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Some new solutions derived from the nonlinear (2+1)-dimensional Toda equation—an efficient method of creating solutions*

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This paper presents a new and efficient approach for constructing exact solutions to nonlinear differential–difference equations (NLDDEs) and lattice equation. By using this method via symbolic computation system MAPLE, we obtained abundant soliton-like and/or period-form solutions to the (2+1)-dimensional Toda equation. It seems that solitary wave solutions are merely special cases in one family. Furthermore, the method can also be applied to other nonlinear differential–difference equations.

Keywords: hyperbolic function method, nonlinear differential–difference equations, soliton-like solutions, period-form solutions

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1. Introduction

It is generally accepted that nonlinear differential–difference equations (NLDDEs) are always important in applications in many fields of natural science. They enter as models for biological systems, atomic chains, solid state physics, electrical networks and Bose–Einstein condensate. In the literature, considerable comprehensive studies have been reported on applications of NLDDEs, including investigation of integrability criteria, computation of densities, Bäcklund transformation and recursion operator.[1] In the past decades, there have been numerous researches in finding exact solutions to nonlinear partial differential equations (NLPDEs). However, to our knowledge, only limited information is available in relation to symbolically computing exact solutions of NLDDEs. For example, Baldwin *et al*[2] presented an algorithm to find exact travelling wave solutions of DDEs in terms of the tanh function and found kink-type solutions in many spatially discrete nonlinear models. Recently, some exact solutions of hybrid lattice are obtained by use of the improved tanh method in Refs.[3–5]. In Ref.[6], the Jacobian elliptic function method was applied to obtain the periodic wave solutions for the discreet mKdV lattice.

In this paper, a new and efficient method is presented aiming at solving nonlinear polynomial differential–difference equations. In contrast to the numerical computation of travelling wave solutions of lattices, our algorithm computes analytical (closed-form) solutions in the form of polynomials in hyperbolic functions. Apart from their physical relevance, the knowledge of closed-form solutions of NLDDEs facilitates stability and accuracy testing of numerical solvers. Assuming that closed-form solutions exist for specific choices of parameters and/or initial conditions, the numerically computed solutions must coincide with the analytic solutions under the same conditions.

The paper is organized as follows. In Section 2, the key idea of the new hyperbolic function method is described. In Section 3, the proposed method is applied to solve the discrete nonlinear (2+1)-dimensional Toda lattice equation. Conclusions are summarized in the last section.

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2. Summary of the hyperbolic function method for NLDDEs

A system of M polynomial DDEs is given as

$$\Delta(u_{n+s_1}(x),\ldots,u_{n+s_l}(x),u'_{n+s_1}(x),\ldots,u'_{n+s_l}(x),\ldots,u^{(r)}_{n+s_1}(x),\ldots,u^{(r)}_{n+s_l}(x))=0, \quad (1)$$

where the dependent variable u_n has M components $u_{i,n}$ and so do its shifts; the continuous variable x has N components x_i; the discrete variable n has P components n_j; the l shift vector $s_i \in Z^P$, and $u^{(r)}(x)$ denotes the collection of mixed derivative terms of order r. We assume that any arbitrary coefficients that parametrize the system are strictly positive.

Step 1 We seek solutions in the following frame of reference:

$$\begin{aligned} u_{i,n}(x) = & \sum_{j=-M_i}^{M_i} a_{ij}(x)\left(\frac{1}{T'_n(kn+h(x))+r}\right)^j \\ & + \sum_{j=-M_i+1}^{M_i} b_{ij}(x)\left(\frac{1}{T'_n(kn+h(x))+r}\right)^{j-1} \\ & \times \frac{T_n(kn+h(x))}{T'_n(kn+h(x))+r}, \end{aligned} \quad (2)$$

where T_n can be selected as sin, cos, sinh, cosh, sn, cn, and $a_{ij}(x)$, $b_{ij}(x)$ and $h(x)$ are arbitrary functions to be determined later.

Repeatedly applying the chain rule

$$\frac{\mathrm{d}\bullet}{\mathrm{d}x_j} = \frac{\partial h(x)}{\partial x_j}\frac{\mathrm{d}T_n}{\mathrm{d}h(x)}\frac{\mathrm{d}\bullet}{\mathrm{d}T_n}, \quad (3)$$

we transform (1) into

$$\Delta(u_{n+s_1}(T_n),\ldots,u_{n+s_l}(T_n),u'_{n+s_1}(T_n),\ldots,u'_{n+s_l}(T_n),\ldots,u^{(r)}_{n+s_1}(T_n),\ldots,u^{(r)}_{n+s_l}(T_n))=0. \quad (4)$$

Step 2 The degree of the polynomial solutions is given according to the homogeneous balance principle.[7] It is important to note that for any $m\,(m=1,\ldots,l)$, u_{n+s_m} is a function of T_n but not of T_{n+s_m} or $\dfrac{\mathrm{d}T_n}{\mathrm{d}h(x)}$. In fact, T_{n+s_m} and $\dfrac{\mathrm{d}T_n}{\mathrm{d}h(x)}$ can be expressed in T_n. For example,

Case 1 If $T_n=\sin(kn+h(x))$, we have

$$\frac{\mathrm{d}T_n}{\mathrm{d}h(x)} = \sqrt{1-T_n^2}. \quad (5)$$

Using the identity

$$\sin(\xi_1\pm\xi_2)=\sin(\xi_1)\cos(\xi_2)\pm\cos(\xi_1)\sin(\xi_2), \quad (6a)$$

$$\cos(\xi_1+\xi_2)=\cos(\xi_1)\cos(\xi_2)\pm\sin(\xi_1)\sin(\xi_2), \quad (6b)$$

we can write

$$T_{n+s_m} = T_n\cos\varphi_{\mathrm{d}} \pm \sqrt{1-T_n^2}\sin\varphi_{\mathrm{d}}, \quad (7a)$$

$$T'_{n+s_m} = \sqrt{1-T_n^2}\cos\varphi_{\mathrm{d}} \pm T_n\sin\varphi_{\mathrm{d}}, \quad (7b)$$

where

$$\varphi_{\mathrm{d}} = S_{\mathrm{d}}K = s_{\mathrm{d}1}k_1 + s_{\mathrm{d}2}k_2 + \ldots + s_{\mathrm{d}P}k_P, \quad (8)$$

and $s_{\mathrm{d}j}$ is the j-th component of shift vector S_{d}.

Case 2 If $T_n=\sinh(kn+h(x))$, we have

$$\frac{\mathrm{d}T_n}{\mathrm{d}h(x)} = \sqrt{1-T_n^2}. \quad (9)$$

Using the identity

$$\sinh(\xi_1\pm\xi_2) = \sinh(\xi_1)\cosh(\xi_2)\pm\cosh(\xi_1)\sinh(\xi_2), \quad (10a)$$

$$\cosh(\xi_1+\xi_2) = \cosh(\xi_1)\cosh(\xi_2)\pm\sinh(\xi_1)\sinh(\xi_2), \quad (10b)$$

we can write

$$T_{n+s_m} = T_n\cosh\varphi_{\mathrm{d}} \pm \sqrt{1-T_n^2}\sinh\varphi_{\mathrm{d}}, \quad (11a)$$

$$T'_{n+s_m} = \sqrt{1-T_n^2}\cosh\varphi_{\mathrm{d}} \pm T_n\sinh\varphi_{\mathrm{d}}, \quad (11b)$$

Case 3 If $T_n=\mathrm{sn}(kn+h(x))$, we have

$$\frac{\mathrm{d}T_n}{\mathrm{d}h(x)} = \sqrt{1-T_n^2}. \quad (12)$$

Using the identity

$$\mathrm{sn}(\xi_1+\xi_2) = \frac{\mathrm{sn}(\xi_1)\mathrm{cn}(\xi_2)\mathrm{dn}(\xi_2)+\mathrm{sn}(\xi_2)\mathrm{cn}(\xi_1)\mathrm{dn}(\xi_1)}{1-m^2\mathrm{sn}^2(\xi_1)\mathrm{sn}^2(\xi_2)}, \quad (13a)$$

$$\mathrm{cn}(\xi_1+\xi_2) = \frac{\mathrm{cn}(\xi_1)\mathrm{cn}(\xi_2)-\mathrm{sn}(\xi_1)\mathrm{dn}(\xi_1)\mathrm{sn}(\xi_2)\mathrm{dn}(\xi_2)}{1-m^2\mathrm{sn}^2(\xi_1)\mathrm{sn}^2(\xi_2)}, \quad (13b)$$

we can write

$$T_{n+s_m} = \frac{T_n\mathrm{cn}(\varphi_{\mathrm{d}})\mathrm{dn}(\varphi_{\mathrm{d}})+\mathrm{sn}(\varphi_{\mathrm{d}})\sqrt{1-T_n^2}\sqrt{1-m^2T_n^2}}{1-m^2T_n^2\mathrm{sn}^2(\varphi_{\mathrm{d}})}, \quad (14a)$$

$$T'_{n+s_m} = \frac{\sqrt{1-T_n^2}\mathrm{cn}(\varphi_{\mathrm{d}})-T_n\sqrt{1-m^2T_n^2}\mathrm{sn}(\varphi_{\mathrm{d}})\mathrm{dn}(\varphi_{\mathrm{d}})}{1-m^2T_n^2\mathrm{sn}^2(\varphi_{\mathrm{d}})}, \quad (14b)$$

Remark 1 It is necessary to point out that the above combinations only require solving the recurrent coefficient relation or derivative relation for the terms of polynomial for computation to be closed. Therefore other functions can be chosen to combine as a new ansätz. For simplicity, we omit them here.

Step 3 Substituting

$$\begin{aligned} & u_{i,n+s_m}(T_n) \\ &= \sum_{j=-M_i}^{M_i} a_{ij} \left(\frac{1}{T'_{n+s_m} + r} \right)^j \\ &\quad + \sum_{j=-M_i+1}^{M_i} b_{ij} \left(\frac{1}{T'_{n+s_m} + r} \right)^{j-1} \frac{T_{n+s_m}}{T'_{n+s_m} + r} \\ &= \sum_{j=-M_i}^{M_i} a_{ij} \left(\frac{1}{g(T_n) + r} \right)^j \\ &\quad + \sum_{j=-M_i+1}^{M_i} b_{ij} \left(\frac{1}{g(T_n) + r} \right)^{j-1} \frac{f(T_n)}{g(T_n) + r} \end{aligned} \quad (15)$$

into Eq.(4) with φ_{d} as in Eq.(8), where $f(T_n)$ admits Eqs.(7a), (11a), (14a), and $g(T_n)$ admits Eqs.(7b), (11b), (14b). By applying (6) (or (10), (13)) one can split $\sin(\varphi_{\mathrm{d}})$ into powers of $\sin(k_i)$ and $\cos(k_i)$. While doing so, we repeatedly clear the denominations, and then result in nonlinear equations for the unknowing $a_{ij}(x)$, $b_{ij}(x)$, r, k and $h(x)$.

Step 4 We use MAPLE to simplify the overdetermined PDEs and obtain $a_{ij}(x)$, $b_{ij}(x)$, r, k and $h(x)$. Substituting the obtained $a_{ij}(x)$, $b_{ij}(x)$, r, k and $h(x)$ back into the ansätz (2), we may get possible solutions of Eq.(1).

3. Application to (2+1)-dimensional Toda equation

Here, we consider the following (2+1)-dimensional Toda lattice equation,[8,9]

$$\frac{\partial^2 \phi_n}{\partial x \partial t} = \exp\left(\phi_{n-1} - \phi_n\right) - \exp\left(\phi_n - \phi_{n+1}\right), \quad (16)$$

where $\phi_n(x,t)$ is the displacement from equilibrium of the n-th unit mass under an exponentially decaying interaction force between nearest neighbours. To write Eq.(16) as a polynomial NLDDEs, set

$$\frac{\partial \phi_n}{\partial t} = \exp(\phi_{n-1} - \phi_n) - 1, \quad (17)$$

then equation (16) becomes

$$\frac{\partial^2 \phi_n}{\partial x \partial t} = \left(\frac{\partial \phi_n}{\partial t} + 1 \right) (\phi_{n-1} - 2\phi_n + \phi_{n+1}). \quad (18)$$

We expand the solution of Eq.(18) in the form of Eqs.(2) and (15). According to the homogeneous balance principle, by balancing the derivative term $\dfrac{\partial^2 \phi_n}{\partial x \partial t}$ with nonlinear terms $\phi_n \dfrac{\partial \phi_n}{\partial t}$ in Eq.(18), we can get $M_i = 1$ in Eq.(2). So the solution of Eq.(18) can be expressed as

$$\begin{aligned} \phi_n(x,t) &= a_0(x,t) + \frac{a_1(x,t)}{\cosh(kn + h(x,t)) + r} + \frac{b_1(x,t)\sinh(kn + h(x,t))}{\cosh(kn + h(x,t)) + r} \\ &\quad + A_1(x,t)\left(\cosh(kn + h(x,t)) + r\right) + B_1(x,t)\sinh(kn + h(x,t)), \quad (19) \\ \phi_{n+1}(x,t) &= a_0(x,t) + \frac{a_1(x,t)}{\cosh(kn + h(x,t) + k) + r} + \frac{b_1(x,t)\sinh(kn + h(x,t) + k)}{\cosh(kn + h(x,t) + k) + r} \\ &\quad + A_1(x,t)\left(\cosh(kn + h(x,t) + k) + r\right) + B_1(x,t)\sinh(kn + h(x,t) + k), \quad (20) \\ \phi_{n-1}(x,t) &= a_0(x,t) + \frac{a_1(x,t)}{\cosh(kn + h(x,t) - k) + r} + \frac{b_1(x,t)\sinh(kn + h(x,t) - k)}{\cosh(kn + h(x,t) - k) + r} \\ &\quad + A_1(x,t)\left(\cosh(kn + h(x,t) - k) + r\right) + B_1(x,t)\sinh(kn + h(x,t) - k). \quad (21) \end{aligned}$$

Remark 2 In this section, for simplicity, we only use the ansätz solutions (19)–(21) in terms of $\sinh(kn + h(x,t))$ and $\cosh(kn + h(x,t))$ to solve nonlinear differential–difference equations. Based on the same argument and similar performance from ansätz solutions in Cases 1 and 3, we can obtain other types of solutions which are similar to sinh- and cosh-type solutions in form. To avoid lengthiness, we omit them in this paper.

Substituting Eqs.(19)–(21) into Eq.(18), clearing the denominator and setting the coefficients of all powers like $\sinh^i(\xi_n)\cosh^i(\xi_n)$ $(i = 0, 1, 2, \ldots)$ to zero, yields a set of PDEs. From the output of Maple, we find 11 types of solutions, namely:

Case 1

$$a_0(x,t)=\frac{-2\cosh(k)t+2t+c_1F_2(t)}{-2+2\cosh(k)}+F_1(x),\quad a_1(x,t)=\pm\frac{1}{2}c_1I,\quad b_1(x,t)=\frac{1}{2}c_1,$$
$$A_1(x,t)=0,\quad B_1(x,t)=0,\quad r=0,\quad h(x,t)=F_2(t)+c_1x+c_2. \tag{22}$$

Case 2

$$a_0(x,t)=\frac{-\cosh(2k)t+t+2c_1F_2(t)}{-1+\cosh(2k)}+F_1(x),\quad a_1(x,t)=0,\quad b_1(x,t)=c_1,$$
$$A_1(x,t)=0,\quad B_1(x,t)=0,\quad r=0,\quad h(x,t)=F_2(t)+c_1x+c_2. \tag{23}$$

Case 3

$$a_0(x,t)=\frac{-2\cosh(k)t+2t+c_1F_2(t)}{-2+2\cosh(k)}+F_1(x),\quad a_1(x,t)=0,\quad b_1(x,t)=\frac{1}{2}c_1,$$
$$A_1(x,t)=0,\quad B_1(x,t)=0,\quad r=\pm1,\quad h(x,t)=F_2(t)+c_1x+c_2. \tag{24}$$

Case 4

$$a_0(x,t)=\frac{-2\cosh(k)t+2t+c_1F_2(t)}{-2+2\cosh(k)}+F_1(x),\quad a_1(x,t)=\pm\frac{1}{4}\frac{\left(\mathrm{e}^k-1\right)c_1}{\mathrm{e}^{k/2}},\quad b_1(x,t)=\frac{1}{2}c_1,$$
$$A_1(x,t)=0,\quad B_1(x,t)=0,\quad r=\pm\frac{1}{2}\frac{\mathrm{e}^k+1}{\mathrm{e}^{k/2}},\quad h(x,t)=F_2(t)+c_1x+c_2. \tag{25}$$

Case 5

$$a_0(x,t)=\frac{-2\cosh(k)t+2t+c_1F_2(t)}{-2+2\cosh(k)}+F_1(x),\quad a_1(x,t)=\pm\frac{1}{4}\frac{\sqrt{2}\left(\mathrm{e}^k-1\right)c_1}{\mathrm{e}^{k/2}},\quad b_1(x,t)=\frac{1}{2}c_1,$$
$$A_1(x,t)=0,\quad B_1(x,t)=0,\quad r=\pm\frac{1}{2}\frac{\sqrt{2}\sqrt{\mathrm{e}^k\left(\left(\mathrm{e}^k\right)^2+1\right)}}{\mathrm{e}^k},$$
$$h(x,t)=F_2(t)+c_1x+c_2. \tag{26}$$

Case 6

$$a_0(x,t)=\frac{-2\cosh(k)t+2t+c_1F_2(t)}{-2+2\cosh(k)}+F_1(x),\quad a_1(x,t)=\pm\frac{1}{2}c_1\sqrt{r^2-1},\quad b_1(x,t)=\frac{1}{2}c_1,$$
$$A_1(x,t)=0,\quad B_1(x,t)=0,\quad r=r,\quad h(x,t)=F_2(t)+c_1x+c_2. \tag{27}$$

Case 7

$$a_0(x,t)=\pm\int F_2'(t)F_2(x)\mathrm{e}^{F_2(t)}\mathrm{d}t-t+F_1(x),\quad a_1(x,t)=0,\quad b_1(x,t)=0,$$
$$A_1(x,t)=\pm F_2(x)\mathrm{e}^{F_2(t)},$$
$$B_1(x,t)=\mp F_2(x)\mathrm{e}^{F_2(t)},\ r=\mp1,\quad h(x,t)=F_2(t)+c_1x+c_2. \tag{28}$$

Case 8

$$a_0(x,t)=-t+F_1(x),\quad a_1(x,t)=0,\quad b_1(x,t)=0,\quad A_1(x,t)=\pm F_2(x)\mathrm{e}^{F_2(t)},$$
$$B_1(x,t)=\mp F_2(x)\mathrm{e}^{F_2(t)},\quad r=0,\quad h(x,t)=F_2(t)+c_1x+c_2. \tag{29}$$

Case 9

$$a_0(x,t)=\int-\frac{1}{2}\frac{2(\mathrm{e}^k)^2+2+2\mathrm{e}^k-\sqrt{-2F_2^2(x)\mathrm{e}^{2F_2(t)}\left(F_2'(t)\right)^2\mathrm{e}^k\left(\left(\mathrm{e}^k\right)^2+1+\mathrm{e}^k\right)}}{\left(\mathrm{e}^k\right)^2+1+\mathrm{e}^k}\mathrm{d}t+F_1(x),$$
$$a_1(x,t)=0,\quad b_1(x,t)=0,\quad A_1(x,t)=\pm F_2(x)\mathrm{e}^{F_2(t)},\quad B_1(x,t)=\mp F_2(x)\mathrm{e}^{F_2(t)},$$
$$r=-\frac{1}{2}\frac{\sqrt{-2\mathrm{e}^k\left(\left(\mathrm{e}^k\right)^2+1+\mathrm{e}^k\right)}}{\left(\left(\mathrm{e}^k\right)^2+1+\mathrm{e}^k\right)},\quad h(x,t)=F_2(t)+c_1x+c_2. \tag{30}$$

Case 10

$$a_0(x,t)=\int\left(-1\pm\frac{1}{2}\sqrt{2}F_2'(t)F_2(x)\mathrm{e}^{F_2(t)}\right)\mathrm{d}t+F_1(x),\quad a_1(x,t)=0,\quad b_1(x,t)=0,$$
$$A_1(x,t)=\pm F_2(x)\mathrm{e}^{F_2(t)},\quad B_1(x,t)=\mp F_2(x)\mathrm{e}^{F_2(t)},\quad r=-\frac{1}{2}\sqrt{2},\quad h(x,t)=F_2(t)+c_1x+c_2. \tag{31}$$

Case 11

$$a_0(x,t)=\int\left(-1+c_3\exp\left(\int\left(F_2'(t)+F_2^{''}(t)/F_2'(t)\right)\mathrm{d}t\right)\exp\left(F_2(x)\right)\right)\mathrm{d}t+F_3(x),$$
$$a_1(x,t)=0,\quad b_1(x,t)=0,\quad A_1(x,t)=\pm F_2(x)\mathrm{e}^{F_2(t)},\quad B_1(x,t)=\mp F_2(x)\mathrm{e}^{F_2(t)},$$
$$r=\mp\frac{c_3\exp\left(\int\left(F_2'(t)+F_2^{''}(t)/F_2'(t)\right)\mathrm{d}t\right)\exp\left(F_2(x)\right)}{F_2(x)\mathrm{e}^{F_2(t)}F_2'(t)},\quad h(x,t)=F_2(t)+c_1x+c_2. \tag{32}$$

Substituting the obtained $a_0(x,t)$, $a_1(x,t)$, $b_1(x,t)$, $A_1(x,t)$, $B_1(x,t)$, $h(x,t)$ and r in Eqs.(22)–(32) back into the ansätz solution (19), then the closed form solutions of Eq.(18) are:

$$\begin{aligned}\phi_{n1}(x,t)&=\frac{-2\cosh(k)t+2t+c_1F_2(t)}{-2+2\cosh(k)}+F_1(x)\pm\frac{1}{2}c_1I\,\mathrm{sec\,h}\left(kn+F_2(t)+c_1x+c_2\right)\\&\quad+\frac{1}{2}c_1\tanh\left(kn+F_2(t)+c_1x+c_2\right),\end{aligned} \tag{33}$$

$$\phi_{n2}(x,t)=\frac{-\cosh(2k)t+t+2c_1F_2(t)}{-1+\cosh(2k)}+F_1(x)+c_1\tanh\left(kn+F_2(t)+c_1x+c_2\right), \tag{34}$$

$$\phi_{n3}(x,t)=\frac{-2\cosh(k)t+2t+c_1F_2(t)}{-2+2\cosh(k)}+F_1(x)+\frac{1}{2}c_1\frac{\sinh\left(kn+F_2(t)+c_1x+c_2\right)}{\cosh(kn+F_2(t)+c_1x+c_2)\pm1}, \tag{35}$$

$$\begin{aligned}\phi_{n4}(x,t)&=\frac{-2\cosh(k)t+2t+c_1F_2(t)}{-2+2\cosh(k)}+F_1(x)\\&\quad+\frac{1}{2}c_1\frac{2\sinh\left(kn+F_2(t)+c_1x+c_2\right)\pm\mathrm{e}^{-k/2}\left(\mathrm{e}^k-1\right)}{2\cosh\left(kn+F_2(t)+c_1x+c_2\right)\pm\mathrm{e}^{-k/2}\left(\mathrm{e}^k+1\right)},\end{aligned} \tag{36}$$

$$\begin{aligned}\phi_{n5}(x,t)&=\frac{-2\cosh(k)t+2t+c_1F_2(t)}{-2+2\cosh(k)}+F_1(x)\\&\quad+\frac{c_1\sinh\left(kn+F_2(t)+c_1x+c_2\right)\pm\dfrac{\sqrt{2}}{2}\mathrm{e}^{-\frac{k}{2}}\left(\mathrm{e}^k-1\right)c_1}{2\cosh\left(kn+F_2(t)+c_1x+c_2\right)\pm\dfrac{\sqrt{2}\sqrt{\mathrm{e}^k\left(\left(\mathrm{e}^k\right)^2+1\right)}}{\mathrm{e}^k}},\end{aligned} \tag{37}$$

$$\phi_{n6}(x,t)=\frac{-2\cosh(k)t+2t+c_1F_2(t)}{-2+2\cosh(k)}+F_1(x)+\frac{c_1\sinh\left(kn+F_2(t)+c_1x+c_2\right)\pm c_1\sqrt{r^2-1}}{2\cosh\left(kn+F_2(t)+c_1x+c_2\right)+2r}, \tag{38}$$

$$\begin{aligned}\phi_{n7}(x,t)&=\pm\int F_2'(t)F_2(x)\mathrm{e}^{F_2(t)}\mathrm{d}t-t+F_1(x)\pm F_2(x)\mathrm{e}^{F_2(t)}\left(\cosh(kn+F_2(t)+c_1x+c_2)\mp1\right)\\&\quad\mp F_2(x)\mathrm{e}^{F_2(t)}\sinh\left(kn+F_2(t)+c_1x+c_2\right),\end{aligned} \tag{39}$$

$$\begin{aligned}\phi_{n8}(x,t)&=-t+F_1(x)\pm F_2(x)\mathrm{e}^{F_2(t)}\cosh\left(kn+F_2(t)+c_1x+c_2\right)\\&\quad\mp F_2(x)\mathrm{e}^{F_2(t)}\sinh\left(kn+F_2(t)+c_1x+c_2\right),\end{aligned} \tag{40}$$

$$\begin{aligned}\phi_{n9}(x,t)&=\int-\frac{1}{2}\frac{2(\mathrm{e}^k)^2+2+2\mathrm{e}^k-\sqrt{-2F_2^2(x)\mathrm{e}^{2F_2(t)}\left(F_2'(t)\right)^2\mathrm{e}^k\left(\left(\mathrm{e}^k\right)^2+1+\mathrm{e}^k\right)}}{\left(\mathrm{e}^k\right)^2+1+\mathrm{e}^k}\mathrm{d}t+F_1(x)\\&\quad\pm F_2(x)\mathrm{e}^{F_2(t)}\left(\cosh\left(kn+F_2(t)+c_1x+c_2\right)-\frac{1}{2}\frac{\sqrt{-2\mathrm{e}^k\left(\left(\mathrm{e}^k\right)^2+1+\mathrm{e}^k\right)}}{\left(\left(\mathrm{e}^k\right)^2+1+\mathrm{e}^k\right)}\right)\\&\quad\mp F_2(x)\mathrm{e}^{F_2(t)}\sinh\left(kn+F_2(t)+c_1x+c_2\right),\end{aligned} \tag{41}$$

$$\phi_{n10}(x,t) = \int \left(-1 \pm \frac{1}{2}\sqrt{2}F_2'(t)F_2(x)\mathrm{e}^{F_2(t)}\right)\mathrm{d}t + F_1(x)$$
$$\pm F_2(x)\mathrm{e}^{F_2(t)}\left(\cosh\left(kn + F_2(t) + c_1x + c_2\right) - \frac{1}{2}\sqrt{2}\right)$$
$$\mp F_2(x)\mathrm{e}^{F_2(t)}\sinh\left(kn + F_2(t) + c_1x + c_2\right), \tag{42}$$
$$\phi_{n11}(x,t) = \int \left(-1 + c_3\exp\left(\int \left(F_2'(t) + F_2^{''}(t)/F_2'(t)\right)\mathrm{d}t\right)\exp\left(F_2(x)\right)\right)\mathrm{d}t + F_3(x)$$
$$\pm F_2(x)\mathrm{e}^{F_2(t)}\left(\cosh\left(kn + F_2(t) + c_1x + c_2\right) \mp \frac{c_3\exp\left(\int \left(F_2'(t) + F_2^{''}(t)/F_2'(t)\right)\mathrm{d}t\right)\exp\left(F_2(x)\right)}{F_2(x)\mathrm{e}^{F_2(t)}F_2'(t)}\right)$$
$$\mp F_2(x)\mathrm{e}^{F_2(t)}\sinh\left(kn + F_2(t) + c_1x + c_2\right), \tag{43}$$

where k, r, c_1, c_3 are arbitrary nonzero constants, c_2 is an arbitrary constant, $F_1(x)$, $F_2(x)$, $F_3(x)$ and $F_2(t)$ are all arbitrary functions of the indicated variables.

Since $F_1(x)$, $F_2(x)$, $F_3(x)$ and $F_2(t)$ are arbitrary functions, the solutions have abundant properties. We show some properties of the solution (34) in Figs.1–5.

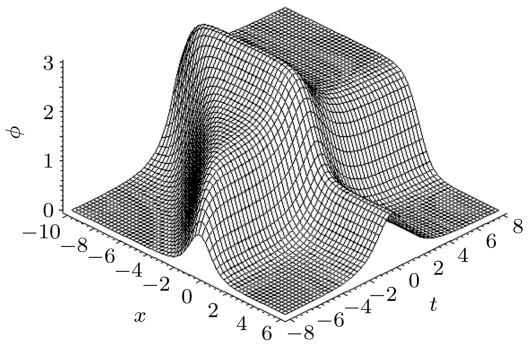


Fig.1. The shape of solution (34) with $F_1(x) = -\tanh(x)$, $F_2(t) = \mathrm{sech}(t) + t$, $n = 10$.

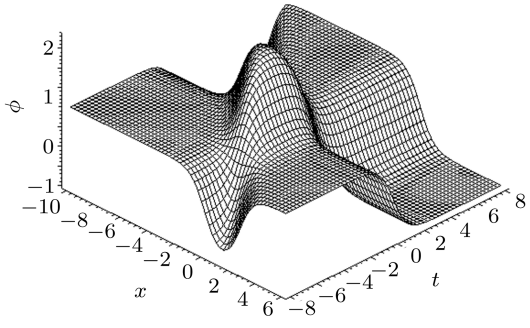


Fig.2. The shape of solution (34) with $F_1(x) = -\tanh(x)$, $F_2(t) = -\tanh(t) + t$, $n = 5$.

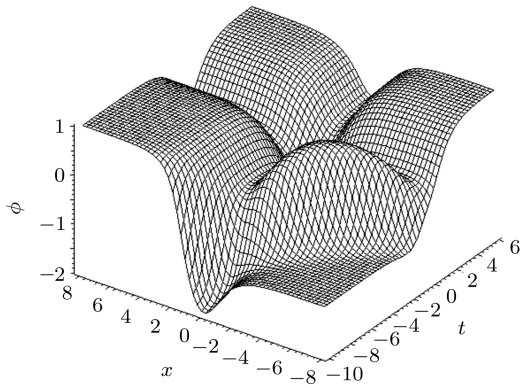


Fig.3. The shape of solution (34) with $F_1(x) = -\mathrm{sech}(x)$, $F_2(t) = -\mathrm{sech}(t) + t$, $n = 10$.

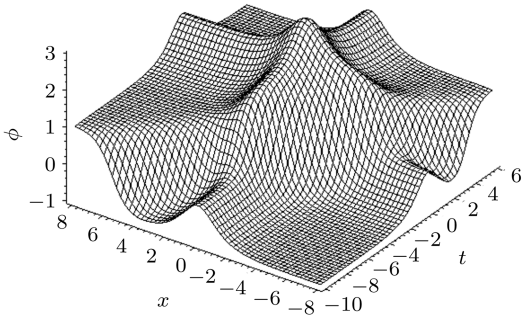


Fig.4. The shape of solution (34) with $F_1(x) = \mathrm{sech}(x)$, $F_2(t) = \mathrm{sech}(t) + t$, $n = 5$.

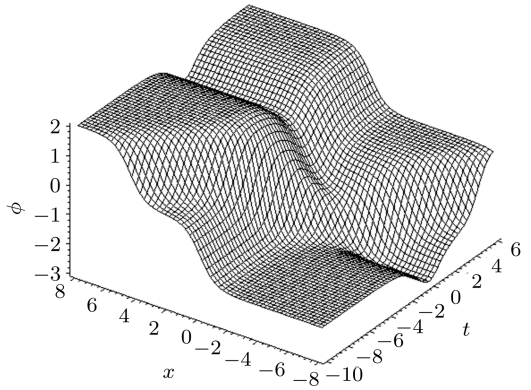


Fig.5. The shape of solution (34) with $F_1(x) = -\mathrm{sech}(x)$, $F_2(t) = \tanh(t) + t$, $n = 5$.

4. Conclusions

Based on the computerized symbolic computation, this paper has provided a new and efficient approach to constructing exact solutions for NLDDEs. Using the method, abundant soliton-like and/or period-form solutions for the (2+1)-dimensional Toda equation are derived. In particular, it was found that solitary wave solutions are merely special cases in one family. It is believed that more applications of this method to other NLDDEs merit further investigation. Moreover, how to explain the exact solutions obtained in the preliminary study in physics is still an open problem.

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