = (a_{m1}, a_{m2}, \ldots, a_{mM})$$

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$$\boldsymbol{b}_m = (b_{m1}, b_{m2}, \ldots, b_{mM}),$$
$$\boldsymbol{c}_m = (c_{m1}, c_{m2}, \ldots, c_{mM})$$

and solving

$$\boldsymbol{V}_x = [\boldsymbol{U}, \boldsymbol{V}]_1, \tag{4}$$

we obtain

$$\begin{aligned}
&\boldsymbol{a}_{mx} = (\boldsymbol{q} - \boldsymbol{I}_M)\boldsymbol{c}_{mx} + \boldsymbol{b}_{mx},\\
&\mathrm{i}[(\boldsymbol{q} - \boldsymbol{I}_M)\boldsymbol{a}_{m+1}]_x + \boldsymbol{b}_{mxx} = -2\mathrm{i}\boldsymbol{b}_{m+1x},\\
&-\mathrm{i}\boldsymbol{a}_{m+1x} + \boldsymbol{c}_{mxx} = 2\mathrm{i}\boldsymbol{c}_{m+1x},\\
&\boldsymbol{a}_0 = -2\mathrm{i}\boldsymbol{q}^{-\frac{1}{2}},\\
&\boldsymbol{b}_0 = \mathrm{i}(\boldsymbol{q} - \boldsymbol{I}_M)\boldsymbol{q}^{-\frac{1}{2}},\\
&\boldsymbol{c}_0 = \mathrm{i}\boldsymbol{q}^{-\frac{1}{2}},\\
&\boldsymbol{a}_1 = -\frac{1}{2}\boldsymbol{q}^{-\frac{3}{2}}\boldsymbol{q}_x,\\
&\boldsymbol{b}_1 = -\frac{1}{2}\boldsymbol{q}^{-\frac{3}{2}}\boldsymbol{q}_x,\\
&\boldsymbol{c}_1 = 0.
\end{aligned} \tag{5}$$

For an arbitrary natural number n, denoting

$$\boldsymbol{V}_+^{(n)} = \sum_{m=0}^{n} \begin{pmatrix} \lambda\boldsymbol{a}_m \\ \boldsymbol{b}_{mx} + \mathrm{i}\lambda(\boldsymbol{q} - \boldsymbol{I}_M)\boldsymbol{a}_m \\ \boldsymbol{c}_{mx} - \mathrm{i}\lambda\boldsymbol{a}_m \end{pmatrix} \times \lambda^{n-m}, \quad \boldsymbol{V}_-^{(n)} = \lambda^n \boldsymbol{V} - \boldsymbol{V}_+^{(n)},$$

we can write Eq.(4) as

$$-\boldsymbol{V}_{+x}^{(n)} + [\boldsymbol{U}, \boldsymbol{V}_+^{(n)}]_1 = \boldsymbol{V}_{-x}^{(n)} - [\boldsymbol{U}, \boldsymbol{V}_-^{(n)}]_1. \tag{6}$$

The terms on the left-hand side of Eq.(6) are of degree ≥ 1, while the terms on the right-hand side are of degree ≤ 2. Therefore,

$$-\boldsymbol{V}_{+x}^{(n)} + [\boldsymbol{U}, \boldsymbol{V}_+^{(n)}]_1 = \begin{pmatrix} \boldsymbol{0} \\ -\boldsymbol{b}_{nxx} \\ -\boldsymbol{c}_{nxx} \end{pmatrix}.$$

Denoting $\boldsymbol{V}^{(n)} = \boldsymbol{V}_+^{(n)} + \boldsymbol{\Delta}_{n1}$, and $\boldsymbol{\Delta}_{n1} = (-\mathrm{i}\boldsymbol{c}_{nx}, (\boldsymbol{q} - \boldsymbol{I}_M)\boldsymbol{c}_{nx} - \mathrm{i}\boldsymbol{c}_{n-1xx}, -\boldsymbol{c}_{nx})^{\mathrm{T}}$, from the following zero-curvature equation

$$\boldsymbol{U}_t - \boldsymbol{V}_x^{(n)} + [\boldsymbol{U}, \boldsymbol{V}^{(n)}]_1 = 0, \tag{7}$$

we obtain the integrable system

$$\boldsymbol{q}_t = -2\mathrm{i}\boldsymbol{c}_{n-1xxx} = -2\mathrm{i}\partial^3\boldsymbol{c}_{n-1} = J_1\boldsymbol{c}_{n-1}, \tag{8}$$

where J_1 is a Hamiltonian operator.

From expression (5), we obtain a recurrence operator

$$L_1 = -\frac{1}{2}(\partial\boldsymbol{q} + \boldsymbol{q}\partial)^{-1}\partial^3$$

which meets $\boldsymbol{c}_{n+2} = L_1\boldsymbol{c}_n$. Therefore, expression (8) can be written as

$$\boldsymbol{q}_t = \begin{cases} J_1 L_1^{\frac{n}{2}}\boldsymbol{c}_1, & \text{when } n \text{ is even,} \\ J_1 L_1^{\frac{n-1}{2}}\boldsymbol{c}_0, & \text{when } n \text{ is odd.} \end{cases} \tag{9}$$

Taking $M = 1$, $n = 1$, we find that the system (9) reduces to the H–D equation, so system (9) is called a multi-component H–D hierarchy. Then it is easy to verify that $J_1 L_1 = L_1^* J_1$, therefore, the system (9) is integrable in the Liouville sense.

3. The Hamiltonian structure of system (9)

Introduce a quadratic-form identity functional

$$\{\boldsymbol{a}, \boldsymbol{b}\}_1 = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{F}\boldsymbol{b}, \qquad \boldsymbol{a}, \boldsymbol{b} \in \tilde{V}_{\mathrm{s}}, \tag{10}$$

and consider a functional

$$\begin{aligned}
&\boldsymbol{W} = \{\boldsymbol{V}, \boldsymbol{U}_\lambda\} + \{\boldsymbol{\Lambda}, \boldsymbol{V}_x - [\boldsymbol{U}, \boldsymbol{V}]\},\\
&\boldsymbol{U}, \boldsymbol{V}, \boldsymbol{\Lambda} \in \tilde{V}_{\mathrm{s}},
\end{aligned} \tag{11}$$

which has the following variation constraint conditions:

$$\nabla_\Lambda \boldsymbol{W} = \boldsymbol{V}_x - [\boldsymbol{U}, \boldsymbol{V}] = 0, \tag{12}$$
$$\nabla_V \boldsymbol{W} = \boldsymbol{U}_\lambda - \boldsymbol{\Lambda}_x + [\boldsymbol{U}, \boldsymbol{\Lambda}] = 0. \tag{13}$$

From expressions (1), (2) and (3), we obtain

$$[\boldsymbol{\Lambda}, \boldsymbol{V}] - \boldsymbol{V}_\lambda = \frac{\gamma}{\lambda}\boldsymbol{V}, \quad \gamma = \text{const.}, \tag{14}$$

then we obtain

$$\frac{\delta}{\delta u_i}\{\boldsymbol{V}, \boldsymbol{U}_\lambda\} = \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\left(\lambda^{-\gamma}\{\boldsymbol{V}, \frac{\partial\boldsymbol{U}}{\partial u_i}\}\right), \quad (1 \leq i \leq l), \tag{15}$$

where γ is a constant to be determined.

From $[\boldsymbol{a}, \boldsymbol{b}]_1$, we have

$$[\boldsymbol{a}, \boldsymbol{b}]_1^{\mathrm{T}} = \boldsymbol{a}^{\mathrm{T}} \begin{pmatrix} 0 & 2\boldsymbol{b}_2 & -2\boldsymbol{b}_3 \\ \boldsymbol{b}_3 & -2\boldsymbol{b}_1 & 0 \\ -\boldsymbol{b}_2 & 0 & 2\boldsymbol{b}_1 \end{pmatrix} = \boldsymbol{a}^{\mathrm{T}}\boldsymbol{R}(\boldsymbol{b}). \tag{16}$$

Solving the matrix equation $\boldsymbol{F}_1^{\mathrm{T}} = \boldsymbol{F}_1$ and $\boldsymbol{R}(\boldsymbol{b})\boldsymbol{F}_1 = -(\boldsymbol{R}(\boldsymbol{b})\boldsymbol{F}_1)^{\mathrm{T}}$ we obtain

$$\boldsymbol{F}_1 = \begin{pmatrix} 2\boldsymbol{I}_M & 0 & 0 \\ 0 & 0 & \boldsymbol{I}_M \\ 0 & \boldsymbol{I}_M & 0 \end{pmatrix}. \tag{17}$$

Then

$$\{\boldsymbol{a},\boldsymbol{b}\}_1=\boldsymbol{a}^{\mathrm{T}}\boldsymbol{F}_1\boldsymbol{b}=2\boldsymbol{a}_1\boldsymbol{b}_1+\boldsymbol{a}_2\boldsymbol{b}_3+\boldsymbol{a}_3\boldsymbol{b}_2. \tag{18}$$

Hence, by using the quadratic-form identity we obtain

$$\begin{aligned}&\frac{\delta}{\delta u}[-2\mathrm{i}\boldsymbol{q}\lambda\boldsymbol{a}-\boldsymbol{b}_x+(\boldsymbol{q}-\boldsymbol{I}_M)\boldsymbol{c}_x]\\ =&\lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma}\left[(-\mathrm{i}\lambda^2\boldsymbol{a}+\lambda\boldsymbol{c}_x)\right].\end{aligned} \tag{19}$$

Comparing the coefficients of λ^{-n+2} we obtain

$$\begin{aligned}&\frac{\delta}{\delta u}[2\mathrm{i}(\boldsymbol{q}-\boldsymbol{I}_M)\boldsymbol{c}_{n-1}+2\mathrm{i}\boldsymbol{b}_{n-1}-2\mathrm{i}\boldsymbol{a}_{n-1}]\\ =&2\mathrm{i}(-n+3+\gamma)\boldsymbol{c}_{n-1}.\end{aligned}$$

Taking $n=2$ in the above equation we obtain $\gamma=-1$. Thus, the Hamiltonian structure of expression (9) is derived as follows:

$$\boldsymbol{u}_t=\boldsymbol{q}_t=J_1\frac{\delta H_n}{\delta u}, \tag{20}$$

where

$$H_n=\frac{(\boldsymbol{q}-\boldsymbol{I}_M)\boldsymbol{c}_{n-1}+\boldsymbol{b}_{n-1}-\boldsymbol{a}_{n-1}}{-n+2}.$$

4. Integrable coupling of the multi-component H–D hierarchy and its Hamiltonian structure

Consider a linear space

$$\begin{aligned}&G_6=\{\boldsymbol{a}=(\boldsymbol{a}_1,\boldsymbol{a}_2,\boldsymbol{a}_3,\boldsymbol{a}_4,\boldsymbol{a}_5,\boldsymbol{a}_6)^{\mathrm{T}},\\ &\boldsymbol{a}_i=(a_{i1},a_{i2},\ldots,a_{iM})\}\end{aligned}$$

with the commuting relations

$$\begin{aligned}[\boldsymbol{a},\boldsymbol{b}]_2=&(\boldsymbol{a}_2\boldsymbol{b}_3-\boldsymbol{a}_3\boldsymbol{b}_2,2(\boldsymbol{a}_1\boldsymbol{b}_2-\boldsymbol{a}_2\boldsymbol{b}_1),2(\boldsymbol{a}_3\boldsymbol{b}_1-\boldsymbol{a}_1\boldsymbol{b}_3),\\ &\boldsymbol{a}_2\boldsymbol{b}_6-\boldsymbol{a}_6\boldsymbol{b}_2+\boldsymbol{a}_5\boldsymbol{b}_3-\boldsymbol{a}_3\boldsymbol{b}_5,\\ &2(\boldsymbol{a}_4\boldsymbol{b}_2-\boldsymbol{a}_2\boldsymbol{b}_4+\boldsymbol{a}_1\boldsymbol{b}_5-\boldsymbol{a}_5\boldsymbol{b}_1),\\ &2(\boldsymbol{a}_3\boldsymbol{b}_4-\boldsymbol{a}_4\boldsymbol{b}_3+\boldsymbol{a}_6\boldsymbol{b}_1-\boldsymbol{a}_1\boldsymbol{b}_6))^{\mathrm{T}}.\end{aligned} \tag{21}$$

It is easy to verify that G_6 is a Lie algebra, which derives the resulting loop algebra

$$\begin{aligned}\tilde{G}_6=\{\boldsymbol{a}(\lambda)=&\boldsymbol{a}\lambda^m,\boldsymbol{a}\in G_6,[\boldsymbol{a}\lambda^m,\boldsymbol{b}\lambda^n]_2\\ =&[\boldsymbol{a},\boldsymbol{b}]_2\lambda^{2m+2n},\boldsymbol{a},\boldsymbol{b}\in G_6\}.\end{aligned} \tag{22}$$

Take the following isospectral problem

$$\begin{aligned}&\boldsymbol{\psi}_x=[\boldsymbol{U},\boldsymbol{\psi}]_2,\ \boldsymbol{U},\boldsymbol{V},\boldsymbol{\psi}\in\tilde{G}_6,\\ &\boldsymbol{\psi}_t=[\boldsymbol{V},\boldsymbol{\psi}]_2,\ \lambda_t=0,\end{aligned} \tag{23}$$

where

$$\begin{aligned}\boldsymbol{U}=&(-\mathrm{i}\lambda\boldsymbol{I}_M,(\boldsymbol{u}_1-\boldsymbol{I}_M)\lambda,-\lambda\boldsymbol{I}_M,\\ &-\mathrm{i}\lambda\boldsymbol{I}_M,(\boldsymbol{u}_2-\boldsymbol{I}_M)\lambda,-\lambda\boldsymbol{I}_M)^{\mathrm{T}},\\ \boldsymbol{V}=&\sum_{m\geq0}(\lambda\boldsymbol{a}_m,\boldsymbol{b}_{mx}+\mathrm{i}\lambda(\boldsymbol{u}_1-\boldsymbol{I}_M)\boldsymbol{a}_m,\\ &\boldsymbol{c}_{mx}-\mathrm{i}\lambda\boldsymbol{a}_m,\lambda\boldsymbol{d}_m,\boldsymbol{f}_{mx}+\mathrm{i}\lambda(\boldsymbol{u}_2-\boldsymbol{I}_M)\boldsymbol{d}_m,\\ &\boldsymbol{g}_{mx}-\mathrm{i}\lambda\boldsymbol{d}_m)^{\mathrm{T}}\lambda^{-m},\end{aligned}$$

$$\begin{aligned}&\boldsymbol{u}_1=(u_{11},u_{12},\ldots,u_{1M})^{\mathrm{T}},\\ &\boldsymbol{u}_2=(u_{21},u_{22},\ldots,u_{2M})^{\mathrm{T}},\\ &\boldsymbol{a}_m=(a_{m1},a_{m2},\ldots,a_{mM}),\\ &\boldsymbol{b}_m=(b_{m1},b_{m2},\ldots,b_{mM}),\\ &\boldsymbol{c}_m=(c_{m1},c_{m2},\ldots,c_{mM}),\\ &\boldsymbol{d}_m=(d_{m1},d_{m2},\ldots,d_{mM}),\\ &\boldsymbol{f}_m=(f_{m1},f_{m2},\ldots,f_{mM}),\\ &\boldsymbol{g}_m=(g_{m1},g_{m2},\ldots,g_{mM}).\end{aligned}$$

Solving

$$\boldsymbol{V}_x=[\boldsymbol{U},\boldsymbol{V}]_2, \tag{24}$$

we obtain

$$\begin{aligned}&\boldsymbol{a}_{mx}=(\boldsymbol{u}_1-1)\boldsymbol{c}_{mx}+\boldsymbol{b}_{mx},\\ &\mathrm{i}[(\boldsymbol{u}_1-1)\boldsymbol{a}_{m+1}]_x+\boldsymbol{b}_{mxx}=-2\mathrm{i}\boldsymbol{b}_{m+1x},\\ &-\mathrm{i}\boldsymbol{a}_{m+1x}+\boldsymbol{c}_{mxx}=2\mathrm{i}\boldsymbol{c}_{m+1x},\\ &\boldsymbol{d}_{mx}=(\boldsymbol{u}_1-1)\boldsymbol{g}_{mx}+\boldsymbol{f}_{mx}+\mathrm{i}(\boldsymbol{u}_2-\boldsymbol{u}_1)\boldsymbol{d}_{m+1}+\mathrm{i}(\boldsymbol{u}_1-\boldsymbol{u}_2)\boldsymbol{a}_{m+1}+(\boldsymbol{u}_2-1)\boldsymbol{c}_{mx}+\boldsymbol{b}_{mx},\\ &\boldsymbol{f}_{mxx}+\mathrm{i}[(\boldsymbol{u}_2-1)\boldsymbol{d}_{m+1}]_x=-2\mathrm{i}\boldsymbol{f}_{m+1x}-2\mathrm{i}\boldsymbol{b}_{m+1x}+2(\boldsymbol{u}_2-\boldsymbol{u}_1)\boldsymbol{d}_{m+2}-2(\boldsymbol{u}_2-u_1)\boldsymbol{a}_{m+2},\\ &\boldsymbol{g}_{mxx}-\mathrm{i}\boldsymbol{d}_{m+1x}=2\mathrm{i}\boldsymbol{g}_{m+1x}+2\mathrm{i}\boldsymbol{c}_{m+1x},\end{aligned}$$

$$\begin{aligned}
&\boldsymbol{a}_0=-2\mathrm{i}\boldsymbol{u}_1^{-\frac{1}{2}},\ \ \boldsymbol{b}_0=\mathrm{i}(\boldsymbol{u}_1-1)\boldsymbol{u}_1^{-\frac{1}{2}},\ \ \boldsymbol{c}_0=\mathrm{i}\boldsymbol{u}_1^{-\frac{1}{2}},\ \ \boldsymbol{d}_0=\mathrm{i}\boldsymbol{u}_1^{\frac{1}{2}},\\
&\boldsymbol{f}_0=-\mathrm{i}\boldsymbol{u}_1^{\frac{1}{2}},\ \ \boldsymbol{g}_0=-2\mathrm{i}\boldsymbol{u}_1^{-\frac{1}{2}}+\boldsymbol{\alpha},\ \ \boldsymbol{a}_1=-\frac{1}{2}\boldsymbol{u}_1^{-\frac{3}{2}}\boldsymbol{u}_{1x},\ \ \boldsymbol{b}_1=-\frac{1}{2}\boldsymbol{u}_1^{-\frac{3}{2}}\boldsymbol{u}_{1x},\\
&\boldsymbol{c}_1=\beta,\ \ \boldsymbol{d}_1=0,\ \ \boldsymbol{f}_1=\boldsymbol{u}_1^{-\frac{1}{2}}\boldsymbol{u}_{1x}+\boldsymbol{u}_2^{\frac{1}{2}}\boldsymbol{u}_{2x},\ \ \boldsymbol{g}_1=-2\boldsymbol{u}_1^{-\frac{3}{2}}\boldsymbol{u}_{1x},
\end{aligned}\tag{25}$$

where $\boldsymbol{\alpha}=(\alpha_1,\alpha_2,...,\alpha_M)$ and $\boldsymbol{\beta}=(\beta_1,\beta_2,...,\beta_M)$ are constants.
And for an arbitrary natural number n, denoting

$$\boldsymbol{V}_+^{(n)}=\sum_{m=0}^{n}\begin{pmatrix}\boldsymbol{\lambda}a_m\\ \boldsymbol{b}_{mx}+\mathrm{i}\lambda(\boldsymbol{u}_1-\boldsymbol{I}_M)\boldsymbol{a}_m\\ \boldsymbol{c}_{mx}-\mathrm{i}\lambda\boldsymbol{a}_m\\ \lambda\boldsymbol{d}_m\\ \boldsymbol{f}_{mx}+\mathrm{i}\lambda(\boldsymbol{u}_2-\boldsymbol{I}_M)\boldsymbol{d}_m\\ \boldsymbol{g}_{mx}-\mathrm{i}\lambda\boldsymbol{d}_m\end{pmatrix}\lambda^{2n-2m},\ \boldsymbol{V}_-^{(n)}=\lambda^n\boldsymbol{V}-\boldsymbol{V}_+^{(n)},$$

we can write Eq.(4) as

$$-\boldsymbol{V}_{+x}^{(n)}+[\boldsymbol{U},\boldsymbol{V}_+^{(n)}]_2=\boldsymbol{V}_{-x}^{(n)}-[\boldsymbol{U},\boldsymbol{V}_-^{(n)}]_2,\tag{26}$$

where the terms on the left-hand side of Eq.(26) are of degree ≥ 1, while the terms on the right-hand side are of degree ≤ 2. Therefore,

$$-\boldsymbol{V}_{+x}^{(n)}+[\boldsymbol{U},\boldsymbol{V}_+^{(n)}]_2=\begin{pmatrix}\boldsymbol{0}\\ -\lambda\boldsymbol{b}_{nxx}\\ -\lambda\boldsymbol{c}_{nxx}\\ \mathrm{i}\lambda^2(\boldsymbol{u}_1-\boldsymbol{u}_2)(\boldsymbol{d}_{n+1}-\boldsymbol{a}_{n+1})\\ 2\lambda^2(\boldsymbol{u}_1-\boldsymbol{u}_2)(\boldsymbol{d}_{n+1}-\boldsymbol{a}_{n+1})-\boldsymbol{f}_{n-1xx}\\ -\lambda\boldsymbol{g}_{nxx}\end{pmatrix}.$$

Denoting $\boldsymbol{V}^{(n)}=\boldsymbol{V}_+^{(n)}+\boldsymbol{\Delta}_{n2}$, and $\boldsymbol{\Delta}_{n2}=(-\mathrm{i}\lambda\boldsymbol{c}_{nx},\lambda[(\boldsymbol{u}_1-\boldsymbol{I}_M)\boldsymbol{c}_{nx}-\mathrm{i}\boldsymbol{c}_{n-1xx}],\ -\lambda\boldsymbol{c}_{nx},\ -\mathrm{i}\lambda\boldsymbol{g}_{nx},\lambda[(\boldsymbol{u}_2-\boldsymbol{u}_1)(\boldsymbol{g}_{nx}-2\mathrm{i}\boldsymbol{g}_{n+1})+\mathrm{i}(\boldsymbol{c}_{n-1xx}-\boldsymbol{g}_{n-1xx})+(\boldsymbol{u}_1-\boldsymbol{I}_M)\boldsymbol{g}_{nx}],-\lambda\boldsymbol{g}_{nx})^{\mathrm{T}}$, from the following zero-curvature equation

$$\boldsymbol{U}_t-\boldsymbol{V}_x^{(n)}+[\boldsymbol{U},\boldsymbol{V}^{(n)}]_2=0,\tag{27}$$

we obtain an integrable system

$$\begin{aligned}\boldsymbol{U}_t=\begin{pmatrix}\boldsymbol{u}_1\\ \boldsymbol{u}_2\end{pmatrix}_t&=\begin{pmatrix}-2\mathrm{i}\boldsymbol{c}_{n-1xx}\\ \boldsymbol{k}+2\boldsymbol{g}_{nx}+\boldsymbol{f}_{nx}\end{pmatrix}_x=\begin{pmatrix}\boldsymbol{0}&-2\mathrm{i}\boldsymbol{I}_M\partial^3\\ -2\mathrm{i}\boldsymbol{I}_M\partial^3&6\mathrm{i}\boldsymbol{I}_M\partial^3\end{pmatrix}\begin{pmatrix}\boldsymbol{g}_{n-1}+2\boldsymbol{c}_{n-1}\\ \boldsymbol{c}_{n-1}\end{pmatrix}\\
&=\boldsymbol{J}_2\begin{pmatrix}\boldsymbol{g}_{n-1}+2\boldsymbol{c}_{n-1}\\ \boldsymbol{c}_{n-1}\end{pmatrix},\end{aligned}\tag{28}$$

where $\boldsymbol{k}=(\boldsymbol{u}_2-\boldsymbol{u}_1)(\boldsymbol{g}_{nx}-2\mathrm{i}\boldsymbol{g}_{n+1})+\mathrm{i}(\boldsymbol{c}_{n-1xx}-\boldsymbol{g}_{n-1xx})+(\boldsymbol{u}_1-\boldsymbol{I}_M)\boldsymbol{g}_{nx}$.
From expression (25), we obtain a recurrence operator

$$\boldsymbol{L}_2=\begin{pmatrix}-(2\partial\boldsymbol{u}_1+\boldsymbol{u}_1\partial)^{-1}\partial^3&\boldsymbol{A}\\ 0&-(2\partial\boldsymbol{u}_1+\boldsymbol{u}_1\partial)^{-1}\partial^3\end{pmatrix},$$

where $\boldsymbol{A}=(2\partial\boldsymbol{u}_1+4\boldsymbol{u}_1\partial)^{-1}(2\partial\boldsymbol{u}_2+4\boldsymbol{u}_2\partial)(2\partial\boldsymbol{u}_1+4\boldsymbol{u}_1\partial)^{-1}\partial^3+(2\partial\boldsymbol{u}_1+\boldsymbol{u}_1\partial)^{-1}\partial^3$, which meets

$$\begin{pmatrix}\boldsymbol{g}_{n+2}+2\boldsymbol{c}_{n+2}\\ \boldsymbol{c}_{n+2}\end{pmatrix}=\boldsymbol{L}_2\begin{pmatrix}\boldsymbol{g}_n+2\boldsymbol{c}_n\\ \boldsymbol{c}_n\end{pmatrix}.$$

Therefore, expression (28) can be written as

$$\boldsymbol{u}_t = \begin{pmatrix} \boldsymbol{u}_1 \\ \boldsymbol{u}_2 \end{pmatrix}_t = \begin{cases} \boldsymbol{J}\boldsymbol{L}^{\frac{n}{2}-1} \begin{pmatrix} \boldsymbol{g}_1 + 2\boldsymbol{c}_1 \\ \boldsymbol{c}_1 \end{pmatrix}, & \text{when } n \text{ is oven,} \\ \boldsymbol{J}\boldsymbol{L}^{\frac{n-1}{2}} \begin{pmatrix} \boldsymbol{g}_0 + 2\boldsymbol{c}_0 \\ \boldsymbol{c}_0 \end{pmatrix}, & \text{when } n \text{ is odd .} \end{cases} \tag{29}$$

From the expression (21), we have

$$\begin{aligned}[\boldsymbol{a}, \boldsymbol{b}]_2^{\mathrm{T}} =& \boldsymbol{a}^{\mathrm{T}} \begin{pmatrix} 0 & 2\boldsymbol{b}_2 & -2\boldsymbol{b}_3 & 0 & 2\boldsymbol{b}_5 & -2\boldsymbol{b}_6 \\ \boldsymbol{b}_3 & -2\boldsymbol{b}_1 & 0 & \boldsymbol{b}_6 & -2\boldsymbol{b}_4 & 0 \\ -\boldsymbol{b}_2 & 0 & 2\boldsymbol{b}_1 & -\boldsymbol{b}_5 & 0 & 2\boldsymbol{b}_4 \\ 0 & 0 & 0 & 0 & 0 & -2\boldsymbol{b}_3 \\ 0 & 0 & 0 & -\boldsymbol{b}_3 & 2\boldsymbol{b}_2 & 0 \\ 0 & 0 & 0 & -\boldsymbol{b}_2 & -2\boldsymbol{b}_1 & 2\boldsymbol{b}_1 \end{pmatrix} \\ =& \boldsymbol{a}^{\mathrm{T}} \boldsymbol{R}(\boldsymbol{b}). \end{aligned} \tag{30}$$

Solving the matrix equation $\boldsymbol{F}^{\mathrm{T}} = \boldsymbol{F}$ and $\boldsymbol{R}(\boldsymbol{b})\boldsymbol{F} = -(\boldsymbol{R}(\boldsymbol{b})\boldsymbol{F})^{\mathrm{T}}$ we obtain

$$\boldsymbol{F} = \begin{pmatrix} 2\boldsymbol{I}_M & 0 & 0 & 2\boldsymbol{I}_M & 0 & 0 \\ 0 & 0 & \boldsymbol{I}_M & 0 & 0 & \boldsymbol{I}_M \\ 0 & \boldsymbol{I}_M & 0 & 0 & \boldsymbol{I}_M & 0 \\ 2\boldsymbol{I}_M & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \boldsymbol{I}_M & 0 & 0 & 0 \\ 0 & \boldsymbol{I}_M & 0 & 0 & 0 & 0 \end{pmatrix}, \tag{31}$$

$$\begin{aligned}\{\boldsymbol{a}, \boldsymbol{b}\}_2 =& 2(\boldsymbol{a}_1 + \boldsymbol{a}_4)\boldsymbol{b}_1 + (\boldsymbol{a}_3 + \boldsymbol{a}_6)\boldsymbol{b}_2 + (\boldsymbol{a}_2 + \boldsymbol{a}_5)\boldsymbol{b}_3 \\ &+ 2\boldsymbol{a}_1\boldsymbol{b}_4 + \boldsymbol{a}_3\boldsymbol{b}_5 + \boldsymbol{a}_2\boldsymbol{b}_6. \end{aligned} \tag{32}$$

Hence, by using the quadratic-form identity we obtain

$$\begin{aligned}&\frac{\delta}{\delta u}(-4\mathrm{i}\boldsymbol{a}\lambda - 2\mathrm{i}\boldsymbol{d}\lambda + (\boldsymbol{u}_1 + \boldsymbol{u}_2 - 2\boldsymbol{I}_M)\boldsymbol{c}_x \\ &- 3\mathrm{i}(\boldsymbol{u}_1 - \boldsymbol{I}_M)\boldsymbol{g}_x - \mathrm{i}(\boldsymbol{u}_1 + \boldsymbol{u}_2 - 2\boldsymbol{I}_M)\lambda\boldsymbol{d} \\ &- 2\boldsymbol{b}_x - \boldsymbol{f}_x - \mathrm{i}(\boldsymbol{u}_2 - \boldsymbol{I}_M)\lambda\boldsymbol{a}) \\ =& \lambda^{-\gamma}\frac{\partial}{\partial\lambda}\lambda^{\gamma} \begin{pmatrix} \lambda\boldsymbol{c}_x - \mathrm{i}\lambda^2\boldsymbol{a} + \lambda\boldsymbol{g}_x - \mathrm{i}\lambda^2\boldsymbol{d} \\ \lambda\boldsymbol{c}_x - \mathrm{i}\lambda^2\boldsymbol{a} \end{pmatrix}. \end{aligned}$$

Comparing the coefficients of λ^{-n+2} we obtain

$$\begin{aligned}&\frac{\delta}{\delta u}(-\mathrm{i}(3\boldsymbol{u}_1 + \boldsymbol{u}_2)\boldsymbol{a}_{n-1} + (\boldsymbol{u}_1 + \boldsymbol{u}_2 - 2\boldsymbol{I}_M)\boldsymbol{c}_{n-2x} \\ &- \mathrm{i}(\boldsymbol{u}_1 + \boldsymbol{u}_2)\boldsymbol{d}_{n-1} + (\boldsymbol{u}_1 - \boldsymbol{I}_M)\boldsymbol{g}_{n-2x} \\ &- 2\boldsymbol{b}_{n-2x} - \boldsymbol{f}_{n-2x}) \\ =& 2\mathrm{i}(3 + \gamma - n) \begin{pmatrix} \boldsymbol{g}_{n-1} + 2\boldsymbol{c}_{n-1} \\ \boldsymbol{c}_{n-1} \end{pmatrix}. \end{aligned}$$

Taking $n = 2$ in above equation we obtain $\gamma = -1$. Thus, the Hamiltonian structure of expression (28) is derived as follows:

$$\boldsymbol{u}_t = \begin{pmatrix} \boldsymbol{u}_{1t} \\ \boldsymbol{u}_{2t} \end{pmatrix} = \boldsymbol{J}\frac{\delta H_n}{\delta u}, \tag{33}$$

where

$$\begin{aligned}H_n =& \frac{1}{2\mathrm{i}(n-2)}[\mathrm{i}(3\boldsymbol{u}_1 + \boldsymbol{u}_2)\boldsymbol{a}_{n-1} \\ &- (\boldsymbol{u}_1 + \boldsymbol{u}_2 - 2\boldsymbol{I}_M)\boldsymbol{c}_{n-2x} + \mathrm{i}(\boldsymbol{u}_1 + \boldsymbol{u}_2)\boldsymbol{d}_{n-1} \\ &- (\boldsymbol{u}_1 - \boldsymbol{I}_M)\boldsymbol{g}_{n-2x} + 2\boldsymbol{b}_{n-2x} - \boldsymbol{f}_{n-2x}]. \end{aligned}$$

It is easy to verify that $\boldsymbol{J}_2\boldsymbol{L}_2 = \boldsymbol{L}_2^*\boldsymbol{J}_2$, so the system (29) is Liouville integrable.

Taking $u_2 = 0$ in system (29), we find that the system reduces to system (9), thus we can conclude that system (29) is the integrable coupling of system (9) according to the definition of integrable coupling.

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