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The generalized Stefan–Boltzmann law of a rectilinear non-uniformly accelerating Kinnersley black hole*

Jiang Ji-Jian(蒋继建), Meng Qing-Miao(孟庆苗)†, and Wang Shuai(王 帅)

Department of Physics, Heze University, Heze 274015, *China*

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Using entropy density of Dirac field near the event horizon of a rectilinear non-uniformly accelerating Kinnersley black hole, the law for the thermal radiation of black hole is studied and the instantaneous radiation energy density is obtained. It is found that the instantaneous radiation energy density of a black hole is always proportional to the quartic of the temperature on event horizon in the same direction. That is to say, the thermal radiation of a black hole always satisfies the generalized Stefan–Boltzmann law. In addition, the derived generalized Stefan–Boltzmann coefficient is no longer a constant, but a dynamic coefficient related to the space–time metric near the event horizon and the changing rate of the event horizon in black holes.

Keywords: Kinnersley black hole, entropy density, thin film model, instantaneous radiation energy density

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1. Introduction

Since Bekenstein and Hawking suggested that the entropy of black holes is proportional to the area of event horizon,[1,2] the entropy of black holes is always an important topic of theoretical physics. Many researchers had used different methods to calculate the entropy of black holes and obtained the same result that the entropy of black hole is proportional to its area of the event horizon.[3−6] The illuminating method of them is the brick-wall model proposed by 't Hooft,[3] which gives a statistical interpretation for the entropy of black holes. Recently, the brick-wall has been developed to the thin film model[7,8] by Liu *et al*. Based on this thin film model, the entropy of black holes proves to come from the quantum field in the infinitesimal layer around event horizon. The thin film model can be used to study the thermal radiation of dynamic black holes using local thermal equilibrium near the event horizon and does not need the large-area thermal equilibrium. So this model supplies more convenience to the study of the thermal radiation of dynamic black holes. Many researchers[9−13] adopted the thin film model to calculate the entropy of dynamic black holes and obtain the same result. Where event horizon is, there should be entropy of black holes and Hawking radiation.[14] With the self-gravitation action of the radiation particles, Parikh and Wilczek considered that the Hawking radiation is a quantum tunnelling process, and obtained that the quantum tunnelling rate is related to the change of entropy of Bekenstein–Hawking.[15] Then some author[16−21] have obtained the same result as that of Parikh. This shows that there must be an internal relation between the entropy and the thermal radiation of black holes. So it is worthy to further study the relation between them. On the basis of black body radiation in a flat space,[22] Ref.[23] defined the radiation energy density for static spherically symmetric black holes (the total radiation energy comes from the per unit area and per unit time on the even horizon). Using the statistic entropy of Dirac field in the static spherically symmetric black holes, we obtain the radiation energy density. Conclusion can be drawn that the radiation energy density is always proportional to quartic of the temperature on event horizon. Black holes are actually all dynamic because of their radiation and absorption. In Ref.[24], the dynamic black hole is treated

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†Corresponding author. E-mail: mengqingmiao@yahoo.com.cn

as an open non-equilibrium thermodynamic system. And this open system continuously exchanges materials and energy with environment. Namely, there should be entropy current through the event horizon. For a dynamic black hole slightly deviating from the thermodynamic equilibrium, in order to study the law of thermal radiation, the instantaneous radiation energy density(the radiation power comes from the per unit area on the event horizon at a time) is introduced. By using the entropy density near the event horizon of black holes, the instantaneous radiation energy density of dynamic black holes is written as

$$M = M_0 - \alpha s r_{hv} T, \tag{1}$$

where M_0 is the radiation energy density of black holes under the condition of thermodynamic equilibrium, α is a coefficient related to both radiation and absorption coefficients, s is the entropy volume density in thin film near the event horizon, $r_{hv} = \left(\frac{\mathrm{d}r}{\mathrm{d}v}\right)_{r=r_\mathrm{h}}$ is the changing rate of the event horizon, and T is the temperature on the event horizon. Equation (1) provides a new method to study the thermal radiation of dynamic black holes. Using this method, Meng *et al*[25,26] studied the instantaneous radiation energy density of both the spherical symmetric dynamic black holes and accelerating black holes, and obtained that the instantaneous radiation energy density is always proportional to the quartic of the temperature on the even horizon in the same direction. And Ment *et al*[27] also have studied the instantaneous radiation energy density of the slowly changing dynamic Kerr–Newman black hole. In order to make the result more meaningful and closely related to the astronomical observation, this paper will go further to study the instantaneous radiation energy density of a rectilinear non-uniformly accelerating Kinnersley black hole and find out the thermal radiation law for dynamic black holes.

2. The event horizon equation of the rectilinearly non-uniformly accelerating Kinnersley black hole

The line element of the rectilinear non-uniformly accelerating Kinnersley black hole,[28] by using the advanced Eddington coordinate, is

$$\begin{aligned}\mathrm{d}s^2 =& g_{00}\mathrm{d}v^2 + 2g_{01}\mathrm{d}v\mathrm{d}r + 2g_{02}\mathrm{d}v\mathrm{d}\theta \\ &+ g_{22}\mathrm{d}\theta^2 + g_{33}\mathrm{d}\phi^2,\end{aligned} \tag{2}$$

where

$$\begin{aligned} g_{00} &= -\left(1 - 2ar\cos\theta - r^2a^2\sin^2\theta - \frac{2M}{r}\right), \\ g_{01} &= 1, \\ g_{02} &= -r^2 a\sin\theta, \quad g_{22} = r^2, \\ g_{33} &= r^2\sin^2\theta, \end{aligned} \tag{3}$$

$M = M(v)$ and $a = a(v)$ are the mass and accelerator of the black hole respectively. Because of the axial symmetry of the space–time, the angle $\theta = 0$ points to the direction of accelerator.

Introducing a coordinate transformation[29]

$$\begin{aligned} R &= r - r_h(v,\theta), \\ \mathrm{d}R &= \mathrm{d}r - r_{hv}\mathrm{d}v - r_{h\theta}\mathrm{d}\theta, \end{aligned} \tag{4}$$

where $r_\mathrm{h}(v,\theta)$ is the radius of the event horizon, we have $r_{hv} = \left(\frac{\partial r}{\partial v}\right)_{r=r_\mathrm{h}}$, $r_{h\theta} = \left(\frac{\partial r}{\partial \theta}\right)_{r=r_\mathrm{h}}$. Equation (2) can be rewritten as

$$\begin{aligned}\mathrm{d}s^2 =& \hat{g}_{00}\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R + \hat{g}_{22}\left(\mathrm{d}\theta + \frac{\hat{g}_{02}}{\hat{g}_{22}}\mathrm{d}v\right)^2 \\ &+ \hat{g}_{33}\mathrm{d}\phi^2,\end{aligned} \tag{5}$$

where

$$\begin{aligned} \hat{g}_{00} =& -\left(1 - 2ar\cos\theta - \frac{2M}{r}\right) + 2r_{hv} \\ &+ 2r_{h\theta}a\sin\theta - \frac{r_{h\theta}^2}{r^2}, \\ \hat{g}_{02} =& -r^2 a\sin\theta + r_{h\theta}, \quad \hat{g}_{22} = r^2, \\ \hat{g}_{33} =& r^2\sin^2\theta. \end{aligned} \tag{6}$$

Introducing another coordinate transformation

$$\mathrm{d}\Theta = \mathrm{d}\theta + \frac{\hat{g}_{02}}{\hat{g}_{22}}\mathrm{d}v. \tag{7}$$

We can rewrite Eq.(5) as

$$\mathrm{d}s^2 = \hat{g}_{00}\mathrm{d}v^2 + 2\mathrm{d}v\mathrm{d}R + \hat{g}_{22}\mathrm{d}\Theta^2 + \hat{g}_{33}\mathrm{d}\phi^2. \tag{8}$$

The infinite red-shift surface coincides with the event horizon after transformation, $\hat{g}_{00} = 0$ is just the event horizon equation. Then the event horizon surface equation of black hole can be easily obtained

$$\begin{aligned} &-\left(1 - 2ar_\mathrm{h}\cos\theta - \frac{2M}{r_\mathrm{h}}\right) + 2r_{hv} \\ &+ 2r_{h\theta}a\sin\theta - \frac{r_{h\theta}^2}{r_\mathrm{h}^2} = 0. \end{aligned} \tag{9}$$

Because $\hat{g}_{00}$ =0 is the event horizon equation of black hole, $\hat{g}_{00}$ can be expressed by

$$\hat{g}_{00} = P(\upsilon, r, \theta)(r - r_{\rm h}). \tag{10}$$

3. The generalized Stefan–Boltzmann law of the rectilinear non-uniformly accelerating Kinnersley black hole

The space–time will not have the global thermal equilibrium any longer because the space–time of the accelerating black hole is dynamic. Reference [30] divided the thin film into many subsystems in which the quantum field of each subsystem holds the thermal equilibrium by adopting the thin film model. Then for the ith subsystem's wavefunction in the Dirac field near the event horizon of the rectilinear non-uniformly accelerating Kinnersley black hole, the contribution to the entropy from its first component is

$$(\Delta S_i)_1 = \frac{7\pi^2 \Delta A_i}{180\beta_{\rm h}^3 P^2(r_{\rm h})} \cdot \frac{\delta}{\varepsilon(\varepsilon+\delta)}, \tag{11}$$

where ΔA_i is the small area on the event horizon of the ith subsystem, $\beta_{\rm h} = \dfrac{1}{k_{\rm B}T}$, $k_{\rm B}$ is the Boltzmann constant, T the temperature on the event horizon, δ is the thinness of thin film, ε is the distance between the thin film and the even horizon of the black hole. When ε and δ are both much smaller than $r_{\rm h}$, the contribution to the entropy of the other three components of the ith subsystem wavefunction can be equal to $(\Delta S_i)_1$ under the condition of the first-order approximation. So, the entropy of the ith subsystem in Dirac field near the event horizon of the black hole is

$$\Delta S_i = \frac{7\pi^2 \Delta A_i k_{\rm B}^3}{45P^2(r_{\rm h})} \cdot \frac{\delta}{\varepsilon(\varepsilon+\delta)} T^3. \tag{12}$$

By considering the thin film model, the entropy volume density of the ith subsystem near the event horizon of the black hole can be obtained from Eq.(12)

$$s = \frac{7\pi^2 k_{\rm B}^3}{45P^2(r_{\rm h})\varepsilon(\varepsilon+\delta)} T^3. \tag{13}$$

It is known that M_0 is the radiation energy density of black holes under the condition of thermodynamic equilibrium. In order to obtain M_0 first, consider a slowly changing rectilinear non-uniformly accelerating massive black hole with its whole state slightly deviating from thermodynamic equilibrium. When its mass approaches a constant and the accelerator approaches zero, the state of the black hole will be in the global thermal equilibrium. So the Schwarzschild black hole can be used as a reference background when the state of the rectilinear non-uniformly accelerating Kinnersley black hole tends to be in the global thermodynamic equilibrium. The corresponding space–time line element is

$$\begin{aligned} {\rm d}s^2 = &-\left(1-\frac{2M}{r}\right){\rm d}t^2 + \left(1-\frac{2M}{r}\right)^{-1}{\rm d}r^2 \\ &+ r^2({\rm d}\theta^2 + \sin^2\theta {\rm d}\phi^2), \end{aligned} \tag{14}$$

where M is the mass of the black hole. We set

$$\hat{g}'_{00} = -\left(1-\frac{2M}{r}\right) = f(r)(r - r_{\rm h}). \tag{15}$$

So $\hat{g}'_{00} = 0$ is the horizon equation of the Schwarzschild black hole. In Ref.[23], the radiation energy density of the Dirac field of the static spherically symmetric black holes is given by

$$M_0^{(D)} = \frac{21\pi^2 c k_{\rm B}^3}{90\varepsilon(\varepsilon+\delta)f_{\rm h}^2} T^4, \tag{16}$$

where c is the speed of light. Substituting Eqs.(13) and (16) into Eq.(1), the instantaneous radiation energy density of the Dirac field of the rectilinear non-uniformly accelerating Kinnersley black hole can be obtained

$$M^{(\rm D)} = \frac{7\pi^2 k_{\rm B}^3}{90\varepsilon(\varepsilon+\delta)}\left[\frac{3c}{f^2(r_{\rm h})} - \frac{2\alpha r_{h\upsilon}}{P^2(r_{\rm h})}\right] T^4, \tag{17}$$

where

$$f(r_{\rm h}) = \left(\frac{\partial \hat{g}'_{00}}{\partial r}\right)_{r=r_{\rm h}} = \frac{1}{r_{\rm h}}, \tag{18}$$

$$P(r_{\rm h}) = \left(\frac{\partial \hat{g}_{00}}{\partial r}\right)_{r=r_{\rm h}} = 2\left(a\cos\theta - \frac{M}{r_{\rm h}^2} + \frac{r_{h\theta}^2}{r_{\rm h}^3}\right). \tag{19}$$

We set

$$\sigma(\upsilon, \theta) = \frac{7\pi^2 k_{\rm B}^3}{90\varepsilon(\varepsilon+\delta)}\left[\frac{3c}{f^2(r_{\rm h})} - \frac{2\alpha r_{h\upsilon}}{P^2(r_{\rm h})}\right]. \tag{20}$$

Equation (17) can be rewritten as

$$M^{(\rm D)} = \sigma(\upsilon, \theta) T^4. \tag{21}$$

It can be easily seen through Eqs.(18–20) that $\sigma(\upsilon,\ \theta)$ changes with both the Eddington time υ and the direction angle θ for a given δ and ε. For a fixed υ_0 and a fixed direction angle θ_0 on the event horizon of black holes, the $\sigma(\upsilon_0,\ \theta_0)$ will be a fixed value. So the instantaneous radiation energy density of the rectilinear non-uniformly accelerating Kinnersley black hole at any time is always proportional to quartic power of the temperature on event horizon of black holes

in the same direction. This result is similar to the Stefan–Boltzmann law of blackbody radiation in a flat space–time. Therefore Eq.(21) can be called the generalized Stefan–Boltzmann law of the Dirac field of the rectilinear non-uniformly accelerating Kinnersley black hole. Equation (20) is called the corresponding generalized Stefan–Boltzmann dynamic proportionate coefficient. And σ is no longer a constant as compared with the blackbody radiation in the flat space–time, but a dynamic proportionate coefficient related to the rate of change of the event horizon, the space–time metric near the event horizon, the absorption and radiation factors of the black hole. It is obvious that the thermal radiation of a curved space–time is different from that of a flat space–time, and the gravitational field around the black hole will affect the thermal radiation. So there must be an internal relation between thermal radiation t and gravitational field.

Reference [11] obtained the conclusion that when the same cut-off factor is obtained, the statistical entropy of Dirac field of the static spherically symmetric black holes is 7/2 times that of a scalar field. In addition, Reference [27] also obtained the conclusion that when the same cut-off factor is obtained, the statistical entropy of Dirac field of the rectilinear non-uniformly accelerating Kinnersley black hole is 7/2 times that of a scalar field. The instantaneous radiation energy density of the scalar field of the rectilinearly non-uniformly accelerating Kinnersley black hole can be obtained from Eq.(17)

$$M^{(\mathrm{B})}=\frac{\pi^2k_{\mathrm{B}}^3}{45\varepsilon(\varepsilon+\delta)}\left[\frac{3c}{f^2(r_{\mathrm{h}})}-\frac{2\alpha r_{hv}}{P^2(r_{\mathrm{h}})}\right]T^4. \tag{22}$$

Equation (22) can be called the generalized Stefan–Boltzmann law of scalar field of the rectilinear non-uniformly accelerating Kinnersley black hole. The corresponding generalized Stefan–Boltzmann dynamic proportionate coefficient is

$$\sigma(v,\theta)=\frac{\pi^2k_{\mathrm{B}}^3}{45\varepsilon(\varepsilon+\delta)}\left[\frac{3c}{f^2(r_{\mathrm{h}})}-\frac{2\alpha r_{hv}}{P^2(r_{\mathrm{h}})}\right]. \tag{23}$$

4. Discussion and conclusion

The Schwarzschild black hole can be used as a reference background when the state of the Vaidya black hole tends to be in the global thermodynamic equilibrium. The parameter $r_{\mathrm{h}}=\dfrac{2M}{1-2r_{hv}}$ denotes the radius of the event horizon of Vaidya black hole and $r_{\mathrm{h}}=2M$ is the radius of the event horizon of Schwarzschild black hole. So when we set $a=r_{h\theta}=0$, the instantaneous radiation energy density of the Dirac field of the Vaidya black hole can be obtained from Eqs.(17–19)

$$M^{(\mathrm{D})}=\frac{14\pi^2k_{\mathrm{B}}^3M^2}{45\varepsilon(\varepsilon+\delta)}\left[3c-\frac{2\alpha r_{hv}}{(1-2r_{hv})^4}\right]T^4. \tag{24}$$

Equation (24) can be called the generalized Stefan–Boltzmann law for the Vaidya black hole. The corresponding generalized Stefan–Boltzmann dynamic proportionate coefficient is

$$\sigma(v)=\frac{14\pi^2k_{\mathrm{B}}^3M^2}{45\varepsilon(\varepsilon+\delta)}\left[3c-\frac{2\alpha r_{hv}}{(1-2r_{hv})^4}\right]. \tag{25}$$

Setting $r_{hv}=0$, we have the radiation energy density of the Dirac field of the Schwarzschild black hole, which can be obtained from Eq.(24)

$$M_0^{(\mathrm{D})}=\frac{14\pi^2ck_{\mathrm{B}}^3M^2}{15\varepsilon(\varepsilon+\delta)}T^4. \tag{26}$$

Equation (26) can be called the generalized Stefan–Boltzmann law for the Schwarzschild black hole. The corresponding generalized Stefan–Boltzmann dynamic proportionate coefficient is

$$\sigma=\frac{14\pi^2ck_{\mathrm{B}}^3M^2}{15\varepsilon(\varepsilon+\delta)}. \tag{27}$$

These results indicate that the thermal radiation of the rectilinear non-uniformly accelerating Kinnersley black hole satisfies the generalized Stefan–Boltzmann law. That is to say, the instantaneous radiation energy density of a black hole is always proportional to the quartic of the temperature on the event horizon in the same direction. Compared with the Stefan–Boltzmann law of blackbody radiation in flat space–time, σ is no longer a constant, but a dynamic proportional coefficient that is related to the event horizon changing rate, space–time metric near the event horizon, the radiation and absorption coefficient of the black hole. For different black holes, because of the different parameters, the derived generalized Stefan–Boltzmann coefficients are different from each other too. The result that the value of σ changes with the parameters of the black hole indicates the thermal radiation of black holes in the curved space–time is different from that of blackbody in the flat space–time. The huge gravitational field near the black hole will have an important effect on the thermal radiation, so there must be an internal relation between gravitational field and the thermal radiation of black holes. When $r_{hv}=0$, the instantaneous radiation energy density of the dynamic black holes naturally becomes that of the static black hole, which is in

good agreement with the known theory. Although the thin film model can describe many thermal properties of black holes, but the cut-off factor is inevitably. This paper provides a new method to study the thermal radiation of black holes, which is closely related to the astronomical observation.

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