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Three-dimensional interfacial wave theory of dendritic growth: (I). multiple variables expansion solutions*

Chen Yong-Qiang(陈永强)[a)b)], Tang Xiong-Xin(唐熊忻)[a)c)], and Xu Jian-Jun(徐鉴君)[a)d)†]

[a)] *School of Mathematical Science, Nankai University, Tianjin* 300071, *China*
[b)] *Department of Fundamental Subject, Tianjin Institute of Urban Construction, Tianjin* 300384, *China*
[c)] *School of materials Science and Engineering, University of Science and Technology Beijing, Beijing* 100083, *China*
[d)] *Department of mathematics and Statistics, McGill University, H3A2K6 Montreal, Canada*

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Dendritic pattern formation at the interface between liquid and solid is a commonly observed phenomenon in crystal growth and solidification process. The theoretical investigation of dendritic growth is one of the most profound and highly challenging subjects in the broad areas of interfacial pattern formation, condensed matter physics and materials science, preoccupying many researchers from various areas. Some longstanding key issues on this subject finally gained a breakthrough in the late of last century, via the 'Interfacial Wave (IFW) Theory' on the ground of systematical global stability analysis of the basic state of dendritic growth.

The original form of the IFW theory mainly focus on the investigation of various axi-symmetric unsteady perturbed modes solutions around the axi-symmetric basic state of system of dendritic growth. In reality, the system may allow various non-axi-symmetric, unsteady perturbed states. Whether or not the system of dendritic growth allows some growing non-axi-symmetric modes? Will the stationary dendritic pattern be destroyed by some of such non-axi-symmetric modes? Or, in one word, what is the stability property of the system, once the non-axi-symmetric modes can be evoked? The answers for these questions are important for the solid foundation of IFW theory. The present work attempts to settle down these issues and develop a three-dimensional (3D) interfacial wave theory of dendritic growth. Our investigations verify that dendritic growth indeed allows a discrete set of non-axi-symmetric unstable global wave modes, which gives rise to a set of multiple arms spiral waves propagating along the Ivantsov's paraboloid.

Keywords: Dendritic growth, pattern formation, interfacial waves, selection criterion
PACC: 6150C, 6470D, 8130F

1. Introduction

Dendritic pattern formation at the interface between liquid and solid is a commonly observed phenomenon in crystal growth and solidification process.[1,2] A typical single free dendritic growth is shown in Fig.1, which is the experimental curves recorded by Dougherty and Gollub[1] for two-dimensional dendritic growth from a supersaturated NH_4Br solution. Theoretical studies of dendritic growth is one of the most profound and highly challenging subjects in the area of interfacial pattern formation. The basic and vital issues are:

- What mechanism determines the tip growth velocity?
- What is the origin and essence of the microstructure?

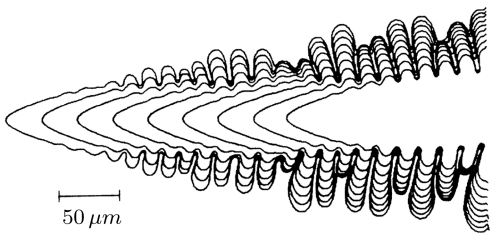


Fig.1. Experimental curve of 2D dendritic growth from a supersaturated (NH_4Br) solution, taken from Ref.[1].

These problems have been at the centre of research activities in the broad areas of condensed matter physics and materials science, preoccupying many researchers from various areas,[2−29] and finally gained a breakthrough via the 'Interfacial Wave (IFW) Theory'[19,29] originally established in 1990's. The ground of the IFW is the systematical global stability analysis of the basic state of dendritic growth. The IFW theory derives that dendritic growth observed in

*Project supported by the Nankai University, China and in part by NSERC Grant of Canada.
†E-mail: xu@math.mcgill.ca

the moving frame fixed at the dendrite-tip is essentially a wave phenomenon. The grand dendritic side-branching structure is described by the axi-symmetric global neutral mode ($n = 0$) and the selection condition for the tip growth velocity is determined by the global stability criterion of the system. The IFW theory is different from the *Marginal Stability Hypothesis* historically proposed by Langer and Müller-Krumbhaar in 1978.[9] The formal is a global stability theory systematically derived from the physico-mathematical model without additional artificial assumptions, while the latter is an artificial hypothesis on the local stability at the dendrite-tip without valid theoretical derivations. The IFW theory is also different from the Microscopic Solvability Condition (MSC) theory proposed by a number of researchers in last century.[8] The latter missed the important global wave instability mechanism discovered by the IFW theory and can be considered as a special case of the IFW theory valid only for the system of dendritic growth with anisotropic surface tension greater than a critical number α_c (refer to Ref.[26]). The theoretical predictions and conclusions drawn by the IFW have been well supported by the available experimental data and observations.

To explore the global stability mechanism, the original form of the IFW theory mainly focused on the behaviours of various axi-symmetric unsteady perturbed modes around the axi-symmetric basic state, which is very close to the famous Ivantsov solution, in the region away from the dendrite-root. In reality, the system may allow various non-axi-symmetric, unsteady perturbed states. Whether or not the system of dendritic growth allow some growing non-axi-symmetric modes? Will the stationary dendritic pattern be destroyed by some of such non-axi-symmetric modes? Or, in one word, what is the stability property of the system, once the non-axi-symmetric modes are evoked? The answers to these questions are important for the solid foundation of the IFW theory. In this sequence of two papers, we attempt to refine the IFW and settle down these issues. Our investigations verify that dendritic growth indeed allows a discrete set of non-axi-symmetric, unstable global wave modes, which gives rise to a set of multiple arms spiral waves propagating along the Ivantsov's paraboloid. However, these spiral waves are less dangerous, compared with the axi-symmetric mode ($n = 0$), explored previously in the IFW theory. So, the presence of these non-axi-symmetric modes does not affect the basic conclusions of the IFW on the selection condition and origin of pattern formation of dendritic growth.

This work consists of two parts. In part (I), this paper deals with the asymptotic solutions of basic state of the system and the multiple variables expansion solutions for the perturbed states. The paper is arranged as follows. In Section 2, we give the mathematical formulation of problem, and in Section 3, we give the basic state of the system, then turn to the study of stability of the basic state. In Section 4, we give the linear system for the three-dimensional (3D) perturbed states; in Section 5, we derive the multiple variables asymptotic expansion (MVE) solution in the region away from the singular points of the system.

2. Mathematical formulation of the general problem

We consider a free dendritic growth at the late stage of growth from the undercooled pure melt, when the average growth speed of dendrite-tip has reached a constant U. Following Refs.[19–21], we adopt the symmetric model and employ the paraboloidal coordinate system (ξ, η, θ) moving with the velocity U, which can be defined through the cylindrical coordinate system (r, θ, z) by the formula (see Fig.2):

$$\begin{aligned} \frac{r}{\eta_0^2} &= \xi\eta, \\ \frac{z}{\eta_0^2} &= \frac{1}{2}(\xi^2 - \eta^2), \end{aligned} \tag{1}$$

where the constant η_0^2 is to be determined by setting the location of the dendrite-tip so that the steady interface shape satisfies

$$\eta_{\rm s}(0) = 1. \tag{2}$$

It is known that the normalization parameter η_0^2 is just the Peclet number of the system with zero surface tension. We choose the growth speed of dendrite-tip U as the scale of velocity, use the thermal diffusion length $\ell_{\rm T}$ as the length scale and, accordingly, use $\ell_{\rm T}/U$ as the time scale. Furthermore, we use $\Delta H/c_{\rm p}\rho$ as the scale of temperature, where ΔH is the latent heat per unit volume of the solid, and define the dimensionless temperature as $\bar{T} = \big[T - T_{\rm M0}\big]c_{\rm p}\rho/\Delta H$, where $T_{\rm M0}$ is the melting temperature at a flat interface.

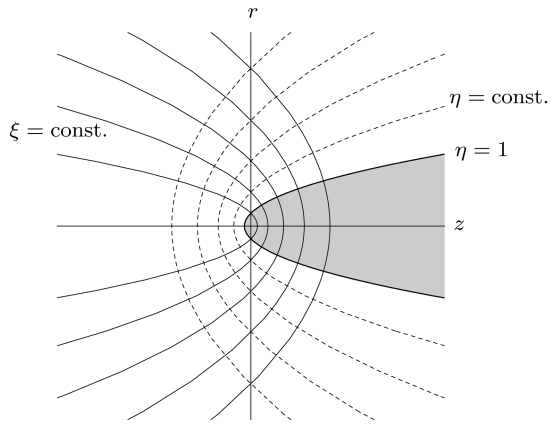


Fig.2. The paraboloidal coordinate system (ξ, η, θ) for 3D dendritic growth.

In the moving paraboloidal system, if we omit the notation '-' on the top of dimensionless quantities, the dimensionless governing equation for the temperature field in both solid and liquid phase can be written in the following forms:

$$\frac{\partial^2 T}{\partial \xi^2} + \frac{\partial^2 T}{\partial \eta^2} + \frac{1}{\xi}\frac{\partial T}{\partial \xi} + \frac{1}{\eta}\frac{\partial T}{\partial \eta} + \left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right)\frac{\partial^2 T}{\partial \theta^2} = \eta_0^2\left(\xi\frac{\partial T}{\partial \xi} - \eta\frac{\partial T}{\partial \eta}\right) + \eta_0^4\left(\xi^2 + \eta^2\right)\frac{\partial T}{\partial t}. \tag{3}$$

The boundary conditions are:

(i) for the far-field

$$T \to T_\infty = \frac{(T_\infty)_D - T_{\mathrm{M0}}}{\Delta H/(c_{\mathrm{p}}\rho)} < 0 \quad \text{as} \quad \eta \to \infty\,, \tag{4}$$

where $(T_\infty)_D$ is the dimensional temperature in the far field ahead of the dendrite-tip, while T_∞ is the dimensionless undercooling parameter;

(ii) for the regularity state

$$T_{\mathrm{S}} = \mathcal{O}(1) \quad \text{as } \eta \to 0; \tag{5}$$

(iii) for the interface at $\eta = \eta_{\mathrm{s}}(\xi, \theta, t)$:

(a) in the thermodynamic equilibrium

$$T = T_{\mathrm{S}}\,, \tag{6}$$

(b) for the Gibbs–Thomson condition with isotropic surface tension:

$$T = -\varepsilon^2\eta_0^2\mathcal{K}_3\Big\{\eta_{\mathrm{s}}(\xi, \theta, \varepsilon)\Big\}, \tag{7}$$

where $\mathcal{K}_3/\eta_0^2$ is twice the local mean curvature of the interface $\eta_{\mathrm{s}}(\xi, \theta)$:

$$\begin{aligned}\mathcal{K}_3\Big\{\eta_{\mathrm{s}}(\xi, \theta, \varepsilon)\Big\} = & -\frac{1}{\left[\xi^2 + \eta_{\mathrm{s}}^2\right]^{3/2}\left[\xi^2\eta_{\mathrm{s}}^2(1 + \eta_{\mathrm{s}}'^2) + \eta_{\mathrm{s},\theta}^2(\xi^2 + \eta_{\mathrm{s}}^2)\right]^{3/2}} \\ & \times\Big\{\Big[(1 + \eta_{\mathrm{s}}'^2)(\xi^2 + \eta_{\mathrm{s}}^2)\Big]\Big[-(\xi\eta_{\mathrm{s}})^2(\eta_s\eta_s' - \xi) + 2\xi\eta_{\mathrm{s},\theta}^2(\xi^2 + \eta_{\mathrm{s}}^2) \\ & + \xi^2\eta_{\mathrm{s}}\eta_{\mathrm{s},\theta\theta}(\eta_s\eta_s' - \xi) - \xi\eta_{\mathrm{s}}(\xi\eta_{\mathrm{s}})'(\eta_{\mathrm{s},\theta}^2 + \eta_{\mathrm{s}}\eta_{\mathrm{s},\theta\theta})\Big] \\ & - 2\Big[\eta_{\mathrm{s},\theta}\eta_{\mathrm{s}}'(\xi^2 + \eta_{\mathrm{s}}^2)\Big]\Big[(\xi\eta_{\mathrm{s}})'\eta_{\mathrm{s},\theta}(\xi^2 + \eta_{\mathrm{s}}^2) + \xi\eta_s(\eta_s\eta_s' - \xi)(\xi\eta_{s,\theta})' \\ & - \xi\eta_s(\eta_s\eta_{s,\theta})'(\xi\eta_s)'\Big] + \xi\eta_{\mathrm{s}}\Big[(\xi\eta_{\mathrm{s}})^2 + \eta_{\mathrm{s},\theta}^2(\xi^2 + \eta_{\mathrm{s}}^2)\Big] \\ & \times\Big[(\xi\eta_{\mathrm{s}})''(\eta_{\mathrm{s}}\eta_{\mathrm{s}}' - \xi) + (\xi\eta_{\mathrm{s}})'(1 - \eta_{\mathrm{s}}'^2 - \eta_{\mathrm{s}}\eta_{\mathrm{s}}'')\Big]\Big\}\,.\end{aligned} \tag{8}$$

The parameter ε is defined as

$$\varepsilon = \frac{\sqrt{\Gamma}}{\eta_0^2}, \tag{9}$$

and called the interfacial stability parameter of isotropic surface tension, while Γ is the surface tension parameter defined as $\Gamma = \ell_{\mathrm{c}}/\ell_{\mathrm{T}}$ and $\ell_{\mathrm{c}} = \gamma c_{\mathrm{p}} T_{\mathrm{M0}}\rho/(\Delta H)^2$ is the capillary length.

For the specific axi-symmetric case, $\eta_{\mathrm{s}} = \eta_{\mathrm{s}}(\xi, \varepsilon)$, one has

$$\begin{aligned}&\mathcal{K}_3\Big\{\eta_{\mathrm{s}}(\xi, \varepsilon)\Big\} \\ &= -\frac{1}{\sqrt{\xi^2 + \eta_{\mathrm{s}}^2}}\left(\frac{\eta_{\mathrm{s}}''}{(1 + \eta_{\mathrm{s}}'^2)^{3/2}} - \frac{1}{\eta_{\mathrm{s}}(1 + \eta_{\mathrm{s}}'^2)^{1/2}}\right. \\ &\quad \left. + \frac{\eta_{\mathrm{s}}'(\eta_{\mathrm{s}}^2 + 2\xi^2) - \xi\eta_{\mathrm{s}}}{\xi(\xi^2 + \eta_{\mathrm{s}}^2)(1 + \eta_{\mathrm{s}}'^2)^{1/2}}\right).\end{aligned} \tag{10}$$

(c) for the heat balance state

$$\left(\frac{\partial}{\partial\eta}-\eta_{\rm s}'\frac{\partial}{\partial\xi}\right)(T-T_{\rm S})$$
$$-\eta_{{\rm s}\theta}\left(\frac{1}{\xi^2}+\frac{1}{\eta_{\rm s}^2}\right)\frac{\partial}{\partial\theta}(T-T_{\rm S})$$
$$+\eta_0^4\left(\xi^2+\eta_{\rm s}^2\right)\frac{\partial\eta_{\rm s}}{\partial t}+\eta_0^2\left(\xi\eta_{\rm s}\right)'=0, \qquad (11)$$

which, in the axi-symmetric case, is reduced to:

$$\left(\frac{\partial}{\partial\eta}-\eta_{\rm s}'\frac{\partial}{\partial\xi}\right)(T-T_{\rm S})$$
$$+\eta_0^4\left(\xi^2+\eta^2\right)\frac{\partial\eta_{\rm s}}{\partial t}+\eta_0^2\left(\xi\eta_{\rm s}\right)'=0. \qquad (12)$$

3. The basic state for dendritic growth with nonzero surface tension

To study the long-term behaviour of a dynamic system, one needs to first specify the basic state of the system, then perform the stability analysis for the basic state. The basic state of dendritic growth may be given by the so-called non-classic needle solutions, which may be a nearly steady solution with a slow time-dependency, or a steady solution with large but finite length of stem (refer to Ref.[19]). The exact analytical forms of these non-classic needle solutions in whole physical space from the tip to the root of dendrite is very hard to obtain. However, in order to understand the selection mechanism and essence of pattern formation of dendritic growth, one is only interested in the behaviour of the system in a finite region behind the dendrite-tip away from the root, not near the root. It is just in that region, that for a given (ξ,η,θ), as $\varepsilon\to 0$ the basic state of dendritic growth can be expanded in the following regular perturbation expansion (RPE):[14,22,24,25]

$$\begin{aligned} T &= T_{\rm B}(\xi,\eta,\theta,\varepsilon)=T_0(\eta)+\varepsilon^2\eta_0^2T_1(\xi,\eta,\theta)+\ldots,\\ T_{\rm S} &= T_{\rm BS}(\xi,\eta,\theta,\varepsilon)=T_{\rm S0}+\varepsilon^2\eta_0^2T_{\rm S1}(\xi,\eta,\theta)+\ldots,\\ \eta &= \eta_{\rm B}(\xi,\theta,\varepsilon)=1+\varepsilon^2\eta_1(\xi,\theta)+\ldots. \end{aligned} \qquad (13)$$

3.1. $O(\varepsilon^0)$

It is known that the zeroth order approximation solution is just the Ivantsov similarity solution for the case with zero surface tension ($\varepsilon=0$) and arbitrary undercooling,[4]

$$\begin{aligned} &T=T_*(\eta)=T_\infty+\frac{\eta_0^2}{2}{\rm e}^{\eta_0^2/2}E_1\left(\frac{\eta_0^2\eta^2}{2}\right),\\ &T_{\rm S}=T_{\rm S*}=T_*(1)=0,\\ &\eta=\eta_*=1,\\ &T_\infty=-\frac{\eta_0^2}{2}{\rm e}^{\eta_0^2/2}E_1\left(\frac{\eta_0^2}{2}\right), \quad (0\le\xi<\infty), \end{aligned} \qquad (14)$$

where $E_1(x)$ is the exponential function defined as

$$E_1(x)=\int_x^\infty\frac{{\rm e}^{-t}}{t}{\rm d}t, \qquad (15)$$

(see Refs.[30] and [31]). In the above, the constant η_0^2 is uniquely determined as a function of the undercooling T_∞. In general, the Peclet number is defined as the ratio of the tip radius to the thermal diffusion length, i.e., ${\rm Pe}=\ell_{\rm t}/\ell_{\rm T}={\rm Pe}(\varepsilon,T_\infty)$. So the above results yields that $\eta_0^2={\rm Pe}_0={\rm Pe}(0,{\rm T}_\infty)$.

The Ivantsov solution describes a steadily growing, axi-symmetric, classic needle crystal, whose interface has an infinitely long stem with no microstructure. Moreover, as a similarity solution, it represents a continuous family of dimensional, physical solutions with arbitrary tip velocity for given growth conditions and material properties. However, experimental observations show that the interface of growing dendrite always exhibits micro-structure (see Fig.1) and, for given material and growth condition, in the limit $t\to\infty$ the dendrite-tip velocity always approaches a certain constant, which appears to be unaffected by the details of initial settings of growth.[32] These experimental results consequently raised the problems about the selection mechanism of tip-growth velocity and origin of the dendritic pattern pointed out in the introduction section. These problems are fundamental in the field of condensed matter physics and materials science and have been the focus of broad theoretical and experimental research activities during the past decades.

3.2. $O(\varepsilon^2)$

The first-order approximation solution can be obtained by using the general Laguerre function expansion method (refer to Refs.[22]–[24], and Ref.[25]). Especially, for the system with isotropic surface tension, the temperature in the liquid is

$$T_1(\xi,\eta)$$

$$= \sum_{n=0}^{\infty} \beta_n L_n\left(\frac{\eta_0^2\xi^2}{2}\right) \frac{\mathrm{e}^{-\eta_0^2\eta^2/2} U\left(n+1, 1, \dfrac{\eta_0^2\eta^2}{2}\right)}{\mathrm{e}^{-\eta_0^2/2} U\left(n+1, 1, \dfrac{\eta_0^2}{2}\right)}, \tag{16}$$

while the temperature in the solid phase is

$$T_{\mathrm{S1}}(\xi,\eta) = \sum_{n=0}^{\infty} \frac{\alpha_n}{L_n\left(-\dfrac{\eta_0^2}{2}\right)} L_n\left(\frac{\eta_0^2\xi^2}{2}\right) L_n\left(-\frac{\eta_0^2\eta^2}{2}\right). \tag{17}$$

Thus the interface shape function $\eta_1(\xi)$ is

$$\eta_1(\xi) = \sum_{n=0}^{\infty} \gamma_n L_n\left(\frac{\eta_0^2\xi^2}{2}\right). \tag{18}$$

For the system with mth fold, anisotropic surface tension or interface kinetics, some non-axi-symmetrical components may be added in the solution. For the temperature field in the liquid and solid phase, these additional terms are

$$T_1(\xi,\eta,\theta) = \frac{\eta_0^{2m}\xi^m\eta^m}{2^m} \sum_{n=0}^{\infty} \beta_n L_n\left(\frac{\eta_0^2\xi^2}{2}\right) \frac{\mathrm{e}^{-\eta_0^2\eta^2/2} U\left(n+m+1, \quad 1+m, \dfrac{\eta_0^2\eta^2}{2}\right)}{\mathrm{e}^{-\eta_0^2/2} U\left(n+m+1, \quad 1+m, \dfrac{\eta_0^2}{2}\right)} \cos m\theta \tag{19}$$

and

$$T_{\mathrm{S1}}(\xi,\eta,\theta) = \left(\frac{\eta_0^{2m}\xi^m\eta^m}{2^m}\right) \sum_{n=0}^{\infty} \alpha_n \frac{L_n^{(m)}\left(\dfrac{\eta_0^2\xi^2}{2}\right) L_n^{(m)}\left(-\dfrac{\eta_0^2\eta^2}{2}\right)}{L_n^{(m)}\left(-\dfrac{\eta_0^2}{2}\right)} \cos m\theta, \tag{20}$$

respectively; whereas for the interface shape function $\eta_1(\xi,\theta)$ they are:

$$\eta_1(\xi,\theta) = \frac{\eta_0^{2m}\xi^m}{2^m} \sum_{n=0}^{\infty} \gamma_n L_n^{(m)}\left(\frac{\eta_0^2\xi^2}{2}\right) \cos m\theta. \tag{21}$$

The coefficients $\alpha_n, \beta_n, \gamma_n, (n = 0, 1, 2, \ldots)$ can be determined uniquely by the corresponding interface conditions.[22−25] The investigation of asymptotic behaviour in the far field further show that as $\xi \to \infty$, one has $\eta_1(\xi,\theta) \to 0$.

The first-order approximation solution does not solve the fundamental problems of the selection of dendrite-tip velocity and microstructure formation, nor any higher order RPE solutions. The key for solving these problems is to explore the linear stability of the basic state of the system.

4. Three-dimensional, linear perturbed states around the basic state of dendritic growth

We write the basic state in the form

$$T_{\mathrm{B}}(\xi,\eta,\theta,\varepsilon) = T_*(\eta) + \varepsilon^2 T_1(\xi,\eta,\theta) + \mathcal{O}(\varepsilon^4),$$
$$T_{\mathrm{SB}}(\xi,\eta,\theta,\varepsilon) = \varepsilon^2 T_{\mathrm{S1}}(\xi,\eta,\theta) + \mathcal{O}(\varepsilon^4),$$
$$\eta_{\mathrm{B}}(\xi,\theta,\varepsilon) = 1 + \varepsilon^2\eta_1(\xi,\theta) + \mathcal{O}(\varepsilon^4), \tag{22}$$

and separate the general unsteady solutions into two parts:

$$T = T_{\mathrm{B}} + \tilde{T}(\xi,\eta,\theta,t,\varepsilon),$$
$$T_{\mathrm{S}} = T_{\mathrm{SB}} + \tilde{T}_{\mathrm{S}}(\xi,\eta,\theta,t,\varepsilon),$$
$$\eta_{\mathrm{s}} = \eta_{\mathrm{B}} + \tilde{h}(\xi,\theta,t,\varepsilon)/\eta_0^2. \tag{23}$$

Consider initially infinitesimal, general 3D perturbations around the basic state. The linearized perturbed system is as follows:

$$\frac{\partial^2\tilde{T}}{\partial\xi^2} + \frac{\partial^2\tilde{T}}{\partial\eta^2} + \left(\frac{1}{\xi} - \eta_0^2\xi\right)\frac{\partial\tilde{T}}{\partial\xi} + \left(\frac{1}{\eta} + \eta_0^2\eta\right)\frac{\partial\tilde{T}}{\partial\eta} + \left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right)\frac{\partial^2\tilde{T}}{\partial\theta^2} = \eta_0^4\left(\xi^2+\eta^2\right)\frac{\partial\tilde{T}}{\partial t},$$
$$\left(0 \le \xi \le \xi_{\max} = \mathcal{O}(1/\varepsilon);\ 0 \le \eta < \infty\right) \tag{24}$$

with the boundary conditions

(i) As $\eta \to \infty$

$$\tilde{T} \to 0. \tag{25}$$

(ii) As $\eta \to 0$

$$\tilde{T}_{\rm S} = \mathcal{O}(1). \tag{26}$$

(iii) The interface conditions: For simplicity, for the case of small anisotropy of surface tension, we neglect the effect of anisotropy of surface tension on the perturbed states. Then, making the Taylor expansions around the interface of the basic state, it follows that at $\eta = \eta_{\rm B}(\xi, \theta, \varepsilon)$:

(a)

$$\tilde{T} - \tilde{T}_{\rm S} = -\left\{ \frac{\partial T_{\rm B}}{\partial \eta} - \frac{\partial T_{\rm SB}}{\partial \eta} \right\} \frac{\tilde{h}}{\eta_0^2}, \tag{27}$$

(b) After the linearization, Eq.(8) is reduced to

$$\begin{aligned} &\mathcal{K}_3\Big\{\eta_{\rm s}(\xi,\theta,\varepsilon)\Big\} \\ &= -\frac{2+\xi^2}{(1+\xi^2)^{3/2}} + \frac{\varepsilon^2}{(1+\xi^2)^{1/2}} \\ &\quad \times\left\{ \eta_1'' + \frac{1+2\xi^2}{\xi(1+\xi^2)}\eta_1' + \frac{\xi^2(2+\xi^2)+4}{(1+\xi^2)^2}\eta_1 \right. \\ &\quad \left. + \frac{(1+\xi^2)}{\xi^2}\eta_{1,\theta\theta} \right\} + \mathcal{O}(\varepsilon^4). \end{aligned} \tag{28}$$

Thus the Gibbs–Thomson condition can be written as

$$\begin{aligned} \tilde{T}_{\rm S} = \frac{\varepsilon^2}{\hat{S}(\xi)} &\left\{ \frac{\partial^2 \tilde{h}}{\partial \xi^2} + \left(1 + \frac{1}{\xi^2}\right) \frac{\partial^2 \tilde{h}}{\partial \theta^2} + \frac{1+2\xi^2}{\xi \hat{S}^2(\xi)} \frac{\partial \tilde{h}}{\partial \xi} \right. \\ &\left. + \frac{\xi^2(2+\xi^2)+4}{\hat{S}^2(\xi)} \tilde{h} \right\} + (\text{h.o.t.}) \,, \end{aligned} \tag{29}$$

(c) Then one has

$$\begin{aligned} &\frac{\partial}{\partial \eta}\left(\tilde{T} - \tilde{T}_{\rm S}\right) + \eta_0^2 \hat{S}^2(\xi) \frac{\partial \tilde{h}}{\partial t} + \xi \frac{\partial \tilde{h}}{\partial \xi} \\ &+ \tilde{h} + \left\{ \frac{\partial^2}{\partial \eta^2}\left(T_{\rm B} - T_{\rm SB}\right) \right\} \frac{\tilde{h}}{\eta_0^2} \\ &- \frac{1}{\eta_0^2} \frac{\partial \tilde{h}}{\partial \xi} \frac{\partial}{\partial \xi}\left(T_{\rm B} - T_{\rm SB}\right) - \frac{\eta_{\rm B}' \tilde{h}}{\eta_0^2} \frac{\partial^2}{\partial \xi \partial \eta}\left(T_{\rm B} - T_{\rm SB}\right) \\ &- \eta_{\rm B}' \frac{\partial}{\partial \xi}\left(\tilde{T} - \tilde{T}_{\rm S}\right) + (\text{h.o.t.}) = 0, \end{aligned} \tag{30}$$

where

$$\hat{S}(\xi) = \sqrt{\xi^2 + \eta_{\rm B}^2} \,. \tag{31}$$

(iv) The root condition: at $\xi = \xi_{\max} = \dfrac{C}{\varepsilon}$,

$$\{\tilde{T}, \tilde{T}_{\rm S}, \tilde{h}\} = 0 \,. \tag{32}$$

(v) The tip smoothness condition: at $\xi = 0$, $\eta = \eta_{\rm B}(0, \varepsilon)$,

$$\frac{\partial}{\partial \xi}\{\tilde{T}, \tilde{T}_{\rm S}, \tilde{h}\} = 0. \tag{33}$$

The above system (24)–(33) leads to a linear eigenvalue problem with two parameters ε, and η_0^2, when one looks for solutions of the type $\tilde{q} = \hat{q} \mathrm{e}^{\sigma t}$. The eigenvalue σ must be a function of (η_0^2, ε). Following the IFW theory, one may solve the above eigenvalue problem in two steps: First, solve the system (24)–(32) for any given $(\sigma, \eta_0^2, \varepsilon)$ by using the multiple variables expansion (MVE) method. The solution is required to satisfy all boundary conditions except the tip smoothness condition (33). Then apply the tip condition (33) to the asymptotic solution obtained, which will lead to the quantization conditions and yield the parameter σ as a function of η_0^2 and ε.

5. Multiple variables expansion solution in the outer region

In the outer region away from all possible singular points, we shall look for the asymptotic expansion of the solutions $\tilde{q}(\xi, \eta, \theta, t, \varepsilon)$ by means of the multiple variables expansion method. The set of fast variables introduced below is slightly different from what given in Refs.[19] and [20], and the derivations of the global asymptotic solutions are refined in the present paper. The fast variable η_+ along η direction is now defined as

$$\eta_+ = \frac{\psi(\xi, \eta, \varepsilon)}{\varepsilon}, \quad \psi(\xi, 1, \varepsilon) = 0. \tag{34}$$

The fast variable η_+ determines a boundary layer with the thickness of $\mathcal{O}(\varepsilon)$ near the interface $\eta = 1$, which we call the Interfacial Wave (IW) layer. It will be seen that the perturbations of temperature field are only restricted inside the IW layer. It vanishes exponentially away from this layer. Furthermore, we introduce the fast variable ξ_+ along ξ-direction as

$$\xi_+ = \frac{\phi(\xi, \eta, \varepsilon)}{\varepsilon}, \quad k(\xi, \eta, \varepsilon) = \frac{\partial \phi}{\partial \xi}. \tag{35}$$

Especially, at the interface $\eta = 1$, this fast variable is given by

$$\begin{aligned} \bar{\xi}_+(\xi, \varepsilon) &= \xi_+(\xi, 1, \varepsilon) = \frac{\bar{\phi}(\xi, \varepsilon)}{\varepsilon} \\ &= \frac{1}{\varepsilon} \int_0^{\xi} \bar{k}(\xi_1, \varepsilon) \mathrm{d}\xi_1, \end{aligned} \tag{36}$$

where

$$\begin{aligned} &\bar{\phi}(\xi, \varepsilon) = \phi(\xi, 1, \varepsilon), \\ &\bar{k}(\xi, \varepsilon) = \frac{\partial \bar{\phi}(\xi, \varepsilon)}{\partial \xi} = k(\xi, 1, \varepsilon). \end{aligned} \tag{37}$$

The fast time variable t_+ can be defined as

$$t_+ = \frac{t}{\eta_0^2 \varepsilon}. \tag{38}$$

We apply the MVE method.[33,34] The fast and slow variables $(\xi, \eta, \xi_+, \eta_+, t_+)$ are treated formally as independent variables. Hence, one must make the following replacements for the derivatives in the system (24)–(30):

$$\begin{aligned}
&\frac{\partial}{\partial \xi} \Longrightarrow \frac{\phi_\xi}{\varepsilon}\frac{\partial}{\partial \xi_+} + \frac{\psi_\xi}{\varepsilon}\frac{\partial}{\partial \eta_+} + \frac{\partial}{\partial \xi},\\
&\frac{\partial}{\partial \eta} \Longrightarrow \frac{\psi_\eta}{\varepsilon}\frac{\partial}{\partial \eta_+} + \frac{\phi_\eta}{\varepsilon}\frac{\partial}{\partial \xi_+} + \frac{\partial}{\partial \eta},\\
&\frac{\partial^2}{\partial \xi^2} \Longrightarrow \frac{\phi_\xi^2}{\varepsilon^2}\frac{\partial^2}{\partial \xi_+^2} + \frac{2\phi_\xi}{\varepsilon}\frac{\partial^2}{\partial \xi_+ \partial \xi} + \frac{\phi_{\xi\xi}}{\varepsilon}\frac{\partial}{\partial \xi_+} + \frac{\partial^2}{\partial \xi^2}\\
&\qquad + \frac{\psi_\xi^2}{\varepsilon^2}\frac{\partial^2}{\partial \eta_+^2} + \frac{2\phi_\xi\psi_\xi}{\varepsilon^2}\frac{\partial^2}{\partial \xi_+ \partial \eta_+} + \frac{\psi_{\xi\xi}}{\varepsilon}\frac{\partial}{\partial \eta_+} + \frac{2\psi_\xi}{\varepsilon}\frac{\partial^2}{\partial \eta_+ \partial \xi},\\
&\frac{\partial^2}{\partial \eta^2} \Longrightarrow \frac{\psi_\eta^2}{\varepsilon^2}\frac{\partial^2}{\partial \eta_+^2} + \frac{2\psi_\eta}{\varepsilon}\frac{\partial^2}{\partial \eta_+ \partial \eta} + \frac{\psi_{\eta\eta}}{\varepsilon}\frac{\partial}{\partial \eta_+} + \frac{\partial^2}{\partial \eta^2}\\
&\qquad + \frac{\phi_\eta^2}{\varepsilon^2}\frac{\partial^2}{\partial \xi_+^2} + \frac{2\phi_\eta\psi_\eta}{\varepsilon^2}\frac{\partial^2}{\partial \xi_+ \partial \eta_+} + \frac{\phi_{\eta\eta}}{\varepsilon}\frac{\partial}{\partial \xi_+} + \frac{2\phi_\eta}{\varepsilon}\frac{\partial^2}{\partial \xi_+ \partial \eta}.
\end{aligned} \tag{39}$$

The converted system with the multiple variables is as follows: The equation in the liquid phase is

$$\begin{aligned}
&\left[\left(\phi_\xi^2 + \phi_\eta^2\right)\frac{\partial^2}{\partial \xi_+^2} + \left(\psi_\xi^2 + \psi_\eta^2\right)\frac{\partial^2}{\partial \eta_+^2} + 2(\phi_\xi\psi_\xi + \phi_\eta\psi_\eta)\frac{\partial^2}{\partial \xi_+ \partial \eta_+}\right]\tilde{T} + \varepsilon^2\left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right)\frac{\partial^2 \tilde{T}}{\partial \theta^2}\\
&= -\varepsilon\left(2\phi_\xi\frac{\partial^2 \tilde{T}}{\partial \xi_+ \partial \xi} + \phi_{\xi\xi}\frac{\partial \tilde{T}}{\partial \xi_+} + \psi_{\xi\xi}\frac{\partial \tilde{T}}{\partial \eta_+} + 2\psi_\xi\frac{\partial^2 \tilde{T}}{\partial \eta_+ \partial \xi}\right) - \varepsilon\left(2\psi_\eta\frac{\partial^2 \tilde{T}}{\partial \eta_+ \partial \eta} + \psi_{\eta\eta}\frac{\partial \tilde{T}}{\partial \eta_+} + \phi_{\eta\eta}\frac{\partial \tilde{T}}{\partial \xi_+} + 2\phi_\eta\frac{\partial^2 \tilde{T}}{\partial \xi_+ \partial \eta}\right)\\
&\quad -\varepsilon\left(\frac{1}{\xi} - \eta_0^2\xi\right)\left(\phi_\xi\frac{\partial \tilde{T}}{\partial \xi_+} + \psi_\xi\frac{\partial \tilde{T}}{\partial \eta_+}\right) - \varepsilon\left(\frac{1}{\eta} + \eta_0^2\eta\right)\left(\phi_\eta\frac{\partial \tilde{T}}{\partial \xi_+} + \psi_\eta\frac{\partial \tilde{T}}{\partial \eta_+}\right) + \varepsilon\eta_0^2\left(\xi^2 + \eta^2\right)\frac{\partial \tilde{T}}{\partial t_+} + \mathcal{O}(\varepsilon^2).
\end{aligned} \tag{40}$$

In the solid phase, for the symmetric model under discussion we assume

$$\phi = \phi_{\mathrm{s}}, \qquad \psi = \psi_{\mathrm{s}}.$$

Thus, the heat equation for the region of solid phase is the same as the above for the region of liquid phase, except that the functions $\tilde{T}$ is replaced by $\tilde{T}_{\mathrm{S}}$.

We assume that the pair of functions (ϕ, ψ) are subject to the following Cauchy–Riemann conditions:

$$\begin{aligned}
&\phi_\xi = \psi_\eta, \qquad \psi_\xi = -\phi_\eta\\
&(\Re\{\phi_\xi > 0\}, \quad \Re\{\psi_\eta\} > 0).
\end{aligned} \tag{41}$$

It follows that

$$\phi_\xi\psi_\xi + \phi_\eta\psi_\eta = 0, \quad \frac{\partial \psi}{\partial \eta} = k(\xi, \eta, \varepsilon). \tag{42}$$

Thus, one may write

$$\eta_+(\xi, \eta, \varepsilon) = \frac{1}{\varepsilon}\int_1^\eta k(\xi, \eta_1, \varepsilon)\mathrm{d}\eta_1. \tag{43}$$

Moreover, we have

$$\nabla^2\phi(\xi, \eta, \varepsilon) = \nabla^2\psi(\xi, \eta, \varepsilon) = 0. \tag{44}$$

Under the condition (42), the multiple variables equation (40) can be simplified to

$$\begin{aligned}
&\left(\phi_\xi^2 + \phi_\eta^2\right)\left(\frac{\partial^2}{\partial \xi_+^2} + \frac{\partial^2}{\partial \eta_+^2}\right)\tilde{T} + \varepsilon^2\left(\frac{1}{\xi^2} + \frac{1}{\eta^2}\right)\frac{\partial^2 \tilde{T}}{\partial \theta^2}\\
&= -\varepsilon\left(2\phi_\xi\frac{\partial^2 \tilde{T}}{\partial \xi_+ \partial \xi} + 2\psi_\xi\frac{\partial^2 \tilde{T}}{\partial \eta_+ \partial \xi}\right)\\
&\quad -\varepsilon\left(2\psi_\eta\frac{\partial^2 \tilde{T}}{\partial \eta_+ \partial \eta} + 2\phi_\eta\frac{\partial^2 \tilde{T}}{\partial \xi_+ \partial \eta}\right)\\
&\quad -\varepsilon\left(\frac{1}{\xi} - \eta_0^2\xi\right)\left(\phi_\xi\frac{\partial \tilde{T}}{\partial \xi_+} + \psi_\xi\frac{\partial \tilde{T}}{\partial \eta_+}\right)\\
&\quad -\varepsilon\left(\frac{1}{\eta} + \eta_0^2\eta\right)\left(\phi_\eta\frac{\partial \tilde{T}}{\partial \xi_+} + \psi_\eta\frac{\partial \tilde{T}}{\partial \eta_+}\right)
\end{aligned}$$

$$+\varepsilon\eta_0^2\left(\xi^2+\eta^2\right)\frac{\partial\tilde{T}}{\partial t_+}+\mathcal{O}(\varepsilon^2). \tag{45}$$

Let us denote

$$\phi_\eta(\xi,\eta,\varepsilon)=g(\xi,\eta,\varepsilon). \tag{46}$$

From the boundary conditions at $\eta=1$, one deduces $\psi(\xi,1,\varepsilon)=0$ and

$$\begin{aligned}&\phi(\xi,1,\varepsilon)=\bar{\phi}(\xi,\varepsilon)=\int_0^\xi \bar{k}(\xi_1,\varepsilon)\mathrm{d}\xi_1,\\ &\phi_\eta(\xi,1,\varepsilon)=-\psi_\xi(\xi,1,\varepsilon)=g(\xi,1,\varepsilon)=0,\end{aligned} \tag{47}$$

where $\bar{k}(\xi,\varepsilon)$ is the wave number function on the interface. We can obtain the solution $\phi(\xi,\eta)$ from Eq.(44) in the whole (ξ,η) plane. For this purpose, we first extend the function $\bar{k}(\xi)$ analytically to the function $\bar{k}(\zeta)$ defined in the complex plane $\zeta=\xi+\mathrm{i}\xi_\mathrm{I}$. Furthermore, we may re-write Eq.(44) in the form

$$\begin{aligned}&\frac{\partial^2\phi(\xi,\eta)}{\partial\xi^2}+\frac{\partial^2\phi(\xi,\eta)}{\partial\eta^2}\\ =&\left(\frac{\partial}{\partial\xi}-\mathrm{i}\frac{\partial}{\partial\eta}\right)\left(\frac{\partial}{\partial\xi}+\mathrm{i}\frac{\partial}{\partial\eta}\right)\phi(\xi,\eta)\\ =&\,0,\end{aligned} \tag{48}$$

and derive the solution

$$\begin{aligned}&\phi(\xi,\eta)\\ =&\frac{1}{2}\Big\{\bar{\phi}\big[\xi+\mathrm{i}(\eta-1)\big]+\bar{\phi}\big[\xi+\mathrm{i}(\eta-1)\big]\Big\}\\ =&\frac{1}{2}\left\{\int_0^{\xi+\mathrm{i}(\eta-1)}\bar{k}(t)\mathrm{d}t+\int_0^{\xi-\mathrm{i}(\eta-1)}\bar{k}(t)\mathrm{d}t\right\}.\end{aligned} \tag{49}$$

It follows that

$$\begin{aligned}&\phi_\xi(\xi,\eta)\\ =&\,\psi_\eta(\xi,\eta)=k(\xi,\eta)\\ =&\frac{1}{2}\Big\{\bar{k}\big[\xi+\mathrm{i}(\eta-1)\big]+\bar{k}\big[\xi-\mathrm{i}(\eta-1)\big]\Big\}.\end{aligned} \tag{50}$$

Furthermore, from the condition (41), we can derive

$$\begin{aligned}&\phi_\eta(\xi,\eta)\\ =&-\psi_\xi(\xi,\eta)=g(\xi,\eta)\\ =&\frac{\mathrm{i}}{2}\Big\{\bar{k}\big[\xi+\mathrm{i}(\eta-1)\big]-\bar{k}\big[\xi-\mathrm{i}(\eta-1)\big]\Big\},\end{aligned} \tag{51}$$

and

$$\psi(\xi,\eta)=-\frac{\mathrm{i}}{2}\int_{\xi-\mathrm{i}(\eta-1)}^{\xi+\mathrm{i}(\eta-1)}\bar{k}(t)\mathrm{d}t. \tag{52}$$

It should be remarked that the functions (ϕ,ψ) and $\bar{k}(\xi,\varepsilon)$ under discussion are not necessarily real functions. They may be complex functions of the real variables (ξ,η).

The boundary conditions for the system with multiple variables are:

(i) As $\eta_+\to\infty$,

$$\tilde{T}\to 0. \tag{53}$$

(ii) As $\eta_+\to-\infty$,

$$\tilde{T}_\mathrm{S}\to 0. \tag{54}$$

(iii) At the interface, $\eta_+=0,\eta=1$:

(a) under the thermodynamic equilibrium condition, one has

$$\tilde{T}=\tilde{T}_\mathrm{S}+\tilde{h}-\varepsilon\eta_1\bar{k}\frac{\partial}{\partial\eta_+}\big(\tilde{T}-\tilde{T}_\mathrm{S}\big)+\mathcal{O}(\varepsilon^2); \tag{55}$$

(b) under the Gibbs–Thomson condition, one has

$$\begin{aligned}&\tilde{T}_\mathrm{S}+\varepsilon\eta_1\bar{k}\frac{\partial\tilde{T}_\mathrm{S}}{\partial\eta_+}\\ =&\frac{1}{S(\xi)}\left\{\left(\bar{k}^2\frac{\partial^2}{\partial\xi_+^2}+2\varepsilon\bar{k}\frac{\partial^2}{\partial\xi_+\partial\xi}+\varepsilon\bar{k}'\frac{\partial}{\partial\xi_+}\right)\right.\\ &\left.+\varepsilon\left(\frac{1}{\xi}+\frac{\xi}{S^2(\xi)}\right)\left(\bar{k}\frac{\partial}{\partial\xi_+}+\varepsilon\frac{\partial}{\partial\xi}\right)\right\}\tilde{h}\\ &+\frac{\varepsilon^2}{\xi^2}\frac{\partial^2\tilde{h}}{\partial\theta^2}+\mathcal{O}(\varepsilon^2),\end{aligned} \tag{56}$$

where we have defined

$$S=\sqrt{1+\xi^2}; \tag{57}$$

(c) under the heat balance condition, one has

$$\begin{aligned}&\bar{k}\left(\frac{\partial\tilde{T}}{\partial\eta_+}-\frac{\partial\tilde{T}_\mathrm{S}}{\partial\eta_+}\right)+\varepsilon\eta_1\bar{k}^2\frac{\partial^2}{\partial\eta_+^2}\big(\tilde{T}-\tilde{T}_\mathrm{S}\big)\\ &+\varepsilon\left(\frac{\partial\tilde{T}}{\partial\eta}-\frac{\partial\tilde{T}_\mathrm{S}}{\partial\eta}\right)+S^2(\xi)\frac{\partial\tilde{h}}{\partial t_+}+\xi\left(k\frac{\partial\tilde{h}}{\partial\bar{\xi}_+}+\varepsilon\frac{\partial\tilde{h}}{\partial\xi}\right)\\ &+\varepsilon\left(2+\eta_0^2\right)\tilde{h}+\mathcal{O}(\varepsilon^2)=0.\end{aligned} \tag{58}$$

Note that in the above, we have applied the boundary conditions (47);

(iv) under the root condition, as $\xi\to\infty$ one has

$$\big\{\tilde{T};\ \tilde{T}_\mathrm{S};\ \tilde{h}\big\}=0\,. \tag{59}$$

In terms of the multiple variables, $(\xi,\eta,\theta,\xi_+,\eta_+,t_+)$, we now make the following multiple variables expansion (MVE) for general, 3D perturbed states:

$$\tilde{T} = \tilde{\mu}_0(\varepsilon)\left\{\tilde{T}_0(\xi,\eta,\xi_+,\eta_+) + \varepsilon\tilde{T}_1(\xi,\eta,\xi_+,\eta_+) + \ldots\right\}\mathrm{e}^{\mathrm{i}m\theta+\sigma t_+},$$
$$\tilde{T}_{\mathrm{S}} = \tilde{\mu}_0(\varepsilon)\left\{\tilde{T}_{\mathrm{S}0}(\xi,\eta,\xi_+,\eta_+) + \varepsilon\tilde{T}_{\mathrm{S}1}(\xi,\eta,\xi_+,\eta_+) + \ldots\right\}\mathrm{e}^{\mathrm{i}m\theta+\sigma t_+},$$
$$\tilde{h} = \tilde{\mu}_0(\varepsilon)\left\{\tilde{h}_0(\xi,\bar{\xi}_+) + \varepsilon\tilde{h}_1(\xi,\bar{\xi}_+) + \ldots\right\}\mathrm{e}^{\mathrm{i}m\theta+\sigma t_+}, \tag{60}$$

where $m = 0, 1, 2, \ldots$ and $\sigma = \sigma_{\mathrm{R}} - \mathrm{i}\omega \quad (\omega \geq 0)$ is, generally, a complex number.

Note that with the expansion form (60), the solution $\tilde{q}(\xi,\eta,\theta,t,\varepsilon)$ has been extended to the complex field, and as a consequence, the physical solution is now the real part of Eq.(60). This can be done, since our system is a linear one. However, at this point, the independent variables of such a complex solution are still real variables. Later, it will be seen that in order to find the analytical form of the solution $\tilde{q}(\xi,\eta,\theta,t,\varepsilon)$, we need further make analytical continuation for the solution from real axis of ξ to the complex ζ-plane. Thus, eventually our solution $\tilde{q}(\xi,\eta,\theta,t,\varepsilon)$ will be considered as a complex function of the complex variable $\zeta = \xi + \mathrm{i}\xi_{\mathrm{I}}$ and the real variables η, θ and t.

In addition to the expansion form (60), we make the following asymptotic expansions for the functions $(\phi, \psi, \bar{k})$ and the parameter σ:

$$\begin{aligned}
&\phi(\xi,\eta,\varepsilon) = \phi_0(\xi,\eta) + \varepsilon\phi_1(\xi,\eta) + \ldots,\\
&\psi(\xi,\eta,\varepsilon) = \psi_0(\xi,\eta) + \varepsilon\psi_1(\xi,\eta) + \ldots,\\
&k(\xi,\eta,\varepsilon) = k_0(\xi,\eta) + \varepsilon k_1(\xi,\eta) + \ldots,\\
&g(\xi,\eta,\varepsilon) = g_0(\xi,\eta) + \varepsilon g_1(\xi,\eta) + \ldots,\\
&\bar{k}(\xi,\varepsilon) = \bar{k}_0(\xi) + \varepsilon\bar{k}_1(\xi) + \ldots,\\
&\sigma = \sigma_0 + \varepsilon\sigma_1 + \varepsilon^2\sigma_2 + \ldots.
\end{aligned} \tag{61}$$

In the first step, we set σ_0 as an arbitrary constant. By substituting Eqs.(60) and (61) into the system (40) and (53)–(59), one can successively derive each order of approximations.

5.1. Zeroth-order approximation

As zeroth-order approximation we derive the quasi-steady heat conduction equation

$$\begin{aligned}
&(k_0^2 + g_0^2)\left(\frac{\partial^2}{\partial\xi_+^2} + \frac{\partial^2}{\partial\eta_+^2}\right)\tilde{T}_0 = 0,\\
&(k_0^2 + g_0^2)\left(\frac{\partial^2}{\partial\xi_+^2} + \frac{\partial^2}{\partial\eta_+^2}\right)\tilde{T}_{\mathrm{S}0} = 0
\end{aligned} \tag{62}$$

with the following boundary conditions:

(i) In the region of liquid phase $\eta > 1$, as $\eta_+ \to \infty$, one has

$$\tilde{T}_0 \to 0\,. \tag{63}$$

(ii) In the region of solid phase $\eta < 1$, as $\eta_+ \to -\infty$, one has

$$\tilde{T}_{\mathrm{S}0} \to 0\,. \tag{64}$$

(iii) At the interface $\eta_+ = 0$ and $\eta = 1$, one has

$$\tilde{T}_0 = \tilde{T}_{\mathrm{S}0} + \tilde{h}_0, \tag{65}$$
$$\tilde{T}_{\mathrm{S}0} = \frac{\bar{k}_0^2}{S(\xi)}\frac{\partial^2\tilde{h}_0}{\partial\bar{\xi}_+^2}, \tag{66}$$
$$\bar{k}_0\frac{\partial}{\partial\eta_+}\left(\tilde{T}_0 - \tilde{T}_{\mathrm{S}0}\right) + \sigma_0 S^2(\xi)\tilde{h}_0 + \bar{k}_0\xi\frac{\partial\tilde{h}_0}{\partial\bar{\xi}_+} = 0. \tag{67}$$

(iv) As $\xi \to \infty$, one has

$$\tilde{h}_0 = 0. \tag{68}$$

We extend the system analytically to the complex plane (ξ_+, η_+). Then the system allows the normal mode solutions:

$$\begin{aligned}
&\tilde{T}_0 = A_0(\xi,\eta)\exp\{\mathrm{i}\xi_+ - \eta_+\},\\
&\tilde{T}_{\mathrm{S}0} = A_{\mathrm{S}0}(\xi,\eta)\exp\{\mathrm{i}\xi_+ + \eta_+\},\\
&\tilde{h}_0 = \hat{D}_0\exp\{\mathrm{i}\bar{\xi}_+\},
\end{aligned} \tag{69}$$

where the coefficient $\hat{D}_0$ is a constant. The far field conditions (63) and (64) are automatically satisfied, due to the condition (41). Setting

$$\begin{aligned}
&\hat{A}_0(\xi) = A_0(\xi,1),\\
&\hat{A}_{\mathrm{S}0}(\xi) = A_{\mathrm{S}0}(\xi,1),
\end{aligned} \tag{70}$$

from Eqs.(65)–(67) we derive the system of homogeneous equations

$$\begin{aligned}
&\hat{A}_0 - \hat{A}_{\mathrm{S}0} - \hat{D}_0 = 0,\\
&\hat{A}_{\mathrm{S}0} + \frac{\bar{k}_0^2}{S}\hat{D}_0 = 0,\\
&-\bar{k}_0(\hat{A}_0 + \hat{A}_{\mathrm{S}0}) + (\sigma_0 S^2 + \mathrm{i}\xi\bar{k}_0)\hat{D}_0 = 0.
\end{aligned} \tag{71}$$

Obviously, for a nontrivial solution, one must have

$$\Delta = \det \begin{pmatrix} 1 & -1 & -1 \\ 0 & 1 & \bar{k}_0^2/S \\ -\bar{k}_0 & -\bar{k}_0 & \sigma_0 S^2 + \mathrm{i}\xi\bar{k}_0 \end{pmatrix} = 0, \quad (72)$$

which gives the local dispersion relation

$$\sigma_0 = \Sigma(\xi, \bar{k}_0) = \frac{\bar{k}_0}{S^2}\left(1 - \frac{2\bar{k}_0^2}{S}\right) - \frac{\mathrm{i}\xi}{S^2}\bar{k}_0, \quad (0 \le \xi < \infty). \quad (73)$$

Then, one solves

$$\hat{A}_0 + \hat{A}_{\mathrm{S}0} = \left(1 - \frac{2\bar{k}_0^2}{S}\right)\hat{D}_0,$$
$$\hat{A}_0 - \hat{A}_{\mathrm{S}0} = \hat{D}_0. \quad (74)$$

The amplitude functions $\hat{A}(\xi), \hat{A}_{\mathrm{S}}(\xi)$ are fully determined up to an arbitrary constant $\hat{D}$.

The local dispersion formula (73) obtained above is the generalization of the well-known Mullins–Sekerka dispersion formula for the system of unidirectional solidification.[35,36]

For a fixed parameter σ_0, one can solve for three wave numbers as functions of ξ (Fig.3). Namely

$$\bar{k}_0^{(1)}(\xi) = M(\xi)\cos\left\{\frac{1}{3}\cos^{-1}\left(\frac{\sigma_0}{N(\xi)}\right)\right\} \quad \text{(short-wave branch)},$$
$$\bar{k}_0^{(2)}(\xi) = M(\xi)\cos\left\{\frac{1}{3}\cos^{-1}\left(\frac{\sigma_0}{N(\xi)}\right) + \frac{2\pi}{3}\right\},$$
$$\bar{k}_0^{(3)}(\xi) = M(\xi)\cos\left\{\frac{1}{3}\cos^{-1}\left(\frac{\sigma_0}{N(\xi)}\right) + \frac{4\pi}{3}\right\} \quad \text{(long-wave branch)}, \quad (75)$$

where

$$M(\xi) = \sqrt{\frac{2S(\xi)}{3}}\,(1 - \mathrm{i}\xi)^{1/2},$$
$$N(\xi) = -\frac{M(\xi)}{3S^2(\xi)}\,(1 - \mathrm{i}\xi). \quad (76)$$

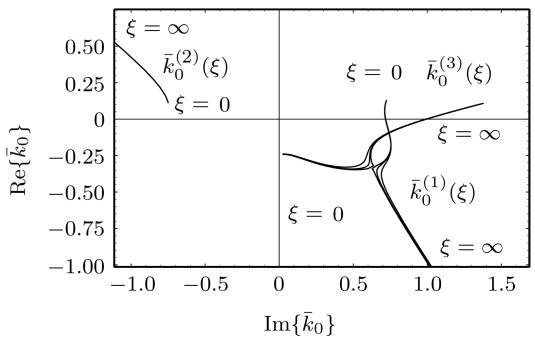


Fig.3. The variations of the wave number functions $\left\{\bar{k}_0^{(1)}(\xi), \bar{k}_0^{(2)}(\xi), \bar{k}_0^{(3)}(\xi)\right\}$ in the complex $\bar{k}$-plane with ξ for given $\sigma_0 = (0.035, -0.265)$; $(0.035, -0.269)$; $(0.035, -0.270)$.

Having obtained the solution for function $\bar{k}_0(\xi)$ defined in the region $(0 \le \xi < \infty)$, we extend it, with the analytical continuation, to the function $\bar{k}_0(\zeta)$ defined in the complex plane $\zeta = \xi + \mathrm{i}\xi_{\mathrm{I}}$. As a consequence, $\bar{k}_0(\zeta)$ is subject to the following extended dispersion formula:

$$\sigma_0 = \Sigma(\zeta, \bar{k}_0) = \frac{\bar{k}_0}{S(\zeta)^2}\left[1 - \frac{2\bar{k}_0^2}{S(\zeta)}\right] - \frac{\mathrm{i}\zeta}{S(\zeta)^2}\bar{k}_0, \quad (77)$$

and can be expressed in the same form as Eq.(75) with the replacement of ξ by ζ.

Moreover, as demonstrated before, one may derive the functions $k_0(\xi, \eta)$, $\phi_0(\xi, \eta)$ and $\psi_0(\xi, \eta)$ on the whole (ξ, η) plane as follows:

$$\phi_0(\xi, \eta) = \frac{1}{2}\left\{\int_0^{\xi + \mathrm{i}(\eta - 1)} \bar{k}_0(t)\mathrm{d}t + \int_0^{\xi - \mathrm{i}(\eta - 1)} \bar{k}_0(t)\mathrm{d}t\right\},$$
$$\psi_0(\xi, \eta) = -\frac{\mathrm{i}}{2}\int_{\xi - \mathrm{i}(\eta - 1)}^{\xi + \mathrm{i}(\eta - 1)} \bar{k}_0(t)\mathrm{d}t, \quad (78)$$

and

$$k_0(\xi,\eta) = \phi_{0,\xi}(\xi,\eta) = \psi_{0,\eta}(\xi,\eta) = \frac{1}{2}\Big\{\bar{k}_0\big[\xi+\mathrm{i}(\eta-1)\big] + \bar{k}_0\big[\xi-\mathrm{i}(\eta-1)\big]\Big\},$$
$$g_0(\xi,\eta) = \phi_{0,\eta}(\xi,\eta) = -\psi_{0,\xi}(\xi,\eta) = \frac{\mathrm{i}}{2}\Big\{\bar{k}_0\big[\xi+\mathrm{i}(\eta-1)\big] - \bar{k}_0\big[\xi-\mathrm{i}(\eta-1)\big]\Big\}. \quad (79)$$

Up to this point, the two fast variables (ξ_+, η_+) are fully determined.

As we have demonstrated before, given σ_0, the dispersion equation has three roots Eq.(75). We derive the three fundamental wave solutions of the system, which respectively correspond to the wave number functions $\{\bar{k}_0^{(1)}, \bar{k}_0^{(2)}, \bar{k}_0^{(3)}\}$:

H_1 wave (short wave branch) with larger $\Re\{\bar{k}_0^{(1)}\} > 0$,

H_2 wave (no physical meaning) with $\Re\{\bar{k}_0^{(2)}\} < 0$,

H_3 wave (long wave branch) with smaller $\Re\{\bar{k}_0^{(3)}\} > 0$.

Among these solutions, the H_2 wave solution must be ruled out because the real part of its wave-number is negative. As $\eta_+ \to \infty$, its corresponding perturbed temperature field $\tilde{T}_0$ in the liquid region ($\eta > 1$) grows exponentially. This violates the boundary condition (63). Consequently, in the zeroth-order approximation, the general H wave solution is

$$\tilde{h} \approx D_1 \mathrm{e}^{\sigma_0 t_+ + \mathrm{i}(\bar{\xi}_+^{(1)} + m\theta)} + D_3 \mathrm{e}^{\sigma_0 t_+ + \mathrm{i}(\bar{\xi}_+^{(3)} + m\theta)}, \quad (80)$$

where

$$\bar{\xi}_+^{(1)} = \frac{1}{\varepsilon}\int_0^{\xi} \bar{k}_0^{(1)} \mathrm{d}\xi_1, \quad \bar{\xi}_+^{(3)} = \frac{1}{\varepsilon}\int_0^{\xi} \bar{k}_0^{(3)} \mathrm{d}\xi_1, \quad (81)$$

and the coefficients $\{D_1, D_3\}$ are arbitrary constants independent of ε, which are to be determined.

5.2. First-order approximation

The first-order approximation solution will fully determine the amplitude $\{A_0(\xi,\eta),\ A_{S0}(\xi,\eta)\}$, the functions $\bar{k}_1(\xi)$, and σ_1. The equation for the first-order approximation can be obtained from Eq.(40) as the following

$$\begin{aligned}
&(k_0^2+g_0^2)\left(\frac{\partial^2}{\partial\xi_+^2}+\frac{\partial^2}{\partial\eta_+^2}\right)\tilde{T}_1 \\
&= -\left(2k_0\frac{\partial^2\tilde{T}_0}{\partial\xi_+\partial\xi} - 2g_0\frac{\partial^2\tilde{T}_0}{\partial\eta_+\partial\xi}\right) \\
&\quad -\left(2k_0\frac{\partial^2\tilde{T}_0}{\partial\eta_+\partial\eta} + 2g_0\frac{\partial^2\tilde{T}_0}{\partial\xi_+\partial\eta}\right) \\
&\quad -\left(\frac{1}{\xi}-\eta_0^2\xi\right)\left(k_0\frac{\partial\tilde{T}_0}{\partial\xi_+} - g_0\frac{\partial\tilde{T}_0}{\partial\eta_+}\right) \\
&\quad -\left(\frac{1}{\eta}+\eta_0^2\eta\right)\left(g_0\frac{\partial\tilde{T}_0}{\partial\xi_+} + k_0\frac{\partial\tilde{T}_0}{\partial\eta_+}\right) \\
&\quad +\eta_0^2\left(\xi^2+\eta^2\right)\frac{\partial\tilde{T}_0}{\partial t_+}. \qquad (82)
\end{aligned}$$

We may write

$$\begin{aligned}
(k_0^2+g_0^2)\left(\frac{\partial^2}{\partial\xi_+^2}+\frac{\partial^2}{\partial\eta_+^2}\right)\tilde{T}_1 &= a_0(\xi,\eta)\mathrm{e}^{\mathrm{i}\xi_+ - \eta_+}, \\
(k_0^2+g_0^2)\left(\frac{\partial^2}{\partial\xi_+^2}+\frac{\partial^2}{\partial\eta_+^2}\right)\tilde{T}_{S1} &= a_{S0}(\xi,\eta)\mathrm{e}^{\mathrm{i}\xi_+ + \eta_+}, \qquad (83)
\end{aligned}$$

where

$$a_0 = 2(k_0 - \mathrm{i}g_0)\left(\frac{\partial A_0}{\partial\eta} - \mathrm{i}\frac{\partial A_0}{\partial\xi}\right) + A_0\Big\{\sigma_0\eta_0^2(\xi^2+\eta^2) - (\mathrm{i}k_0+g_0)\left[\left(\frac{1}{\xi}-\eta_0^2\xi\right) + \mathrm{i}\left(\frac{1}{\eta}+\eta_0^2\eta\right)\right]\Big\} \quad (84)$$

and

$$a_{S0} = -2(k_0 + \mathrm{i}g_0)\left(\frac{\partial A_{S0}}{\partial\eta} + \mathrm{i}\frac{\partial A_{S0}}{\partial\xi}\right) + A_{S0}\Big\{\sigma_0\eta_0^2(\xi^2+\eta^2) - (\mathrm{i}k_0-g_0)\left[\left(\frac{1}{\xi}-\eta_0^2\xi\right) - \mathrm{i}\left(\frac{1}{\eta}+\eta_0^2\eta\right)\right]\Big\}. \quad (85)$$

To eliminate the secular terms on the right-hand side of Eq.(83), one must set

$$a_0 = a_{S0} = 0. \quad (86)$$

Then, from Eq.(86) it follows that

$$\left(\frac{\partial}{\partial\eta}-\mathrm{i}\frac{\partial}{\partial\xi}\right)\ln A_0(\xi,\eta)=\Phi_0(\xi,\eta),$$
$$\left(\frac{\partial}{\partial\eta}+\mathrm{i}\frac{\partial}{\partial\xi}\right)\ln A_{\mathrm{S}0}(\xi,\eta)=\Phi_{\mathrm{S}0}(\xi,\eta), \tag{87}$$

where

$$\Phi_0(\xi,\eta)=-\frac{\sigma_0\eta_0^2(\xi^2+\eta^2)}{2\bar{k}_0\big[\xi+\mathrm{i}(\eta-1)\big]}+\frac{\mathrm{i}}{2}\left[\left(\frac{1}{\xi}-\eta_0^2\xi\right)+\mathrm{i}\left(\frac{1}{\eta}+\eta_0^2\eta\right)\right],$$
$$\Phi_{\mathrm{S}0}(\xi,\eta)=\frac{\sigma_0\eta_0^2(\xi^2+\eta^2)}{2\bar{k}_0\big[\xi-\mathrm{i}(\eta-1)\big]}-\frac{\mathrm{i}}{2}\left[\left(\frac{1}{\xi}-\eta_0^2\xi\right)-\mathrm{i}\left(\frac{1}{\eta}+\eta_0^2\eta\right)\right]. \tag{88}$$

Equations (87) are first-order hyperbolic equations. $A_0(\xi,\eta)$ and $A_{\mathrm{S}0}(\xi,\eta)$ can be solved in the ξ-η plane as an initial value problem, given the initial conditions on the line $\eta=1$:

$$A_0(\xi,1)=\hat{A}_0(\xi);\quad A_{\mathrm{S}0}(\xi,1)=\hat{A}_{\mathrm{S}0}(\xi)$$

and

$$\bar{k}_0\big[\xi+\mathrm{i}(\eta-1)\big]=\bar{k}_0\big[\xi-\mathrm{i}(\eta-1)\big]=\bar{k}_0(\xi).$$

Moreover, with the above results, we may derive that at $\eta=1$,

$$\begin{aligned}\hat{D}_0Q_0 &= \frac{\partial}{\partial\eta}\big(A_0-A_{\mathrm{S}0}\big)\Big|_{\eta=1}\\ &= \mathrm{i}\frac{\partial}{\partial\xi}\big(\hat{A}_0+\hat{A}_{\mathrm{S}0}\big)+\hat{A}_0\Phi_0-\hat{A}_{\mathrm{S}0}\Phi_{\mathrm{S}0}\\ &= \mathrm{i}\frac{\partial}{\partial\xi}\big(\hat{A}_0+\hat{A}_{\mathrm{S}0}\big)-(\hat{A}_0+\hat{A}_{\mathrm{S}0})\frac{\sigma_0\eta_0^2(1+\eta^2)}{2\bar{k}_0(\xi)}\\ &\quad+\frac{\mathrm{i}}{2}(\hat{A}_0+\hat{A}_{\mathrm{S}0})\left(\frac{1}{\xi}-\eta_0^2\xi\right)\\ &\quad-\frac{1}{2}\big(1+\eta_0^2\big)(\hat{A}_0-\hat{A}_{\mathrm{S}0}).\end{aligned} \tag{89}$$

It follows that

$$\begin{aligned}Q_0(\xi) &= -\mathrm{i}\frac{\partial}{\partial\xi}\left(\frac{2\bar{k}_0^2}{S}\right)+\left(1-\frac{2\bar{k}_0^2}{S}\right)\\ &\quad\times\left[-\frac{\sigma_0\eta_0^2S^2}{2\bar{k}_0}+\frac{\mathrm{i}}{2}\left(\frac{1}{\xi}-\eta_0^2\xi\right)\right]\\ &\quad-\frac{1}{2}\big(1+\eta_0^2\big).\end{aligned} \tag{90}$$

These formulas will be needed later to solve for $k_1(\xi)$.

We now turn to solve the equation of the temperature field under the conditions (86):

$$\left(\frac{\partial^2}{\partial\xi_+^2}+\frac{\partial^2}{\partial\eta_+^2}\right)\tilde{T}_1=0,$$
$$\left(\frac{\partial^2}{\partial\xi_+^2}+\frac{\partial^2}{\partial\eta_+^2}\right)\tilde{T}_{S1}=0. \tag{91}$$

The interface conditions are: at the interface, $\eta_+=0,\eta=1$,

(i) for the thermodynamic equilibrium condition, one has

$$\tilde{T}_1=\tilde{T}_{\mathrm{S}1}+\tilde{h}_1-\eta_1\bar{k}_0\frac{\partial}{\partial\eta_+}\big(\tilde{T}_0-\tilde{T}_{\mathrm{S}0}\big); \tag{92}$$

(ii) for the Gibbs–Thomson condition, one has

$$\begin{aligned}&\tilde{T}_{\mathrm{S}1}+\eta_1\bar{k}_0\frac{\partial\tilde{T}_{\mathrm{S}0}}{\partial\eta_+}\\ &=\frac{1}{S(\xi)}\left\{\bar{k}_0^2\frac{\partial^2\tilde{h}_1}{\partial\xi_+^2}+2\bar{k}_0\bar{k}_1\frac{\partial^2\tilde{h}_0}{\partial\xi_+^2}+2\bar{k}_0\frac{\partial^2\tilde{h}_0}{\partial\xi_+\partial\xi}\right.\\ &\quad\left.+\bar{k}_0'\frac{\partial\tilde{h}_0}{\partial\xi_+}+\bar{k}_0\left(\frac{1}{\xi}+\frac{\xi}{S^2(\xi)}\right)\frac{\partial\tilde{h}_0}{\partial\xi_+}\right\};\end{aligned} \tag{93}$$

(iii) for the heat balance condition, one has

$$\begin{aligned}&\bar{k}_0\left(\frac{\partial\tilde{T}_1}{\partial\eta_+}-\frac{\partial\tilde{T}_{\mathrm{S}1}}{\partial\eta_+}\right)+\bar{k}_1\left(\frac{\partial\tilde{T}_0}{\partial\eta_+}-\frac{\partial\tilde{T}_{\mathrm{S}0}}{\partial\eta_+}\right)\\ &+\eta_1\bar{k}_0^2\frac{\partial^2}{\partial\eta_+^2}\big(\tilde{T}_0-\tilde{T}_{\mathrm{S}0}\big)+\left(\frac{\partial\tilde{T}_0}{\partial\eta}-\frac{\partial\tilde{T}_{\mathrm{S}0}}{\partial\eta}\right)\\ &+S^2(\xi)(\sigma_0\tilde{h}_1+\sigma_1\tilde{h}_0)+\xi\left(\bar{k}_0\frac{\partial\tilde{h}_1}{\partial\bar{\xi}_+}+\bar{k}_1\frac{\partial\tilde{h}_0}{\partial\xi_+}\right)\\ &+\big(2+\eta_0^2\big)\,\tilde{h}_0=0.\end{aligned} \tag{94}$$

The first-order approximate solutions can be obtained in the form of normal modes:

$$\tilde{T}_1=A_1(\xi,\eta)\exp\{\mathrm{i}\xi_+-\eta_+\},$$
$$\tilde{T}_{S1}=A_{\mathrm{S}1}(\xi,\eta)\exp\{\mathrm{i}\xi_++\eta_+\},$$
$$\tilde{h}_1=\hat{D}_1\exp\{\mathrm{i}\xi_+\}. \tag{95}$$

By setting

$$\hat{A}_1(\xi) = A_1(\xi, 1),$$
$$\hat{A}_{S1}(\xi) = A_{S1}(\xi, 1), \tag{96}$$

from Eqs.(92)–(94) it is derived that

$$\hat{A}_1 - \hat{A}_{S1} - \hat{D}_1 = I_1\hat{D}_0 = \bar{k}_0\eta_1\left(1 - \frac{2\bar{k}_0^2}{S}\right)\hat{D}_0,$$
$$\hat{A}_{S1} + \frac{\bar{k}_0^2}{S}\hat{D}_1 = I_2\hat{D}_0,$$
$$-\bar{k}_0(\hat{A}_1 + \hat{A}_{S1}) + (\sigma_0 S^2 + \mathrm{i}\xi\bar{k}_0)\hat{D}_1 = I_3\hat{D}_0, \tag{97}$$

where we have defined

$$I_1 = \bar{k}_0\eta_1\left(1 - \frac{2\bar{k}_0^2}{S}\right), \tag{98}$$
$$I_2 = \frac{1}{S}\left\{\mathrm{i}\bar{k}_0' + \mathrm{i}\bar{k}_0\left(\frac{1}{\xi} + \frac{\xi}{S^2}\right) - 2\bar{k}_0\bar{k}_1\right\} + \frac{\bar{k}_0^3}{S}\eta_1, \tag{99}$$

and

$$I_3 = -\sigma_1 S^2 + \bar{k}_1\left(1 - \mathrm{i}\xi - \frac{2\bar{k}_0^2}{S}\right) - Q_0 - (2 + \eta_0^2) - \eta_1\bar{k}_0^2. \tag{100}$$

Since the determinant Δ of the coefficient matrix of the above inhomogeneous system is zero, the necessary condition for the existence of a nontrivial solution for $\{\hat{A}_1, \hat{A}_{S1}, \hat{D}_1\}$ is

$$\det\begin{pmatrix} 1 & -1 & I_1 \\ 0 & 1 & I_2 \\ -\bar{k}_0 & -\bar{k}_0 & I_3 \end{pmatrix} = 0, \tag{101}$$

which results in the solvability condition:

$$I_3 + 2\bar{k}_0 I_2 + I_1\bar{k}_0 = 0, \tag{102}$$

or

$$\bar{k}_1\left[1 - \mathrm{i}\xi - \frac{6\bar{k}_0^2}{S}\right] = S^2\sigma_1 + (2 + \eta_0^2) - \mathrm{i}\frac{2\bar{k}_0\bar{k}_0'}{S} - \mathrm{i}\frac{2\bar{k}_0^2}{S}\left(\frac{1}{\xi} + \frac{\xi}{S^2}\right) + Q_0.$$

With Eq.(90), one may rewrite the above formula in the following form:

$$\bar{k}_1(\xi) = \frac{\mathrm{i}}{2\xi} + \frac{R_1(\xi)}{F(\xi)} - \mathrm{i}\frac{R_2(\xi)}{F(\xi)}\frac{\bar{k}_0'(\xi)}{\bar{k}_0(\xi)}, \tag{103}$$

where we have denoted

$$F(\xi) = \left[1 - \mathrm{i}\xi - \frac{6\bar{k}_0^2}{S}\right],$$
$$R_1(\xi) = S^2\sigma_1 + \left(1 + \frac{\eta_0^2}{2}\right) - \frac{\eta_0^2}{2}\left(1 - \frac{2\bar{k}_0^2}{S}\right)^2.$$
$$R_2(\xi) = \frac{6\bar{k}_0^2}{S}. \tag{104}$$

It should be noted that all the terms involving $\eta_1(\xi, \theta)$ in the formula (103) are cancelled out. As before, one may extend the function $\bar{k}_1(\xi)$ to the $\bar{k}_1(\zeta)$ defined on the complex plane $\zeta = \xi + \mathrm{i}\xi_{\mathrm{I}}$ by analytical continuation. This can be done by replacing all the variable ξ by ζ in the formulas (103) and (104). Moreover, with the formulas:

$$k_0(\zeta, \eta) = \frac{1}{2}\left\{[\bar{k}_0[\zeta + \mathrm{i}(\eta - 1)] + \bar{k}_0[\zeta - \mathrm{i}(\eta - 1)]\right\},$$
$$k_1(\zeta, \eta) = \frac{1}{2}\left\{\bar{k}_1[\zeta + \mathrm{i}(\eta - 1)] + \bar{k}_1[\zeta - \mathrm{i}(\eta - 1)]\right\},$$

and

$$\phi(\zeta, \eta) = \int_0^{\zeta}(k_0 + \varepsilon k_1 + \ldots)\mathrm{d}\zeta_1,$$
$$\psi(\zeta, \eta) = \int_1^{\eta}(k_0 + \varepsilon k_1 + \ldots)\mathrm{d}\eta_1,$$

one may extend the MVE solution (60) and (80) to the solution depending on the complex variable ξ, namely the variable ζ, and the real variable η.

From Eq.(103), it is seen that the solution $\bar{k}_1(\zeta)$ has an isolated point singularity at the tip $\zeta = 0$ and

$$\bar{k}_1(\zeta) \sim \frac{\mathrm{i}}{2\zeta}, \tag{105}$$

as $\zeta \to 0$. Furthermore, from the dispersion formula (77), one finds that

$$F(\zeta) = S^2\left(\frac{\partial\Sigma}{\partial\bar{k}_0}\right). \tag{106}$$

Therefore, it is seen that at the root ζ_c of the equation

$$\left(\frac{\partial\Sigma}{\partial\bar{k}_0}\right) = 0 \quad (\text{or} \quad \mathrm{F}(\zeta) = 0), \tag{107}$$

the solution $\bar{k}_1$, as well as the functions A_0 and A_{S0}, also has a singularity. Given σ_0, from the local dispersion formula (73) we have

$$\frac{\bar{k}_0'(\zeta)}{\bar{k}_0(\zeta)} = \frac{R_0(\zeta)}{F(\zeta)},$$
$$R_0(\zeta) = \mathrm{i} + \frac{2\zeta(1 - \mathrm{i}\zeta)}{S^2} - \frac{6\zeta\bar{k}_0^2}{S^3}. \tag{108}$$

Moreover, one can derive that as $\zeta \to \zeta_c$,

$$F(\zeta) \sim 2^{1/2} m_0^{1/2} (\zeta - \zeta_c)^{1/2}, \tag{109}$$

and

$$\bar{k}_1(\zeta) \sim \frac{R_1(\zeta_c)}{2^{1/2} m_0^{1/2} (\zeta - \zeta_c)^{1/2}} + \frac{\mathrm{i}}{4(\zeta - \zeta_c)} + \mathcal{O}(1) \tag{110}$$

where

$$m_0 = -\frac{12\bar{k}_0^2(\zeta_c)}{S(\zeta_c)} R_0(\zeta_c). \tag{111}$$

Notice that the function $\bar{k}_1(\zeta)$ must be an analytic function in the complex ζ-plane except some isolated singular points ζ_c. Since near the isolated singular point, the analytical function can be expanded into the Laurent series. It implies that Eq.(110),

$$R_1(\zeta_c) = 0 \tag{112}$$

must hold. It is therefore deduced that the point $\zeta = \zeta_c$, as well as $\zeta = 0$, is actually a simple pole of the function $\bar{k}_1(\zeta)$ in the complex ζ-plane.

From condition (112), one obtains:

$$\sigma_1 = -\frac{1}{(1+\zeta_c^2)} \left\{ \left(1 + \frac{\eta_0^2}{2}\right) - \frac{\eta_0^2}{2} \left[1 - \frac{2\bar{k}_0^2(\zeta_c)}{S(\zeta_c)}\right]^2 \right\}. \tag{113}$$

From $F(\zeta_c) = 0$, it follows that

$$\left[1 - \frac{2\bar{k}_0^2(\zeta_c)}{S(\zeta_c)}\right] = \frac{2 + \mathrm{i}\zeta_c}{3}. \tag{114}$$

Hence, Eq.(113) may be re-written as

$$\sigma_1 = -\frac{1}{(1+\zeta_c^2)} \left[\left(1 + \frac{\eta_0^2}{2}\right) - \frac{\eta_0^2}{18} \left(2 + \mathrm{i}\zeta_c\right)^2 \right]. \tag{115}$$

From Eq.(115), it is seen that up to the approximation of $\mathcal{O}(\varepsilon)$, the eigenvalue σ, as well as the eigenfunctions are not affected by the interface shape correction, $\eta_1(\xi, \theta)$, which may be induced by isotropic/anisotropic surface tension. From this result, it is concluded that up to the first approximation, one can use the Ivantsov's solution with zero surface tension as the basic state of dendritic growth.

Up to this point, the MVE solution is not fully determined yet. The complex parameter σ_0 still remains free. The tip smoothness condition is also not applied yet. However, once σ_0 is determined, σ_1 follows.

In the next part, we shall study the singular point ζ_c in more details, and complete the derivation of the global wave modes solutions.

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