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Complete and phase synchronization in a heterogeneous small-world neuronal network*

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Synchronous firing of neurons is thought to be important for information communication in neuronal networks. This paper investigates the complete and phase synchronization in a heterogeneous small-world chaotic Hindmarsh–Rose neuronal network. The effects of various network parameters on synchronization behaviour are discussed with some biological explanations. Complete synchronization of small-world neuronal networks is studied theoretically by the master stability function method. It is shown that the coupling strength necessary for complete or phase synchronization decreases with the neuron number, the node degree and the connection density are increased. The effect of heterogeneity of neuronal networks is also considered and it is found that the network heterogeneity has an adverse effect on synchrony.

Keywords: small-world neuronal network, complete synchronization, phase synchronization, heterogeneity

PACC: 0545, 0547

1. Introduction

Starting with the pioneering work of Watts and Strogatz in 1998, a lot of research has been done on the theory and applications of small-world networks. A small-world network realizes both dense clustering and short average path length. Many real complex networks have been found to have small-world properties. The human brain is a complex network consisting of more than 10^{11} neurons, and each neuron in the cortex connects to more than 10,000 neurons via synapses.[1] Although the real structure of the brain network is not yet clear, small-world properties have also been found in some neuronal networks.[2,3] Thus small-world structure may be instructive for the essence of neuronal networks of human brain. Reference [4] investigated the role played by different connectivity regimes in the dynamics of a small-world network of Hodgkin–Huxley neurons. Coherence resonance of small-world Hodgkin–Huxley networks was investigated in Ref.[5]. The authors found that the randomness of the network enhanced the temporal coherence and the spatial synchronization. Reference [6] studied the self-sustained activity of excitable integrate-and-fire neurons in a small-world network. Reference [7] explored the effects of small-world connectivity on noise-induced temporal and spatial order in neural media.

Chaotic systems are able to synchronize in various senses. Complete synchronization[8] is the strictest form of synchronization. Weaker forms of synchronization are generalized synchronization, phase synchronization[9] and lag synchronization. Neurons in the human brain often decode and transmit information by means of chaos and the synchronous firing of neurons is thought to play a key role in information processing in the brain. Hence, many investigations were devoted to the study of synchronization of coupled neurons. Except for the study on synchronization in coupled neurons with conventional open-ended, ring, star and global coupling schemes,[10−13] there are also several publications about synchronization of small-world systems. Reference [14] studied the synchronization of excitatory recurrent networks and found that small-world networks with a higher proportion of long connections could sustain a higher degree of synchronization. Reference [15] studied network self-organization of small-world networks of Hindmarsh–Rose (HR) neurons and found that 10% of far connections were needed to achieve perfect synchronization. Reference [16] studied the global and

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local synchronization of pulse-coupled leaky integrate-and-fire neurons in small-world networks. The authors found that the rewiring probability parameter bridged precise local synchrony and rough global synchrony, and that the cooperation of the global connections and the local clustering property forced synchronization of distant neuronal groups received coherent inputs. However, less mentions were made for the effects of network characteristic parameters on various types of synchronization in small-world neuronal networks. Especially, the effect of heterogeneity of small-world neuronal networks on synchronization has not been explored.

The aim of this study is to investigate the influence of the network topology and the parameters on complete and phase synchronization of a heterogeneous small-world chaotic HR neuronal network. This article is organized as follows. Section 2 introduces the model of small-world HR neuronal networks, as well as the concepts of the average path length and the heterogeneity of networks. Complete and phase synchronization in the network are discussed in Sections 3 and 4, respectively. Some biological explanations are made after the analysis. The conclusions are given in Section 5.

2. Network model

The HR neuronal model is considered in this paper as a typical example of real neurons, expressed by the following equations:[17]

$$\begin{aligned} \dot{x} &= y - ax^3 + bx^2 - z + I, \\ \dot{y} &= c - dx^2 - y, \\ \dot{z} &= r[s(x - \chi) - z]. \end{aligned} \tag{1}$$

where x is the membrane potential, y is associated with the fast current of the Na^+ or K^+ ions, and z is associated with the slow current of, for example, the Ca^{2+} ions. We choose the parameters as $a = 1$, $b = 3$, $c = 1$, $d = 5$, $s = 4$, $r = 0.006$ and $\chi = -1.6$. The quantity I is the stimulus current which is delivered to the neuron from its external environment. The dynamics of the HR system (1) depends strongly on the choice of I. System (1) exhibits a multiple time-scale chaotic bursting behaviour for $2.92 < I < 3.40$. In this paper, we choose $I = 3.0$ such that each single neuron is chaotic, as shown in Fig.1.

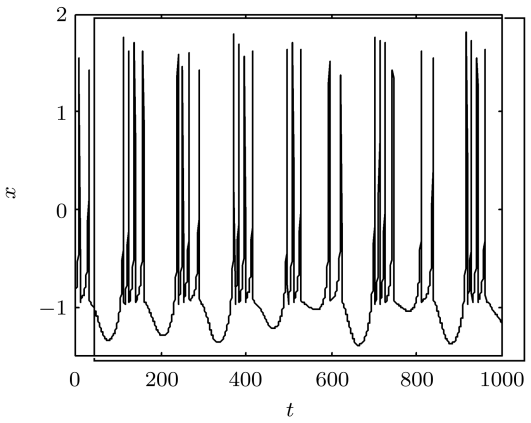


Fig.1. Chaotic bursting of the HR neuron model.

Then we set up the neuronal network by coupling electrically N HR neurons:

$$\begin{aligned} \dot{x}_i &= y_i - ax_i^3 + bx_i^2 - z_i + I_i \\ &\quad + \sigma \sum_{j=1}^{N} g_{ij}(x_j - x_i), \\ \dot{y}_i &= c - dx_i^2 - y_i, \\ \dot{z}_i &= r[s(x_i - \chi) - z_i], \quad i = 1, 2, \ldots N, \end{aligned} \tag{2}$$

where σ is the coupling strength, $\boldsymbol{G} = \{g_{ij}\}_{N\times N}$ is the coupling matrix. N is the overall number of neurons and i denotes the index of the neuron as a node in the network. If neuron i is connected with neuron j, then g_{ij} equals 1; otherwise, g_{ij} equals 0. Since the coupling is diffusive and bilateral, the matrix $\boldsymbol{G}$ is symmetric, negative semi-definite and irreducible. The eigenvalues of $\boldsymbol{G}$ satisfy $0 = \gamma_0 > \gamma_1 \geq \ldots \geq \gamma_{N-1}$.

The coupling matrix $\boldsymbol{G}$ determines the network structure. Here, the Newman–Watts (NW) small-world structure is constructed as follows.[18,19] We start with a ring of N neurons, in which each is coupled diffusively to its k nearest neighbours, that is, the initial node degree of the network equals k. Then we add shortcuts between pairs of nodes with probability p, which actually is the connection density of the network. As special cases, we have original regular networks for $p = 0$ or globally coupled networks for $p = 1$.

The average path length, which measures the typical separation between two vertices in the graph, is an important characteristic measure in small-world network. One expects that biological evolution minimizes the average path length in order to facilitate communication among the neurons.[3,20,21] We estimate the average path length using an approximate formula that

was derived by Newman in Ref.[19]:

$$l = \frac{N}{k} f((pk)^{1/d} N), \tag{3}$$

where l is the average path length, N is the node number, p is the connection density, k is the node degree and d is the dimension of small-world network (we take $d = 1$ in this paper). The function $f(x)$ has the approximate form as $f(x) = f(0)\dfrac{1 + a_1 x}{1 + b_1 x + b_2 x^2}$, where $f(0) = \dfrac{1}{4}$, $a_1 = 1.825 \pm 0.075$, $b_1 = 1.991 \pm 0.075$, $b_2 = 0.301 \pm 0.012$.

Neuronal networks in human brain are controlled by some central neurons and then are heterogeneous in general. For example, in the study of activation patterns of collective neuronal activity in self-organized cortical networks, it was found that as a small subset of recorded neurons, leader neurons, which form a mono-synaptically connected primary circuit, fire the first spike in the network burst and initiate a majority of network bursts. Hence, it is necessary to consider heterogeneity in real neuronal networks. Taking account of the effect of network heterogeneity on synchronization of neuronal networks, a modification to the NW network model is considered. We randomly select n_c neurons of the network as central nodes and add m shortcuts in the network and require that each shortcut contains at least one central node. Thus, the heterogeneity of network is controlled by the parameter $\tau = n_c/N$. A small τ leads to a higher connectivity of the central nodes, which in turn results in a strong heterogeneity. As special cases, for $\tau = 1/N$, all shortcuts are connected to a single central node, making the network similar to the configuration that minimizes the average network distance;[22,23] for $\tau = 1$, we have the homogeneous NW model originally introduced by Newman and Watts.[18,19]

3. Complete synchronization

At first, we study the complete synchronization of a network of N identical chaotic HR neurons. We set the stimulus current $I_i = 3$ for all neurons such that each single neuron exhibits chaotic dynamics. Complete synchronization means that all the neurons fire exactly in the same way. The synchronization subspace can be denoted as $M = \{X_1 = X_2 = \ldots = X_N\}$.

The local transverse stability of the synchronization subspace can be studied by using the master stability function introduced by Pecora and Carroll in Ref.[24]. We can diagonalize the linearization of system (2) in its synchronized state into n blocks of the form $\dot{\zeta}_k = [\boldsymbol{A} + \sigma\gamma_k \boldsymbol{H}]\zeta_k$ for $k = 0, 1, \ldots, N-1$, where $\boldsymbol{A}$ is the Jacobian matrix of a single HR neuron along a trajectory in the synchronization subspace and $\boldsymbol{H} = \mathrm{diag}\{1, 0, 0\}$. This reduces the synchronization problem of system (2) to the three-dimensional problem $\dot{\zeta} = [\boldsymbol{A} + \alpha \mathbf{H}]\zeta$ depending on the parameter α. The master stability function is defined by the maximal Lyapunov exponent $\lambda_{\max}$ of this problem, and its dependence on the parameter α for HR neurons is depicted in Fig.2.

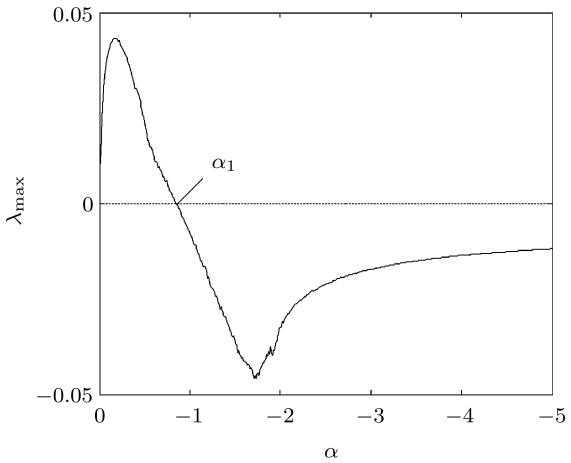


Fig.2. The master stability function of coupled chaotic HR neurons.

Whenever the coupling strength σ and the network topology are such that $\sigma\gamma_k < \alpha_1$ ($\alpha_1 \approx -0.85$) for all negative eigenvalues γ_k of $\boldsymbol{G}$, the synchronization subspace is transversely stable. As the master stability function is negative for all $\alpha < \alpha_1$ and the eigenvalues for the diffusive coupling matrix $\boldsymbol{G}$ satisfy $0 = \gamma_0 > \gamma_1 \geq \ldots \geq \gamma_{N-1}$, synchronization will be stable if only if $\sigma\gamma_1 < \alpha_1$, that is:

$$\sigma > \frac{\alpha_1}{\gamma_1} \approx -\frac{0.85}{\gamma_1}. \tag{4}$$

That is, the critical coupling strength to achieve synchronization only depends on γ_1, the maximum nonzero eigenvalue of the network. The larger the value of γ_1, the larger the coupling strength that is needed to achieve synchronization and the more difficult for the network to be synchronous.

According to the synchronization condition (4), we can compute the critical coupling strength σ_c for various small-world topologies. Figures 3 and 4 show how the critical coupling strength depends on N, p, k and τ (all the results were obtained by averaging over 50 operations).

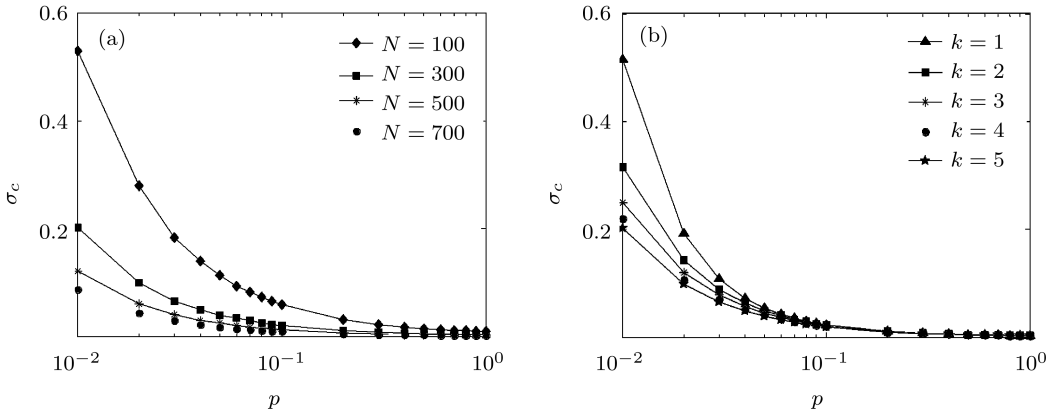


Fig.3. The critical coupling strength σ_c vs. the connection density p for (a) different node number N ($k = 5$); (b) different initial node degree $k(N = 300)$.

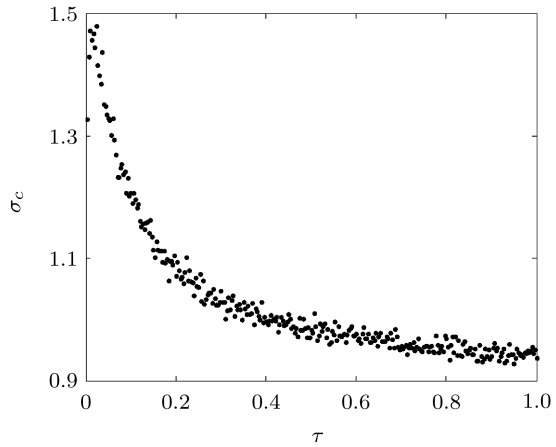


Fig.4. The critical coupling strength σ_c vs. the heterogeneity parameter $\tau(N = 300,\ k = 5,\ m = 200)$.

As expected the critical coupling strength decreases rapidly when the connection density increases. Because introducing shortcuts reduces the average path length of network and information can be transmitted fast by these shortcuts, the ability for synchronization of the network is improved. Moreover, it can be seen from Fig.3(a) that the bigger the node number, the smaller coupling strength is needed to synchronize for the network and only smaller fraction p is enough to reduce the critical coupling strength to a certain value for network with very large node number. It is known that human brain has a great number of neurons, which will greatly facilitate synchronization of neurons and only a very small fraction of shortcuts is needed for synchronization of neurons in human brain. However, it seems conflicting with the deduction that larger average path length will go against synchronization, for bigger node number will enhance the average path length according to formula (3). So the average path length may properly not be the only concerning factor for the evolution of biological networks. Also, it can be seen from Fig.3(b) that the more neighbours a neuron initially has, the more easier for the neuronal network to synchronize, even when the number of shortcuts is not large. This can be explained by formula (3) since larger k will result in smaller l and then facilitate synchronization in the network. However, for a large enough value of p, the effect of k will become trivial. This result can also be deduced by formula (3). For a neuronal network with a very large amount of neurons, even a small fraction of shortcuts will cost very much, so the neuronal network probably tends to have a large node degree in order to synchronize easily. This may be the reason why modularity can be found in brain network and local synchronization may also be important for the information processing.

Now we consider the effect of network heterogeneity. As seen from Fig.4, when τ increases, the critical coupling strength σ_c first increases and then decreases. We have checked the relationship between the critical coupling strength and the heterogeneity parameter for different values of parameters N, k and m and found that the plots of σ_c with respect to τ are similar to that in Fig.4. Namely, when heterogeneity decreases, the network is difficult to achieve synchrony at first but becomes easier and easier later. When there are more specified central nodes, it is easier for the network to synchronize. In other words, synchronization is easily attained with more homogeneous network. Reference [25] shows that the average path length of small-world networks gradually increases when n_c increases. The reason lies in the fact that each neuron has a load capacity limit and when there are less inputs toward a neuron, the increase of n_c results in a

bigger average path length which decreases the ability of synchronization of the network; however, when more inputs are applied to a neuron, the neuron may get overloaded and cannot transmit information efficiently. The results obtained here also demonstrate that the average path length is not the only concerning factor during the biological evolution and heterogeneity may have a great impact on the dynamical process of neuronal networks. So natural selection might tend to make neuronal networks homogeneous to enhance the ability of synchronization, and the real structure of biological neuronal networks would be a compromise between short average path length and high extent of homogeneity. In fact, it has been reported that the neurons in some layers of cortical columns are relatively homogeneous in soma size.[26]

Numerical simulations of the maximum synchronization error are also performed in neuronal networks with various values of parameters of N, k, p and n_c in Fig.5. As another characteristic measure of complete synchronization, the maximum synchronization error is defined by:

$$e_{\max} = \max(\langle e_i \rangle),$$

where $\langle e_i \rangle = \langle \sqrt{(x_i - x_1)^2 + (y_i - y_1)^2 + (z_i - z_1)^2} \rangle$, $(i = 2, 3, \ldots, N)$ and $\langle \cdot \rangle$ denote the average with respect to time over a long time interval. Theoretically, $e_{\max}$ is zero when synchronization in the network is achieved.

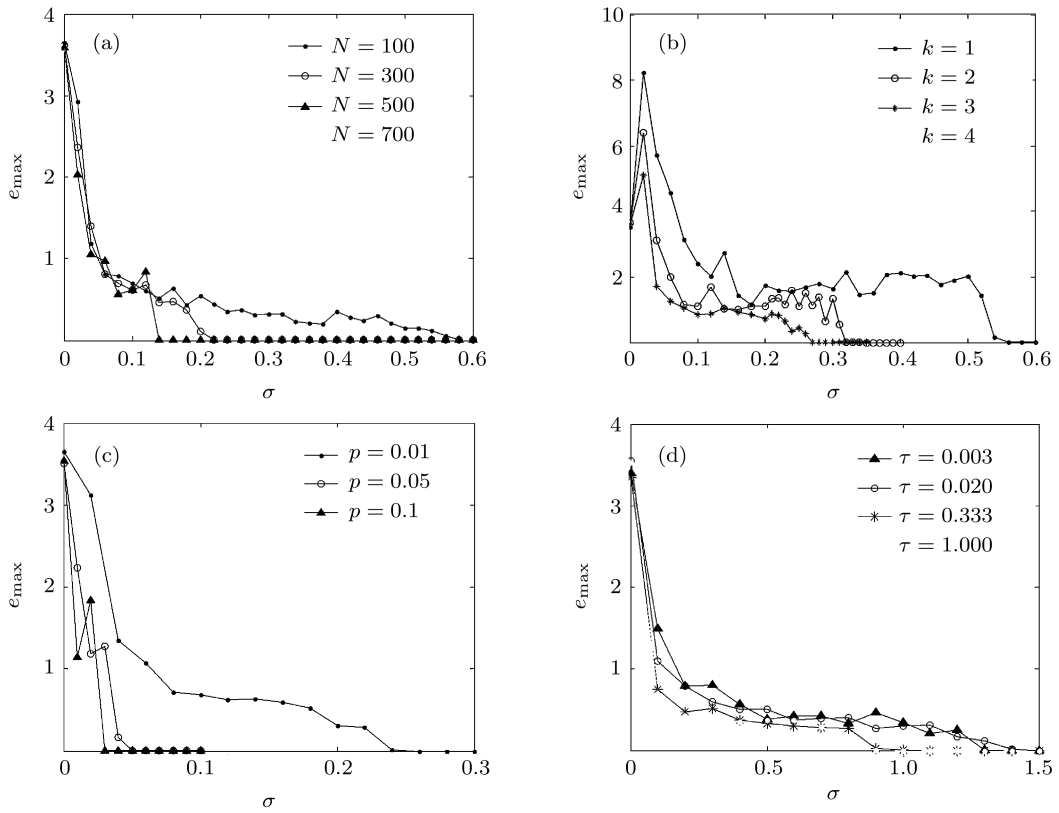


Fig.5. The maximum synchronization error $e_{\max}$ vs. the coupling strength σ with different values of (a) N ($k = 5$, $p = 0.01$), (b) k ($N = 300$, $p = 0.01$), (c) p ($N = 300$, $k = 4$) and (d) τ ($N = 300$, $k = 5$, $m = 200$).

It can be seen from Fig.5 that the maximum errors vanish when the coupling strength increases to the critical value for various cases. The results agree well with the above analysis by the master stability function.

4. Phase synchronization

Then we study phase synchronization of the HR neuronal network. To set the neurons in chaotic states, we set $I_i = 3.125 \pm 0.125 \times \text{rand}$, $(i = 1, 2, \ldots N)$ in system (2), where rand is a random number in the interval $[0, 1]$.

There are a few different methods to calculate

phases of chaotic oscillators. Here, phases of neurons are defined by the Hilbert transformation in general.[27] The Hilbert transformation of signal $x(t)$ can be expressed as $x_h(t) = \frac{1}{\pi}\int_{-\infty}^{\infty}\frac{x(\tau)}{t-\tau}\mathrm{d}\tau$, then an analytical signal can be obtained $s(t) = x(t)+\mathrm{i}x_h(t) = A(t)\exp(\mathrm{i}\phi(t))$, where $\mathrm{i} = \sqrt{-1}$, $A(t)$ and $\phi(t)$ are the instantaneous amplitude and the instantaneous phase of signal $x(t)$, respectively. The phase of a single neuron with chaotic firing is shown in Fig.6.

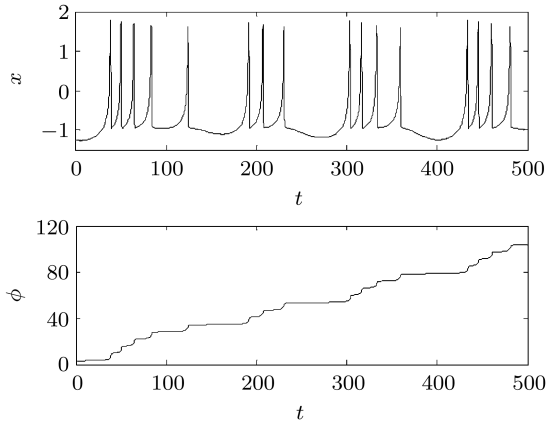


Fig.6. Chaotic firing (top) and the phase (bottom) of a single HR neuron.

It can be seen that whenever a spike happens, the phase increases 2π. Hence, phase synchronization of neurons is actually spike synchronization, which happens to be complete synchronization.

The maximum average phase difference of the network with N chaotic HR neurons within a time interval is defined as $\phi_{\max} = \max(\Delta\phi(t))$, where $\Delta\phi(t) = \frac{1}{N-1}\sum_{i=2}^{N}|\phi_i(t)-\phi_1(t)|$, and it is used to indicate the phase synchronization of neuronal networks. When $\phi_{\max} < 2\pi$, phase synchronization of the small-world neuronal network is thought to happen. Simulation is performed in different network configurations when network parameters vary.

It is shown from Fig.7(a) that the critical coupling strengths for phase synchronization are approximately 0.54, 0.20, 0.11 and 0.07 for $N = 100$, 300, 500 and 700, respectively. So larger neuron number facilitates phase synchronization as the same as complete synchronization. It can also be inferred from Figs.7(b) and 7(c) that bigger coordination number and bigger connection density make the neuronal network much easier to achieve phase synchronization. As for the effect of heterogeneity on phase synchronization, we can see from Fig.7(d) that the critical coupling strengths

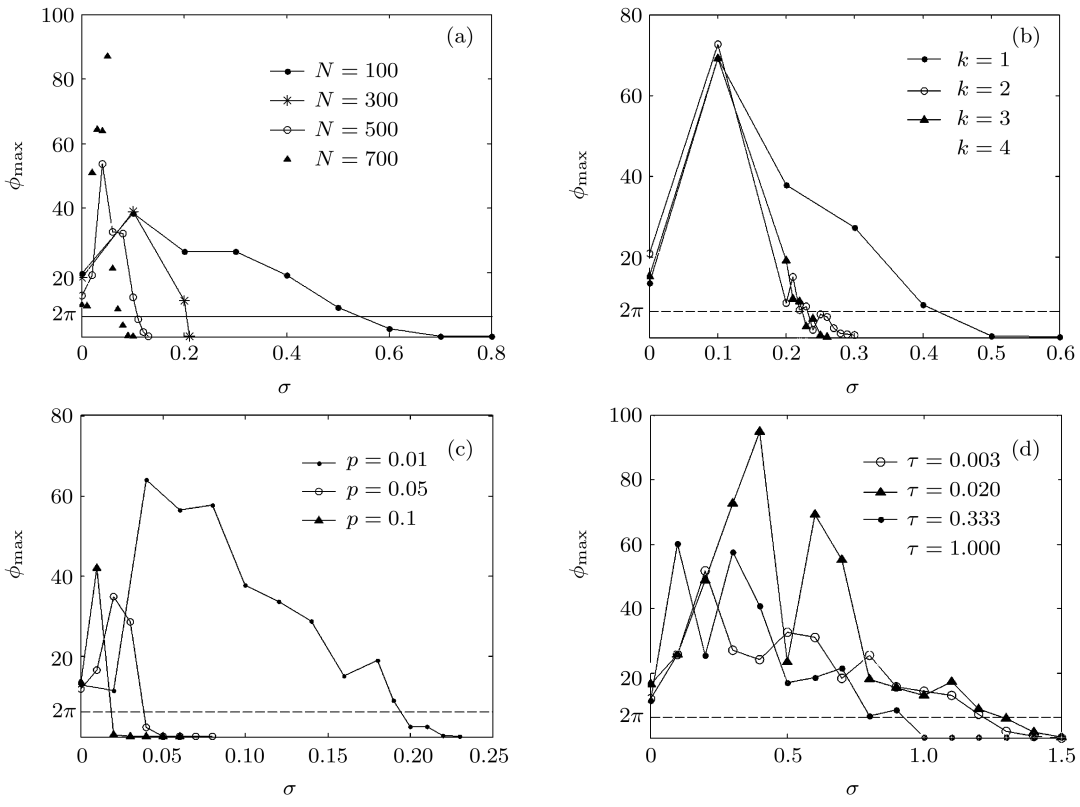


Fig.7. The maximum average phase difference $\phi_{\max}$ vs. the coupling strength σ with different values of (a) N ($k = 5$, $p = 0.01$), (b) k ($N = 300$, $p = 0.01$), (c) p ($N = 300$, $k = 4$) and (d) τ ($N = 300$, $k = 5$, $m = 200$).

for phase synchronization are approximately 1.2, 1.3, 0.95 and 0.93 for heterogeneity parameter $\tau = 0.003$, 0.020, 0.333 and 1 respectively. It can be deduced that the situation is also similar to complete synchronization, that is, phase synchronization becomes difficult to be attained at first and then becomes easier and easier to be achieved later. Comparing the critical coupling strengths for complete synchronization and phase synchronization, we can find that the former ones are larger than the latter ones, so it can be concluded that phase synchronization happens earlier than complete synchronization as the coupling strengths being increased.

5. Conclusion

In this paper, we investigated the effects of various network parameters on complete and phase synchronization of a small-world chaotic HR neuronal network and gave some biological explanations. The master stability function method is used for the analysis of complete synchronization. It is found that larger neuron number, coordination number or connection density can facilitate both complete and phase synchronization of neuronal networks. As for the effect of heterogeneity on network behaviour, generally speaking, the network with lower extent of heterogeneity is easier to achieve synchronization. Through the analysis it is found that the average path length is apparently not the only factor for which the natural selection concerns and neuronal networks tend to have larger neuron number and bigger extent of homogeneity at the expense of larger average path length.

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