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Effective first law of thermodynamics of black holes with two horizons

Wei Yi-Huan(魏益焕)†

Department of Physics, Bohai University, Jinzhou 121000, *China*

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For a black hole with two horizons, the effective entropy is assumed to be a linear combination of the two entropies of the outer and inner horizons. In terms of the effective thermodynamic quantities the effective Bekenstein-Smarr formula and the effective first law of thermodynamics are derived.

Keywords: outer and inner horizons, effective thermodynamic quantities, first law of thermodynamics

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Black hole is a thermodynamic system, which obeys the four laws of thermodynamics.[1−5] The temperature of black hole is defined by the surface gravity on event horizon.[3] The Bekenstein-Hawking entropy of event horizon is equal to one quarter of the horizon area.[1,2] The entropy proportional to the event horizon area of the black hole may be checked by the quantum statistical method.[6] The Arnowitt–Deser–Misner (ADM) mass of a black hole may be given by the Bekenstein–Smarr formula expressed in terms of the thermodynamical quantities defined on the event horizon, and the differential Bekenstein–Smarr formula for δM is the first law of thermodynamics. For a black hole spacetime with two horizons, M and δM may be given in terms of the thermodynamic quantities defined on the outer horizon or those on the inner horizon.[7−10] Since the Bekenstein–Smarr formula and the first law of thermodynamics may also given in terms of the thermodynamic quantities associated with the inner horizon, it is always possible to express the Bekenstein–Smarr formula and the first law of thermodynamics in terms of the effective thermodynamic quantities. As is shown in Ref.[10], the effective temperature may be taken as the harmonic mean of the temperatures of the inner and outer horizons.

In this paper, we will approach the expressions for the effective Bekenstein–Smarr formula and the effective first law of thermodynamics for the black holes with two horizons. Assuming that the effective entropy is a linear combination of the entropies of the outer and inner horizons, then we obtain the effective temperature and other effective thermodynamic quantities. In terms of the effective thermodynamic quantities the effective Bekenstein–Smarr formula and first law of thermodynamics are derived.

Consider a black hole with two horizons, for which the set of thermodynamic quantities takes $(Z_+^A) = (\{X^A\},\{Y_+^A\}) = (S_+,T_+;\{X^a\},\{Y_+^a\})$ with the indices $A = 1,2,...$ and $a = 2,...$, where S_+ and T_+ are respectively the entropy and temperature of the outer horizon, and $\{Y_+^a\}$ denote the values of the intensive quantities $\{Y^a\}$ taken on the outer horizon. For such a system, the first law of thermodynamics is given by

$$\delta M = T_+\delta S_+ + \sum_a Y_+^a \delta X^a, \tag{1}$$

where M is the ADM mass of the black hole. The first law of thermodynamics may also be expressed in terms of $(Z_-^A) = (S_-,T_-;\{X^a\},\{Y_-^a\})$ as

$$\delta M = T_-\delta S_- + \sum_a Y_-^a \delta X^a, \tag{2}$$

where S_- and T_- are respectively the entropy and temperature of the inner horizon, and $\{Y_-^a\}$ are the values of $\{Y^a\}$ on the inner horizon. Combining Eqs.(1) and (2) gives

$$\begin{aligned}\delta M =& \chi(T_+\delta S_+ + \sum_a Y_+^a \delta X^a) \\ &+ (1-\chi)(T_-\delta S_- + Y_-^a \delta X^a),\end{aligned} \tag{3}$$

where χ is a parameter to be determined.

For the Kerr–Newmann (KN) black hole spacetime, the ADM mass M may be given by the

†E-mail: weiyihuan@263.net

Bekenstein–Smarr formula as[4,7−9]

$$M = 2T_{\pm}S_{\pm} + \Phi_{\pm}Q + 2\Omega_{\pm}J, \tag{5}$$

where Q and J denote the electric charge and angular momentum, $\Phi_{\pm}$ and $\Omega_{\pm}$ are the electric potentials and angular velocities of the outer and inner horizons, respectively. Generically, the Bekenstein–Smarr formula may be generalized to the following form

$$M = \eta_T T_{\pm}S_{\pm} + \sum_a \eta_a Y^a_{\pm}X^a, \tag{5}$$

where η_T and η_a are some constant coefficients associated with the number of the spacetime dimensions and the quantities $\{X^A\}$.

From Eq.(5) we have

$$\begin{aligned} M =&\chi(\eta_T T_+S_+ + \sum_a \eta_a Y^a_+X^a) \\ &+ (1-\chi)(\eta_T T_-S_- + \sum_a \eta_a Y^a_-X^a). \end{aligned} \tag{6}$$

Equations (3) and (6) can be reduced to the effective forms by introducing the effective quantities $\{Z^A_{\rm eff}\} = (S_{\rm eff}, T_{\rm eff}; \{X^a\}, \{Y^a_{\rm eff}\})$. Defining the effective quantities by

$$T_{\rm eff}S_{\rm eff} = \chi T_+S_+ + (1-\chi)T_-S_-, \tag{7}$$

$$T_{\rm eff}\delta S_{\rm eff} = \chi T_+\delta S_+ + (1-\chi)T_-\delta S_-, \tag{8}$$

$$Y_{\rm eff} = \chi Y_+ + (1-\chi)Y_-, \tag{9}$$

then Eqs.(6) and (3) are rearranged as

$$M = \eta_T T_{\rm eff}S_{\rm eff} + \sum_a \eta_a Y^a_{\rm eff}X^a, \tag{10}$$

$$\delta M = T_{\rm eff}\delta S_{\rm eff} + \sum_a Y^a_{\rm eff}\delta X^a. \tag{11}$$

We call Eqs.(10) and (11) the effective Bekenstein-Smarr formula and the effective first law of thermodynamics of the black hole with two horizons, respectively.

In what follows, let us determine χ, $T_{\rm eff}$ and $S_{\rm eff}$ by using Eqs.(7) and (8). Define the effective entropy as

$$S_{\rm eff} = S_+ + \alpha S_-, \tag{12}$$

where α is a constant. Substituting the effective entropy into Eqs.(7) and (8), then we obtain the effective temperature

$$T_{\rm eff} = \frac{T_+T_-}{\alpha T_+ + T_-}, \tag{13}$$

with

$$\chi = \frac{T_-}{T_- + \alpha T_+}. \tag{14}$$

Defining the effective surface gravity by $T_{\rm eff} = \dfrac{\kappa_{\rm eff}}{2\pi}$, then one can have

$$\kappa^{-1}_{\rm eff} = \kappa^{-1}_+ + \alpha\kappa^{-1}_-, \tag{15}$$

where $\kappa_{\pm}$ denote the surface gravity of the outer and inner horizons, respectively. For $\alpha = \pm 1$, Eq.(15) will give the right and left temperatures of the black hole in the string theory.[11−16] We have given a general proof that, for a black hole with two horizons, the thermodynamic quantities associated with the outer and inner horizons may be mapped into those of the left-moving and right-moving systems.

As an example, let us discuss the case of the KN black hole. For the KN black hole, the ADM mass M and the electric charge Q are given by

$$M = \frac{1}{2}(r_+ + r_-), \quad Q^2 + a^2 = r_+r_-, \tag{16}$$

where a is rotating angular momentum per unit mass, $r_{\pm}$ are radii of the outer and inner horizons, respectively. The temperatures,† entropies, electric potentials and the angular velocities of the outer and inner horizons are[10]

$$T_{\pm} = \frac{r_{\pm} - r_{\mp}}{4\pi\mathcal{A}_{\pm}}, \quad S_{\pm} = \pi\mathcal{A}_{\pm} \tag{17}$$

$$\Phi_{\pm} = \frac{Qr_{\pm}}{\mathcal{A}_{\pm}}, \quad \Omega_{\pm} = \frac{a}{\mathcal{A}_{\pm}}, \tag{18}$$

with $\mathcal{A}_{\pm} = r^2_{\pm} + a^2$.

The Bekenstein-Smarr formula for the KN black hole is given by Eq.(4), and the first law of thermodynamics reads

$$\delta M_{\pm} = T_{\pm}\delta S_{\pm} + \Phi_{\pm}\delta Q + \Omega_{\pm}\delta J. \tag{19}$$

Noting that for the KN black hole, $\{Y^a\} = \{\Phi,\ \Omega\}$, then according to Eqs.(9), (12) and (13) the effective quantities are

$$\Phi_{\rm eff} = \frac{(r_+ - \alpha r_-)Q}{\mathcal{A}_+ - \alpha\mathcal{A}_-}, \quad \Omega_{\rm eff} = \frac{(1-\alpha)a}{\mathcal{A}_+ - \alpha\mathcal{A}_-}, \tag{20}$$

$$S_{\rm eff} = \pi(\mathcal{A}_+ + \alpha\mathcal{A}_-), \quad T_{\rm eff} = \frac{r_+ - r_-}{4\pi(\mathcal{A}_+ - \alpha\mathcal{A}_-)}. \tag{21}$$

† M may be expressed as $M = \left[\dfrac{S_{\pm}}{4\pi} + \dfrac{\pi}{4S_{\pm}}(4J^2 + Q^4) + \dfrac{Q^2}{2}\right]^{1/2}$. The temperatures of the outer and inner horizons may be given by $T_{\pm} = \dfrac{\partial M}{\partial S_{\pm}}|_{J,Q}$.

Consequently, for the KN black hole the effective Bekenstein–Smarr formula and the effective first law of thermodynamics are obtained as

$$M = 2T_{\rm eff}S_{\rm eff} + \Phi_{\rm eff}Q + 2\Omega_{\rm eff}J, \tag{22}$$

$$\delta M = T_{\rm eff}\delta S_{\rm eff} + \Phi_{\rm eff}\delta Q + \Omega_{\rm eff}\delta J. \tag{23}$$

For $\alpha = 1$, the effective electric potential, angular velocity, temperature and entropy are given by

$$\Phi_{\rm eff} = \frac{Q}{r_+ + r_-}, \quad \Omega_{\rm eff} = 0, \tag{24}$$

$$T_{\rm eff} = \frac{1}{4\pi(r_+ + r_-)}, \quad S_{\rm eff} = \pi(\mathcal{A}_+ + \mathcal{A}_-). \tag{25}$$

For $\alpha = -1$, they are

$$\Phi_{\rm eff} = \frac{(r_+ + r_-)Q}{\mathcal{A}_+ + \mathcal{A}_-}, \quad \Omega_{\rm eff} = \frac{2a}{\mathcal{A}_+ + \mathcal{A}_-}, \tag{26}$$

$$T_{\rm eff} = \frac{r_+ - r_-}{4\pi(\mathcal{A}_+ + \mathcal{A}_-)}, \quad S_{\rm eff} = \pi(\mathcal{A}_+ - \mathcal{A}_-). \tag{27}$$

One can see that the thermodynamic quantities in the two above cases correspond to those of the right-moving and left-moving systems.[10]

As another example, we consider the Banados–Teitelboim–Zanelli (BTZ) black hole.[17,18,13] For the rotating BTZ black hole, radii of the outer and inner horizon are

$$r_\pm^2 = \frac{l^2}{2}\left(M \pm \sqrt{M^2 - \frac{J^2}{l^2}}\right), \tag{28}$$

where l is the characteristic BTZ length scale, M and J are the mass and angular momentum of the BTZ spacetime, respectively. The mass of the BTZ spacetime may be expressed in terms of S_+ and J as[19]

$$M = \frac{S_+^2}{16\pi^2 l^2} + \frac{4\pi^2 J^2}{S_+^2}, \tag{29}$$

where $S_+ = 4\pi r_+$ and $J = \dfrac{2r_+r_-}{l}$ are the entropy of the outer horizon and the angular momentum,[17,18,13] respectively. It can be seen that under the transformation $S_+ \to S_- = 4\pi r_-$, the form of M holds invariant. This implies that $S_- = 4\pi r_-$ may be defined as the entropy of the inner horizon.[20] With $T_\pm = \dfrac{\partial M}{\partial S_\pm}|_{J,l}$, $\Omega_\pm = \dfrac{\partial M}{\partial J}|_{S_\pm,l}$ and $\Theta_\pm = \dfrac{\partial M}{\partial S_\pm}|_{S_\pm,J}$, one can have

$$T_\pm = \frac{r_\pm^2 - r_\mp^2}{2\pi r_\pm l^2}, \quad \Omega_\pm = \frac{r_\mp}{r_\pm l}, \quad \Theta_\pm = -\frac{2r_\pm^2}{l^3}, \tag{30}$$

where $\Theta_\pm$ is the generalized BTZ forces on the outer and inner horizons associated with l,[19] $T_\pm$ and $\Omega_\pm$ are the temperatures and angular velocities of them, respectively.

For the BTZ black hole, there is $\{Y^a\} = \{\Omega, \Theta\}$. According to Eqs.(12), (13) and (9), the effective entropy, temperature, angular momentum and generalized force associated with l of the BTZ black hole are given by

$$S_{\rm eff} = 4\pi(r_+ + \alpha r_-), \quad T_{\rm eff} = \frac{r_+^2 - r_-^2}{2\pi(r_+ - \alpha r_-)l^2}, \tag{31}$$

$$\Omega_{\rm eff} = \frac{r_- - \alpha r_+}{(r_+ - \alpha r_-)l}, \quad \Theta_{\rm eff} = -\frac{2(r_+^3 - \alpha r_-^3)}{(r_+ - \alpha r_-)l^3}. \tag{32}$$

For $\alpha = 1$, Eqs.(31) and (32) reduce to

$$S_{\rm eff} = 4\pi(r_+ + r_-), \quad T_{\rm eff} = \frac{r_+ + r_-}{2\pi l^2}, \tag{33}$$

$$\Omega_{\rm eff} = -\frac{1}{l}, \quad \Theta_{\rm eff} = -\frac{2(r_+^2 + r_+r_- + r_-^2)}{l^3}; \tag{34}$$

and for $\alpha = -1$ they give

$$S_{\rm eff} = 4\pi(r_+ - r_-), \quad T_{\rm eff} = \frac{r_+ - r_-}{2\pi l^2}, \tag{35}$$

$$\Omega_{\rm eff} = \frac{1}{l}, \quad \Theta_{\rm eff} = -\frac{2(r_+^2 - r_+r_- + r_-^2)}{l^3}. \tag{36}$$

The effective quantities for $\alpha = \pm 1$ correspond to those of the right-moving and left-moving systems,[10] respectively.

For a black hole with two horizons, assuming that the Bekenstein-Smarr formula and the first law of thermodynamics may be expressed in terms of the thermodynamic quantities associated with the outer and inner horizons, then one can obtain the effective forms of the Bekenstein–Smarr formula and the first law of thermodynamics. For a number of other black holes with two horizons the effective Bekenstein–Smarr formula and the first law of thermodynamics can be valid.

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