Vol 18 No 2, February 2009
1674-1056/2009/18(02)/0611-05

Chinese Physics B

© 2009 Chin. Phys. Soc.
and IOP Publishing Ltd

Relation between Fresnel transform of input light field and the two-parameter Radon transform of Wigner function of the field*

Fan Hong-Yi(范洪义) and Hu Li-Yun(胡利云)†

Department of Physics, Shanghai Jiao Tong University, Shanghai 200030, *China*

(Received 21 August 2008; revised manuscript received 11 September 2008)

This paper proves a new theorem on the relationship between optical field Wigner function's two-parameter Radon transform and optical Fresnel transform of the field, i.e., when an input field $\psi(x')$ propagates through an optical $[D\,(-B)\,(-C)\,A]$ system, the energy density of the output field is equal to the Radon transform of the Wigner function of the input field, where the Radon transform parameters are D, B. It prove this theorem in both spatial-domain and frequency-domain, in the latter case the Radon transform parameters are A, C.

Keywords: Fresnel transform, two-parameter Radon transform, Wigner function
PACC: 4250, 0450

In optical communication theory every signal or image can be uniquely and indirectly described by a Wigner distribution function (WDF).[1−3] The WDF (or named Wigner transform) of an optical signal field $\psi(x')$ is defined as

$$W_\psi(\nu', x') = \int_{-\infty}^{+\infty} \frac{\mathrm{d}u}{2\pi} \mathrm{e}^{\mathrm{i}\nu' u} \times \psi^*\left(x' + \frac{u}{2}\right) \psi\left(x' - \frac{u}{2}\right), \quad (1)$$

where $W_\psi(\nu', x')$ involves both spatial distribution information and space–frequency distribution information of the signal, ν is named space frequency. $W_\psi(\nu', x')$ is said to be bilinear in the signal because the signal enters twice in its definition. The WDF undergoes certain variations if something happens to the signal. For examples, passage through a lens corresponds to a vertical shearing of the WDF, propagation in free space means a horizontal shearing of the WDF.[4] However, the WDF preserves space and space frequency marginal properties of any signal,

$$\int_{-\infty}^{+\infty} \mathrm{d}\nu' W_\psi(\nu', x') = |\psi(x')|^2, \quad (2)$$

$$\int_{-\infty}^{+\infty} \mathrm{d}x' W_\psi(\nu', x') = |\tilde{\psi}(\nu')|^2, \quad (3)$$

where $\tilde{\psi}(\nu) = \int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{\sqrt{2\pi}} \psi(x)\, \mathrm{e}^{\mathrm{i}x\nu}$. If one wants to reconstruct the Wigner function by using various probability distribution, obviously the position density $|\psi(x')|^2$ and the space–frequency density $|\tilde{\psi}(\nu')|^2$ were not enough, so the Radon transform[5,6] of the Wigner function is introduced,[7]

$$R(x) \equiv \iint_{-\infty}^{\infty} \mathrm{d}x' \mathrm{d}\nu' \delta(x - Dx' + B\nu') W_\psi(\nu', x'), \quad (4)$$

where $R(x)$ is also a probability distribution along an infinitely thin phase space strip denoted by the real parameters D, B. The inverse relation of Eq.(4) is the foundation of optical tomographic imaging techniques (the techniques derive two-dimensional data from a three-dimensional object to obtain a slice image of the internal structure and thus have the ability to peer inside the object noninvasively.)

On the other hand, an optical system can be analysed by using either diffraction theory or ray optics. It is worth setting up a connection between ray transfer matrix and diffraction theory, i.e., the diffraction integration written in terms of ray transfer matrix—named Collins' diffraction integral formula[8−12] which describes the propagation of a general beam $\psi(x')$ through an $(ABCD)$ optical paraxial system,

$$\phi(x) = \int_{-\infty}^{\infty} \mathcal{K}(x, x') \psi(x') \mathrm{d}x', \quad (5)$$

*Project supported by the National Natural Science Foundation of China (Grant Nos 10775097 and 10874174).
†Corresponding author. E-mail: hlyun@sjtu.edu.cn; hlyun2008@126.com

where $\phi(x)$ is the output field, $AD-BC=1$, and $\mathcal{K}(x,x')$ is the integral kernel

$$\mathcal{K}(x,x')=\frac{1}{\sqrt{2\pi \mathrm{i}B}}\exp\left[\frac{\mathrm{i}}{2B}\left(Ax'^2-2x'x+Dx^2\right)\right]. \tag{6}$$

If the energy in the initial beam is normalized, $\int_{-\infty}^{+\infty}|\psi(x')|^2\,\mathrm{d}x'=1$, then the output beam is normalized too, $\int_{-\infty}^{+\infty}|\phi(x)|^2\,\mathrm{d}x=1$. Clearly, if the $[ABCD]$ system is changed to $[D\,(-B)\,(-C)\,A]$ system, then Eq.(5) should read

$$\phi(x)=\int_{-\infty}^{\infty}\mathcal{K}(x,x')\,\psi(x')\,\mathrm{d}x', \tag{7}$$

where the integral kernel $\mathcal{K}(x,x')$ is

$$\mathcal{K}(x,x')=\frac{1}{\sqrt{-2\pi \mathrm{i}B}}\exp\left[\frac{-\mathrm{i}}{2B}\left(Dx'^2-2x'x+Ax^2\right)\right]. \tag{8}$$

In this paper, we shall reveal the following theorem:

When an input field $\psi(x')$ propagates through an optical $[D\,(-B)\,(-C)\,A]$ system, the energy density of the output field $\phi(x)$ is equal to the Radon transform of the Wigner function of the input field, where the Radon transform parameters are D, B. The proof is demonstrated as follows.

With the use of Dirac δ-function, we can re-express the Wigner function of input field $\psi(x')$ in Eq.(1) as

$$\begin{aligned}W_{\psi}(\nu',x')=&\iint_{-\infty}^{\infty}\mathrm{d}x''\mathrm{d}x'''\int_{-\infty}^{+\infty}\frac{\mathrm{d}u}{2\pi}\mathrm{e}^{\mathrm{i}\nu' u}\psi^*(x'')\,\psi(x''')\,\delta\left(x''-x'-\frac{u}{2}\right)\delta\left(x'-\frac{u}{2}-x'''\right)\\ =&\frac{1}{\pi}\iint_{-\infty}^{\infty}\mathrm{d}x''\mathrm{d}x'''\psi^*(x'')\,\psi(x''')\,\mathrm{e}^{\mathrm{i}2\nu'\left(x''-x'\right)}\delta\left(2x'-x''-x'''\right).\end{aligned} \tag{9}$$

Substituting Eq.(9) into Eq.(4) we can rewrite the Radon transform of $W_{\psi}(\nu',x')$ as

$$\begin{aligned}R(x)=&\frac{1}{\pi}\iint_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}\nu'\delta\left(x-Dx'+B\nu'\right)\iint_{-\infty}^{\infty}\mathrm{d}x''\mathrm{d}x'''\psi^*(x'')\,\psi(x''')\,\mathrm{e}^{\mathrm{i}2\nu'\left(x''-x'\right)}\delta\left(2x'-x''-x'''\right)\\ =&\frac{1}{2\pi B}\iint_{-\infty}^{\infty}\mathrm{d}x''\mathrm{d}x'''\psi^*(x'')\,\psi(x''')\exp\left\{\frac{\mathrm{i}}{2B}[D\left(x''^2-x'''^2\right)-2x\left(x''-x'''\right)\right\}.\end{aligned} \tag{10}$$

On the other hand, when the beam $\psi(x')$ propagates through the $[D\,(-B)\,(-C)\,A]$ optical system, according to Eqs.(7) and (8), we have

$$\begin{aligned}|\phi(x)|^2=&\int_{-\infty}^{\infty}\mathrm{d}x'\mathcal{K}(x,x')\,\psi(x')\int_{-\infty}^{\infty}\mathrm{d}x''\psi^*(x'')\,\mathcal{K}^*(x,x'')\\ =&\frac{1}{2\pi B}\int_{-\infty}^{\infty}\mathrm{d}x'\exp\left[\frac{-\mathrm{i}}{2B}\left(Dx'^2-2x'x+Ax^2\right)\right]\psi(x')\int_{-\infty}^{\infty}\mathrm{d}x''\exp\left[\frac{\mathrm{i}}{2B}\left(Dx''^2-2x''x+Ax^2\right)\right]\psi^*(x'')\\ =&\frac{1}{2\pi B}\iint_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}x''\psi(x')\,\psi^*(x'')\exp\left\{\frac{\mathrm{i}}{2B}\left[D\left(x''^2-x'^2\right)+2x\left(x'-x''\right)\right]\right\},\end{aligned} \tag{11}$$

which is the same as $R(x)$ in Eq.(10). So combining Eqs.(4), (10), (11), (7) and (8) we reach the conclusion

$$\begin{aligned}&\left|\frac{1}{\sqrt{-2\pi \mathrm{i}B}}\int_{-\infty}^{\infty}\exp\left[\frac{-\mathrm{i}}{2B}\left(Dx'^2-2x'x+Ax^2\right)\right]\psi(x')\,\mathrm{d}x'\right|^2\\ =&\iint_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}\nu'\delta\left(x-Dx'+B\nu'\right)W_{\psi}(\nu',x'),\end{aligned} \tag{12}$$

where $DA-BC=1$. The physical meaning of Eq.(12) is: when an input field propagates through an optical $[D\,(-B)\,(-C)\,A]$ system, the energy density of the output field is equal to the Radon transform of the Wigner function of the input field. So far as our knowledge is concerned, this conclusion seems new.

Equation (8) is the relationship between the input amplitude and output one in spatial-domain. Now we turn the above discussion to the case of the space–frequency domain.

For a $[D\,(-B)\,(-C)\,A]$ optical system, the Collins' diffraction integral formula in space–frequency (angle spectrum[13]) domain is[14]

$$\widetilde{\phi}\,(v)=\int_{-\infty}^{\infty}\widetilde{\mathcal{K}}\,(\nu,\nu')\,\widetilde{\psi}\,(\nu')\,\mathrm{d}\nu', \tag{13}$$

where the kernel is

$$\widetilde{\mathcal{K}}\,(\nu,\nu')=\frac{1}{\sqrt{2\pi\mathrm{i}C}}\exp\left[\frac{\mathrm{i}}{2C}\left(D\nu^2-2\nu'\nu+A\nu'^2\right)\right]. \tag{14}$$

On the other hand, the Wigner function expressed in terms of the space–frequency field is

$$W_{\tilde{\psi}}(\nu',x')=\int_{-\infty}^{+\infty}\frac{\mathrm{d}s}{2\pi}\mathrm{e}^{-\mathrm{i}x's}\tilde{\psi}^*\left(\nu'+\frac{s}{2}\right)\times\tilde{\psi}\left(\nu'-\frac{s}{2}\right). \tag{15}$$

In the space–frequency domain the Radon transform is along an infinitely thin phase space strip denoted by the real parameters A, C,

$$R\,(\nu)\equiv\iint_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}p'\delta\,(\nu-A\nu'+Cx')\,W_{\tilde{\psi}}(\nu',x'), \tag{16}$$

rather than the parameters D, B of the Radon transform (4) in spatial-domain. We want to examine if the above conclusion still holds in the space–frequency domain. For this propose, we rewrite Eq.(15) as

$$\begin{aligned}W_{\tilde{\psi}}(\nu',x')&=\iint_{-\infty}^{\infty}\mathrm{d}\nu''\mathrm{d}\nu'''\int_{-\infty}^{+\infty}\frac{\mathrm{d}s}{2\pi}\mathrm{e}^{-ix's}\tilde{\psi}^*\,(\nu'')\,\tilde{\psi}\,(\nu''')\,\delta\left(\nu''-\nu'-\frac{s}{2}\right)\delta\left(\nu'-\frac{s}{2}-\nu'''\right)\\&=\iint_{-\infty}^{\infty}\mathrm{d}\nu''\mathrm{d}\nu'''\tilde{\psi}^*\,(\nu'')\,\tilde{\psi}\,(\nu''')\,\mathrm{e}^{-\mathrm{i}2x'\left(\nu''-\nu'\right)}\delta\,(2\nu'-\nu''-\nu'''). \end{aligned}\tag{17}$$

Then we substitute Eq.(17) into Eq.(16), and the result is

$$\begin{aligned}R\,(\nu)=&\frac{1}{\pi}\iint_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}\nu'\delta\,(\nu-A\nu'+Cx')\iint_{-\infty}^{\infty}\mathrm{d}\nu''\mathrm{d}\nu'''\tilde{\psi}^*\,(\nu'')\,\tilde{\psi}\,(\nu''')\,\mathrm{e}^{-\mathrm{i}2x'\left(\nu''-\nu'\right)}\delta\,(2\nu'-\nu''-\nu''')\\=&\frac{1}{2\pi C}\iint_{-\infty}^{\infty}\mathrm{d}\nu''\mathrm{d}\nu'''\tilde{\psi}^*\,(\nu'')\,\tilde{\psi}\,(\nu''')\exp\left\{-\frac{\mathrm{i}}{2C}\left[A\left(\nu''^2-\nu'''^2\right)-2\nu\left(\nu''-\nu'''\right)\right]\right\}. \end{aligned}\tag{18}$$

On the other hand, from Eq.(13) we get

$$\begin{aligned}|\tilde{\phi}\,(\nu)\,|^2=&\frac{1}{2\pi C}\int_{-\infty}^{+\infty}\tilde{\psi}\,(\nu')\exp\left[\frac{\mathrm{i}}{2C}\left(D\nu^2-2\nu'\nu+A\nu'^2\right)\right]\mathrm{d}\nu'\int_{-\infty}^{+\infty}\tilde{\psi}^*\,(\nu'')\exp\left[\frac{-\mathrm{i}}{2C}\left(D\nu^2-2\nu''\nu+A\nu''^2\right)\right]\mathrm{d}\nu''\\=&\frac{1}{2\pi C}\int\int_{-\infty}^{+\infty}\mathrm{d}\nu'\mathrm{d}\nu''\tilde{\psi}\,(\nu')\,\tilde{\psi}^*\,(\nu'')\exp\left[-\frac{\mathrm{i}}{2C}\left(A\left(\nu''^2-\nu'^2\right)-2\nu\left(\nu''-\nu'\right)\right)\right]. \end{aligned}\tag{19}$$

Comparing Eq.(19) with Eq.(18) and using Eqs.(13), (14) we obtain

$$\begin{aligned}&\left|\frac{1}{\sqrt{2\pi\mathrm{i}C}}\int\exp\left[\frac{\mathrm{i}}{2C}\left(D\nu^2-2\nu'\nu+A\nu'^2\right)\right]\tilde{\psi}\,(\nu')\,\mathrm{d}\nu'\right|^2\\=&\iint_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}p'\delta\,(\nu-A\nu'+Cx')\,W_{\tilde{\psi}}(\nu',x'), \end{aligned}\tag{20}$$

this is the theorem expressed in the space–frequency domain.

We now take an example to confirm the theorem. When the input field is described by a Gaussian function (Gaussian chirplet)

$$\psi_0(x') \equiv \sqrt[4]{\frac{\epsilon}{\pi}}\exp\left\{-(\epsilon-\mathrm{i}\beta)\frac{x'^2}{2}\right\},\ \epsilon>0, \tag{21}$$

which depicts a Gaussian windowed linear chirp signal.[15] Using Eq.(1) we obtain its Wigner function

$$\begin{aligned} W_{\psi_0}(\nu',x') =& \mathrm{e}^{-\epsilon x'^2}\sqrt{\frac{\epsilon}{\pi}}\int_{-\infty}^{+\infty}\frac{\mathrm{d}u}{2\pi}\mathrm{e}^{-\frac{1}{4}\epsilon u^2+\mathrm{i}(\nu'-\beta x')u} \\ =& \frac{1}{\pi}\exp\left\{-\left[\epsilon x'^2+\frac{1}{\epsilon}(\nu'-\beta x')^2\right]\right\}, \end{aligned} \tag{22}$$

which shows that the energy of Gaussian chirplet is concentrated at $\nu'=\beta x'$. According to Eq.(4), the Radon transform of Eq.(22) with the parameters D, B is

$$\begin{aligned} &\iint_{-\infty}^{\infty}\mathrm{d}x'\mathrm{d}\nu'\delta(x-Dx'+B\nu')W_{\psi_0}(\nu',x') \\ =&\frac{1}{\pi D}\int_{-\infty}^{+\infty}\mathrm{d}\nu'\exp\left\{-\frac{\epsilon}{D^2}(x+B\nu')^2-\frac{1}{\epsilon D^2}\left[(D-\beta B)\nu'-\beta x\right]^2\right\} \\ =&\sqrt{\frac{\epsilon/\pi}{(D-B\beta)^2+B^2\epsilon^2}}\exp\left[\frac{-\epsilon x^2}{(D-B\beta)^2+B^2\epsilon^2}\right]\equiv R_0(x). \end{aligned} \tag{23}$$

On the other hand, according to Eqs.(7) and (8), the Fresnel transform of input field $W_{\psi_0}(\nu',x')$ through a $[D\ (-B)\ (-C)\ A]$ optical system is given by

$$\begin{aligned} &\frac{1}{\sqrt{-2\pi\mathrm{i}B}}\sqrt[4]{\frac{\epsilon}{\pi}}\int_{-\infty}^{\infty}\mathrm{d}x'\exp\left[-(\epsilon-\mathrm{i}\beta)\frac{x'^2}{2}\right]\exp\left[\frac{-\mathrm{i}}{2B}\left(Dx'^2-2x'x+Ax^2\right)\right] \\ =&\sqrt[4]{\frac{\epsilon}{\pi}}\frac{1}{\sqrt{-\mathrm{i}}}\mathrm{e}^{-\frac{\mathrm{i}A}{2B}x^2}\sqrt{\frac{1}{\epsilon B-\mathrm{i}(\beta B-D)}}\exp\left[\frac{-x^2}{2B(B\epsilon+\mathrm{i}D-\mathrm{i}\beta B)}\right]\equiv\phi_0(x). \end{aligned} \tag{24}$$

Comparing Eq.(24) with Eq.(23) and noticing

$$\frac{1}{2B(B\epsilon+\mathrm{i}D-\mathrm{i}\beta B)}+\frac{1}{2B(B\epsilon-\mathrm{i}D+\mathrm{i}\beta B)}=\frac{\epsilon}{(D-B\beta)^2+B^2\epsilon^2} \tag{25}$$

we get

$$|\phi_0(x)|^2=R_0(x), \tag{26}$$

as expected.

In summary, we have derived a new theorem governing the connection between optical field's Fresnel transform and its Wigner function's Radon transformation, since both the Wigner function and the Fresnel transform are widely used in optical propagation, and the fractional Fourier transformation is related to Fresnel transform,[16−18] we hope this theorem would have new applications in the analysis of optical communication and optical tomography. For the application of Wigner function in deriving the quantum-mechanical photocount formula, we refer to Ref.[19].

References

[1] Wigner E P 1932 *Phys. Rev.* **40** 749

[2] Bastiaans M J 1978 *Opt. Commun.* **25** 26

[3] Bastiaans M J 1979 *J. Opt. Soc. Am.* **69** 1710

[4] Lohmann A W 1993 *J. Opt. Soc. Am.* A **10** 2181
Lohmann A W and Soffer B H 1994 *J .Opt. Soc. Am.* A **11** 1798

[5] Radon J 1917 *Ber. Verh. Saechs. Akad. Wiss. Leipzig*

Math. Phys. K1 **69** 262
[6] Zhang Y, Gu B, Dong B and Yang G 1998 *Opt. Lett.* **23** 1126
[7] For quantum states' reconstruction, see e.g., Wolfgang P. Schleich 2001 *Quantum Optics in Phase Space* (Berlin: Wiley-VCH) and references therein
[8] Collins S A 1970 *J. Opt. Soc. Am.* A **60** 1168
[9] Arnaud J A 1971 *J. Opt. Soc. Am.* **61** 751
[10] Fan H Y and Lu H L 2006 *Opt. Lett.* **31** 2622
[11] Fan H Y and Hu L Y 2008 *Chin. Phys.* B **17** 1640
[12] Wang S M and Zhao D M 2000 *Matrix Optics* (Beijing: China Higher Education Press) Chap.3, p38
[13] Goodman J W 1968 *Introduction to Fourier Optics* (New York: McGraw-Hill)
[14] Liu Z, Xiu X and Fan D 1998 *Opt. Commun.* **155** 7
[15] Papoulis A 1977 *Signal Analysis* (New York: McGraw-Hill)
[16] Hu L Y and Fan H Y 2008 *J. Mod. Opt.* **55** 2429
Hu L Y and Fan H Y 2008 *J. Mod. Opt.* **55** 835
[17] Fan H Y and Hu L Y 2007 *Chin. Phys. Lett.* **24** 2238
Fan H Y and Hu L Y 2008 *Chin. Phys. Lett.* **25** 513
[18] Fan H Y 2003 *Opt. Lett.* **28** 2177
[19] Fan H Y and Hu L Y 2008 *Opt. Lett.* **33** 443