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# The constructive technique and its application in solving a nonlinear reaction diffusion equation*

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A mathematical technique based on the consideration of a nonlinear partial differential equation together with an additional condition in the form of an ordinary differential equation is employed to study a nonlinear reaction diffusion equation which describes a real process in physics and in chemistry. Several exact solutions for the equation are acquired under certain circumstances.

**Keywords:** additional generating condition method, exact solutions, nonlinear reaction-diffusion equation

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## 1. Introduction

Many analytical and numerical techniques have been employed to seek exact solutions of nonlinear partial differential equations. Rosenau and Hyman[1] and Rosenau[2] used the pseudo spectral method in space and a variable order, variable time-step Adams–Basford–Moulton method in time to study a family of nonlinear Korteweg–de Vries (KdV) equations, and obtained a class of solitary waves with compact support, which are called compactons. Li and Zhang[3] developed the symbolic computation method and employed it to study a class of nonlinear evolution equations. Some important discoveries for the nonlinear equations were acquired in Ref.[3]. Fan and Zhang,[4] and Fan[5] extended the hyperbolic tangent (tanh) method and used it to investigate the generalized modified Korteweg–de Vries (mKdV) equation and the generalized Zakharov–Kuznetsov (ZK) equation. This extended method is shown to be a powerful tool to seek exact solutions of nonlinear equations so that many solitary wave solutions were obtained (see Refs.[4–6]). By decomposing the time and the space variables of nonlinear partial differential equations into two integrable ordinary differential equations, Ma and Wu[7] have found some exact solutions for KdV mKdV and Kolmogorov–Petrovsky–Piscounov (KPP) equations. Other approaches, such as the inverse scattering method, the Bäcklund transformation, the Darboux transformation, the Painlevé analysis, the tri-Hamiltonian operators, the finite difference method, the Adomian decomposition method, the sin–cos ansatz method and the Jacobi elliptic function expansion method, may be found from Refs.[8–14] and the references therein.

Recently, a new constructive method based on the consideration of a fixed nonlinear partial differential equation (PDE) together with an additional condition in the form of linear ordinary differential equation (ODE) was developed in Ref.[15] by Cherniha. This method is called the additional generating condition method (see Refs.[15–17]). Using the method, Cherniha[15] acquired the exact solution for the following generalized Burger equation:

$$u_t = (1 + au)u_{xx} + buu_x + \mu u_x + \nu u^2, \qquad (1)$$

where $a, b, \mu$ and $\nu$ are arbitrary constants, $u_t = \partial u/\partial t$, $u_x = \partial u/\partial x$, and $u_{xx} = \partial^2 u/\partial x^2$.

Using the new Q-conditional (nonclassical) symmetry technique together with the additional generating condition method, Cherniha[17] obtained new ex-

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act solutions for the following equation:

$$u_t = u_{xx} + auu_x + bu - \mu u^2 - \nu u^3, \tag{2}$$

where $a, b, \mu$ and $\nu$ are arbitrary constants.

Cherniha[18] mentioned the following reaction diffusion equation:

$$u_t = u_{xx} + \lambda_1 uu_x + u_x^2 + \lambda_2 + \lambda_3 u + \frac{1}{4}\lambda_1^2 u^2, \tag{3}$$

where the coefficients $\lambda_1, \lambda_2$, and $\lambda_3$ are arbitrary constants, and $u = u(t, x)$ is the unknown function. However, Eq.(3) was not solved in Ref.[18]. In fact, Eq.(3) is used as a model to describe a real process in physics and in chemistry (see Ref.[18]). The objective of this paper is to use the additional generating condition method to study Eq.(3). Several exact solutions for the equation are acquired under certain circumstances.

## 2. Description of the method

We briefly state the additional generating condition method presented in Ref.[19] by Cherniha. For Eq.(3), we assume additional condition to be in the form of the linear homogeneous system as follows:

$$\alpha_1(t)\frac{\mathrm{d}u}{\mathrm{d}x} + \alpha_2(t)\frac{\mathrm{d}^2u}{\mathrm{d}x^2} + \alpha_3(t)\frac{\mathrm{d}^3u}{\mathrm{d}x^3} = 0, \tag{4}$$

where $\alpha_1(t), \alpha_2(t)$ and $\alpha_3(t)$ are functions of variable $t$.

From the theory of solving ordinary differential equations, we know that Eq.(4) has the general solution in the form

$$u = \varphi_0(t)g_0(t, x) + \varphi_1(t)g_1(t, x) + \varphi_2(t)g_2(t, x), \tag{5}$$

where $\varphi_0(t), \varphi_1(t)$ and $\varphi_2(t)$ are functions of variable $t$ and $g_0(t, x), g_1(t, x)$, and $g_2(t, x)$ are fixed functions that form a fundamental solution system of Eq.(4).

Substituting expression (5) into Eq.(3) yields the following identity:

$$\begin{aligned}
&\varphi_{0,t}g_0 + \varphi_{1,t}g_1 + \varphi_{2,t}g_2 \\
&= \varphi_0(g_{0,xx} + \lambda_3 g_0 - g_{0,t}) \\
&\quad +\varphi_1(g_{1,xx} + \lambda_3 g_1 - g_{1,t}) \\
&\quad +\varphi_2(g_{2,xx} + \lambda_3 g_2 - g_{2,t}) \\
&\quad +\varphi_0^2\left(\lambda_1 g_0 g_{0,x} + g_0^2 x + \frac{1}{4}\lambda_1^2 g_0^2\right) \\
&\quad +\varphi_1^2\left(\lambda_1 g_1 g_{1,x} + g_1^2 x + \frac{1}{4}\lambda_1^2 g_1^2\right) \\
&\quad +\varphi_2^2\left(\lambda_1 g_2 g_{2,x} + g_2^2 x + \frac{1}{4}\lambda_1^2 g_2^2\right) \\
&\quad +\varphi_0\varphi_1\left(\lambda_1 g_0 g_{1,x} + \lambda_1 g_{0,x} g_1 + 2g_{0,x}g_{1,x}\right. \\
&\quad \left.+\frac{1}{2}\lambda_1^2 g_0 g_1\right) + \varphi_0\varphi_2\left(\lambda_1 g_{0,x} g_2 + \lambda_1 g_0 g_{2,x}\right. \\
&\quad \left.+2g_{0,x}g_{2,x} + \frac{1}{2}\lambda_1^2 g_0 g_2\right) + \varphi_1\varphi_2\left(\lambda_1 g_1 g_{2,x}\right. \\
&\quad \left.+\lambda_1 g_{1,x} g_2 + 2g_{1,x}g_{2,x} + \frac{1}{2}\lambda_1^2 g_1 g_2\right) + \lambda_2,
\end{aligned} \tag{6}$$

where $\varphi_{i,t} = \partial\varphi_i/\partial t$, $g_{i,t} = \partial g_i/\partial t$, $g_{i,x} = \partial g^i/\partial t$, and $g_{i,xx} = \partial^2 g^i/\partial x^2$, $i = 0, 1, 2$. The sufficient conditions for reducing Eq.(6) to a system of ordinary differential equations (ODEs) possess the form

$$g_{i,xx} + \lambda_3 g_i - g_{i,t} = \sum_{j=0}^{2} g_j Q_{ij}(t), \tag{7}$$

$$\lambda_1 g_i g_{i,x} + g_i^2 x + \frac{1}{4}\lambda_1^2 g_i^2 = \sum_{j=0}^{2} g_j R_{ij}(t), \tag{8}$$

and

$$\begin{aligned}
&\lambda_1 g_i g_{i_1,x} + \lambda_1 g_{i,x} g_{i_1} + 2g_{i,x}g_{i_1,x} + \frac{1}{2}\lambda_1^2 g_i g_{i_1} \\
&= \sum_{j=0}^{2} g_j T_{ii_1}^j(t), \quad i < i_1,
\end{aligned} \tag{9}$$

where $Q_{ij}(t)$, $R_{ij}(t)$ and $T_{ii_1}^j(t)$ on the right-hand side are defined by the expressions on the left-hand side, $i_1, i = 0, 1, 2$.

It follows from Eqs.(6)–(9) that

$$\begin{aligned}
\varphi_{0,t} &= \sum_{i=0}^{2} Q_{i0}\varphi_i + \sum_{i=0}^{2} R_{i0}(\varphi_i)^2 \\
&\quad + \sum_{0\le i_1<i_2\le 2} T_{i_1 i_2}^0 \varphi_{i_1}\varphi_{i_2} + \lambda_2,
\end{aligned} \tag{10}$$

$$\begin{aligned}
\varphi_{1,t} &= \sum_{i=0}^{2} Q_{i1}\varphi_i + \sum_{i=0}^{2} R_{i1}(\varphi_i)^2 \\
&\quad + \sum_{0\le i_1<i_2\le 2} T_{i_1 i_2}^1 \varphi_{i_1}\varphi_{i_2},
\end{aligned} \tag{11}$$

and

$$\varphi_{2,t} = \sum_{i=0}^{2} Q_{i2}\varphi_i + \sum_{i=0}^{2} R_{i2}(\varphi_i)^2 + \sum_{0\le i_1<i_2\le 2} T_{i_1 i_2}^2 \varphi_{i_1}\varphi_{i_2}. \tag{12}$$

Therefore, we can conclude that any solution of the systems (10)–(12) generates an exact solution in the form of Eq.(5) for Eq.(13), if functions $g_i$, $i = 0, 1, 2$, satisfy conditions (7)–(9).

# 3. Construction of solutions

Without loss of generality, for Eq.(3), we consider an additional condition in the following third order differential equation:

$$\alpha_1(t)\frac{\mathrm{d}u}{\mathrm{d}x}+\alpha_2(t)\frac{\mathrm{d}^2u}{\mathrm{d}x^2}+\frac{\mathrm{d}^3u}{\mathrm{d}x^3}=0, \tag{13}$$

where $\alpha_1(t)$ and $\alpha_2(t)$ are arbitrary functions.

## 3.1. Ansatz 1 and exact solutions

From the theory of ordinary differential equation, Eq.(13) has the following form of solution:

$$u=\varphi_0(t)+\varphi_1(t)\exp(\gamma_1 x)+\varphi_2(t)\exp(\gamma_2 x), \tag{14}$$

where $\gamma_{1,2}(t)=\frac{1}{2}\left[\pm(\alpha_2^2-4\alpha_1)^{1/2}-\alpha_2\right]$ and $\gamma_1\neq\gamma_2$.

Substituting functions $g_0=1,\quad g_1=\exp(\gamma_1 x)$ and $g_2=\exp(\gamma_2 x)$ into Eq.(7) results in

$$g_{i,xx}+\lambda_3 g_i-g_{i,t}=g_0Q_{i0}+g_1Q_{i1}+g_2Q_{i2}. \tag{15}$$

If $i=0$, we obtain

$$\lambda_3=Q_{00}+\exp(\gamma_1 x)Q_{01}+\exp(\gamma_2 x)Q_{02}, \tag{16}$$

from which we have

$$Q_{00}=\lambda_3,\quad Q_{01}=Q_{02}=0. \tag{17}$$

If $i=1$, we have

$$\begin{aligned}&\gamma_1^2\exp(\gamma_1 x)+\lambda_3\exp(\gamma_1 x)\\ &=Q_{10}+\exp(\gamma_1 x)Q_{11}+\exp(\gamma_2 x)Q_{12},\end{aligned} \tag{18}$$

which yields

$$Q_{10}=Q_{12}=0,\ \text{ and }\ Q_{11}=\gamma_1^2+\lambda_3. \tag{19}$$

If $i=2$, we have

$$\begin{aligned}&\gamma_2^2\exp(\gamma_2 x)+\lambda_3\exp(\gamma_2 x)\\ &=Q_{20}+\exp(\gamma_1 x)Q_{21}+\exp(\gamma_2 x)Q_{22}.\end{aligned} \tag{20}$$

It follows from Eq.(20) that

$$Q_{20}=Q_{21}=0,\ \text{ and }\ Q_{22}=\gamma_2^2+\lambda_3. \tag{21}$$

Applying $g_0=1,\quad g_1=\exp(\gamma_1 x)$ and $g_2=\exp(\gamma_2 x)$ to Eq.(8) results in

$$\lambda_1 g_i g_{i,x}+g_i^2 x+\frac{1}{4}\lambda_1^2 g_i^2=g_0R_{i0}+g_1R_{i1}+g_2R_{i2}. \tag{22}$$

If $i=0$, it has

$$\frac{1}{4}\lambda_1^2=R_{00}+\exp(\gamma_1 x)R_{01}+\exp(\gamma_2 x)R_{02}, \tag{23}$$

which leads to

$$R_{00}=\frac{1}{4}\lambda_1^2,\quad R_{01}=R_{02}=0. \tag{24}$$

In the case $i=1$, we have

$$\begin{aligned}&\lambda_1\gamma_1\exp(2\gamma_1 x)+\gamma_1^2\exp(2\gamma_1 x)+\frac{1}{4}\lambda_1^2\exp(2\gamma_1 x)\\ &=R_{10}+\exp(\gamma_1 x)R_{11}+\exp(\gamma_2 x)R_{12}.\end{aligned} \tag{25}$$

Requiring $\gamma_2=2\gamma_1$ gives rise to

$$R_{12}=\gamma_1^2+\lambda_1\gamma_1+\frac{1}{4}\lambda_1^2,\quad R_{10}=R_{11}=0. \tag{26}$$

If $i=2$, we have

$$\begin{aligned}&\lambda_1\gamma_2\exp(2\gamma_2 x)+\gamma_2^2\exp(2\gamma_2 x)+\frac{1}{4}\lambda_1^2\exp(2\gamma_2 x)\\ &=R_{20}+\exp(\gamma_1 x)R_{21}+\exp(\gamma_2 x)R_{22}.\end{aligned} \tag{27}$$

Assuming $\gamma_2^2+\lambda_1\gamma_2+\frac{1}{4}\lambda_1^2=0$ leads to

$$R_{20}=R_{21}=R_{22}=0. \tag{28}$$

Applying $g_0=1,\quad g_1=\exp(\gamma_1 x)$ and $g_2=\exp(\gamma_2 x)$ to Eq.(9) yields

$$\begin{aligned}&\lambda_1 g_i g_{i_1,x}+\lambda_1 g_{i,x}g_{i_1}+2g_{i,x}g_{i_1,x}+\frac{1}{2}\lambda_1^2 g_i g_{i_1}\\ &=g_0T_{ii_1}^0+g_1T_{ii_1}^1+g_2T_{ii_1}^2.\end{aligned} \tag{29}$$

If $i=0$ and $i_1=1$, we have

$$\begin{aligned}&\lambda_1\gamma_1\exp(\gamma_1 x)+\frac{1}{2}\lambda_1^2\exp(\gamma_1 x)\\ &=T_{01}^0+\exp(\gamma_1 x)T_{01}^1+\exp(\gamma_2 x)T_{01}^2,\end{aligned} \tag{30}$$

from which we obtain

$$T_{01}^0=T_{01}^2=0,\ \text{ and }\ T_{01}^1=\frac{1}{4}\lambda_1^2. \tag{31}$$

If $i=0$ and $i_1=2$, we have

$$\begin{aligned}&\lambda_1\gamma_2\exp(\gamma_2 x)+\frac{1}{2}\lambda_1^2\exp(\gamma_2 x)\\ &=T_{02}^0+\exp(\gamma_1 x)T_{02}^1+\exp(\gamma_2 x)T_{02}^2,\end{aligned} \tag{32}$$

and

$$T_{02}^0=T_{02}^1=T_{02}^2=0. \tag{33}$$

If $i=1$ and $i_1=2$, we obtain

$$\begin{aligned}&\left(\lambda_1\gamma_2+\lambda_1\gamma_2+2\gamma_1\gamma_2+\frac{1}{2}\lambda_1^2\right)\exp\left[(\gamma_1+\gamma_2)x\right]\\ &=T_{12}^0+\exp(\gamma_1 x)T_{12}^1+\exp(\gamma_2 x)T_{12}^2,\end{aligned} \tag{34}$$

which yields

$$T_{12}^0=T_{12}^1=T_{12}^2=0. \tag{35}$$

Making use of Eqs.(17), (19), (21), (24), (26), (28), (31), (33) and (35), we obtain

$$\begin{aligned}
\frac{\mathrm{d}\varphi_0}{\mathrm{d}t} &= \frac{1}{4}\lambda_1^2\varphi_0^2 + \lambda_3\varphi_0 + \lambda_2,\\
\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} &= \left(\frac{\lambda_1^2}{16} + \lambda_3\right)\varphi_1 + \frac{1}{4}\lambda_1^2\varphi_0\varphi_1,\\
\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} &= \left(\frac{\lambda_1^2}{4} + \lambda_3\right)\varphi_2 + \frac{\lambda_1^2}{16}\varphi_1^2,
\end{aligned} \tag{36}$$

in which we have required that $\gamma_2 = 2\gamma_1$ and $\gamma_2^2 + \lambda_1\gamma_2 + \frac{1}{4}\lambda_1^2 = 0$.

**Case 1** $D = \lambda_3^2 - \lambda_1^2\lambda_2 = 0$. Solving system (36), we have the following solution:

$$\begin{aligned}
\varphi_0 &= \frac{2}{\lambda_1^2}\left(\frac{2}{c_0 - t} - \lambda_3\right),\\
\varphi_1 &= c_1 \exp\left[\left(\frac{\lambda_1^2}{16} + \frac{\lambda_3}{2}\right)t - \ln|c_0 - t|\right]\\
\varphi_2 &= c_2(t)\exp\left[\left(\frac{\lambda_1^2}{4} + \lambda_3\right)t\right],
\end{aligned} \tag{37}$$

where $c_0$ and $c_1$ are arbitrary constants, and $c_2(t) = \int \frac{\lambda_1^2 c_1^2}{16}\frac{1}{(c_0 - t)^2}\exp\left(-\frac{\lambda_1^2 t}{8}\right)\mathrm{d}t$. Using Eqs.(14) and (37), we obtain the exact solution of Eq.(3) in the form

$$\begin{aligned}
u = {} & \frac{4}{\lambda_1^2(c_0 - t)} - \frac{2\lambda_3}{\lambda_1^2} + \frac{c_1}{c_0 - t}\\
& \times \exp\left[\left(\frac{\lambda_1^2}{16} + \frac{\lambda_3}{2}\right)t - \frac{\lambda_1}{4}x\right]\\
& + c_2(t)\exp\left[\left(\frac{\lambda_1^2}{4} + \lambda_3\right)t - \frac{\lambda_1}{2}x\right].
\end{aligned} \tag{38}$$

**Case 2** $D = \lambda_3^2 - \lambda_1^2\lambda_2 < 0$. We have the following solutions of system (36):

$$\begin{aligned}
\varphi_0 &= \frac{2}{\lambda_1^2}\left\{\sqrt{-D}\tan\left[\frac{1}{2}\sqrt{-D}\,(t - c_0)\right] - \lambda_3\right\},\\
\varphi_1 &= c_3\exp\left\{\left(\frac{\lambda_1^2}{16} + \frac{\lambda_3}{2}\right)t\right.\\
&\quad \left. - \ln\left|\cos\left[\frac{1}{2}\sqrt{-D}(t - c_0)\right]\right|\right\},\\
\varphi_2 &= c_4(t)\exp\left[\left(\frac{\lambda_1^2}{4} + \lambda_3\right)t\right]
\end{aligned} \tag{39}$$

and

$$\begin{aligned}
\varphi_0 &= \frac{2}{\lambda_1^2}\left\{\sqrt{-D}\cot\left[\frac{1}{2}\sqrt{-D}\,(t - c_0)\right] - \lambda_3\right\},\\
\varphi_1 &= c_5\exp\left\{\left(\frac{\lambda_1^2}{16} + \frac{\lambda_3}{2}\right)t\right.\\
&\quad \left. - \ln\left|\sin\left[\frac{1}{2}\sqrt{-D}(t - c_0)\right]\right|\right\},\\
\varphi_2 &= c_6(t)\exp\left[\left(\frac{\lambda_1^2}{4} + \lambda_3\right)t\right],
\end{aligned} \tag{40}$$

where $c_3$ and $c_5$ are constants, and

$$\begin{aligned}
c_4(t) = {} & \int \frac{\lambda_1^2}{16}c_3^2\frac{1}{\cos^2\left[\frac{1}{2}\sqrt{-D}(t - c_0)\right]}\\
& \times \exp\left(-\frac{\lambda_1^2 t}{8}\right)\mathrm{d}t,\\
c_6(t) = {} & \int \frac{\lambda_1^2}{16}c_3^2\frac{1}{\sin^2\left[\frac{1}{2}\sqrt{-D}(t - c_0)\right]}\\
& \times \exp\left(-\frac{\lambda_1^2 t}{8}\right)\mathrm{d}t.
\end{aligned}$$

From Eqs.(39) and (40), the solutions for Eq.(3) are expressed as follows:

$$\begin{aligned}
u = {} & \frac{2\sqrt{-D}}{\lambda_1^2}\tan\left[\frac{1}{2}\sqrt{-D}(t - c_0)\right] - \frac{2\lambda_3}{\lambda_1^2}\\
& + \frac{c_3}{\left|\cos\left[\frac{1}{2}\sqrt{-D}(t - c_0)\right]\right|}\\
& \times \exp\left[\left(\frac{\lambda_1^2}{16} + \frac{\lambda_3}{2}\right)t - \frac{\lambda_1}{4}x\right]\\
& + c_4(t)\exp\left\{\left[\left(\frac{\lambda_1^2}{4} + \lambda_3\right)t\right] - \frac{\lambda_1}{2}x\right\}
\end{aligned} \tag{41}$$

and

$$\begin{aligned}
u = {} & \frac{2\sqrt{-D}}{\lambda_1^2}\tan\left[\frac{1}{2}\sqrt{-D}(t - c_0)\right] - \frac{2\lambda_3}{\lambda_1^2}\\
& + \frac{c_5}{\left|\sin\left[\frac{1}{2}\sqrt{-D}(t - c_0)\right]\right|}\\
& \times \exp\left[\left(\frac{\lambda_1^2}{16} + \frac{\lambda_3}{2}\right)t - \frac{\lambda_1}{4}x\right]\\
& + c_6(t)\exp\left\{\left[\left(\frac{\lambda_1^2}{4} + \lambda_3\right)t\right] - \frac{\lambda_1}{2}x\right\}.
\end{aligned} \tag{42}$$

**Case3** $D = \lambda_3^2 - \lambda_1^2\lambda_2 > 0$. Solving system (36) gives rise to

$$\begin{aligned}
\varphi_0 &= \frac{2}{\lambda_1^2}\left\{\sqrt{D}\tanh\left[\frac{1}{2}\sqrt{D}(c_0 - t)\right] - \lambda_3\right\},\\
\varphi_1 &= c_7\exp\left\{\left(\frac{\lambda_1^2}{16} + \frac{\lambda_3}{2}\right)t\right.\\
&\quad \left. - \ln\cosh\left[\frac{1}{2}\sqrt{D}(c_0 - t)\right]\right\},\\
\varphi_2 &= c_8(t)\exp\left[\left(\frac{\lambda_1^2}{4} + \lambda_3\right)t\right],
\end{aligned} \tag{43}$$

and

$$\begin{aligned}
\varphi_0 &= \frac{2}{\lambda_1^2}\left\{\sqrt{D}\coth\left[\frac{1}{2}\sqrt{D}(c_0 - t)\right] - \lambda_3\right\},\\
\varphi_1 &= c_9\exp\left\{\left(\frac{\lambda_1^2}{16} + \frac{\lambda_3}{2}\right)t\right.
\end{aligned}$$

$$-\ln\left|\sinh\left[\frac{1}{2}\sqrt{D}(c_0-t)\right]\right|\Bigg\},$$

$$\varphi_2 = c_{10}(t)\exp\left[\left(\frac{\lambda_1^2}{4}+\lambda_3\right)t\right], \tag{44}$$

where $c_7$ and $c_9$ are constants and

$$c_8(t) = \int \frac{\lambda_1^2 c_5^2}{16}\frac{1}{\cosh^2\left[\frac{1}{2}\sqrt{D}(c_0-t)\right]} \times \exp\left(-\frac{\lambda_1^2 t}{8}\right)\mathrm{d}t,$$

$$c_{10}(t) = \int \frac{\lambda_1^2 c_5^2}{16}\frac{1}{\sinh^2\left[\frac{1}{2}\sqrt{D}(c_0-t)\right]} \times \exp\left(-\frac{\lambda_1^2 t}{8}\right)\mathrm{d}t.$$

Therefore, solutions of Eq.(3) take the following forms:

$$u = \frac{2\sqrt{D}}{\lambda_1^2}\tanh\left[\frac{1}{2}\sqrt{D}(c_0-t)\right] - \frac{2\lambda_3}{\lambda_1^2} + \frac{c_7}{\cosh\left[\frac{1}{2}\sqrt{D}(c_0-t)\right]}\exp\left[\left(\frac{\lambda_1^2}{16}+\frac{\lambda_3}{2}\right)t - \frac{\lambda_1}{4}x\right] + c_8(t)\exp\left[\left(\frac{\lambda_1^2}{4}+\lambda_3\right)t - \frac{\lambda_1}{2}x\right] \tag{45}$$

and

$$u = \frac{2\sqrt{D}}{\lambda_1^2}\cosh\left[\frac{1}{2}\sqrt{D}(c_0-t)\right] - \frac{2\lambda_3}{\lambda_1^2} + \frac{c_9}{\left|\sinh\left[\frac{1}{2}\sqrt{D}(c_0-t)\right]\right|}\exp\left[\left(\frac{\lambda_1^2}{16}+\frac{\lambda_3}{2}\right)t - \frac{\lambda_1}{4}x\right] + c_{10}(t)\exp\left[\left(\frac{\lambda_1^2}{4}+\lambda_3\right)t - \frac{\lambda_1}{2}x\right]. \tag{46}$$

## 3.2. Ansatz 2 and exact solutions

If $\alpha_1 = \alpha_2 = 0$, Eq.(13) generates the following ansatz:

$$u = \varphi_0(t) + \varphi_1(t)x + \varphi_2(t)x^2. \tag{47}$$

Substituting $g_0 = 1$, $g_1 = x$ and $g_2 = x^2$ into Eqs.(7)–(9) yields

$$Q_{00} = Q_{11} = Q_{22} = \lambda_3, \quad Q_{20} = 2,$$
$$R_{10} = 1, \quad R_{22} = 4, \quad T_{12}^1 = 4, \quad \lambda_1 = 0, \tag{48}$$

and the relations

$$Q_{ii_1} = R_{ii_1} = T_{ii_1}^j = 0 \tag{49}$$

for different combinations of the indices $i, i_1$ and $j$ which are not listed in expression (48).

Making use of Eqs.(48) and (49) leads Eqs.(10)–(12) to reduce to the following form:

$$\frac{\mathrm{d}\varphi_0}{\mathrm{d}t} = \lambda_3\varphi_0 + \varphi_1^2 + 2\varphi_2 + \lambda_2,$$
$$\frac{\mathrm{d}\varphi_1}{\mathrm{d}t} = \lambda_3\varphi_1 + 4\varphi_1\varphi_2,$$
$$\frac{\mathrm{d}\varphi_2}{\mathrm{d}t} = \lambda_3\varphi_2 + 4\varphi_2^2, \tag{50}$$

from which we obtain

$$\varphi_0 = b_1(t)\exp(\lambda_3 t),$$
$$\varphi_1 = \frac{b_2\exp\left(\frac{\lambda_3}{2}t\right)}{\sinh\left[\frac{\lambda_3}{2}(b_0-t)\right]},$$
$$\varphi_2 = \frac{\lambda_3}{8}\coth\left[\frac{\lambda_3}{2}(b_0-t)\right] - \frac{\lambda_3}{8}, \tag{51}$$

and

$$\varphi_0 = b_3(t)\exp(\lambda_3 t),$$
$$\varphi_1 = \frac{b_4\exp\left(\frac{\lambda_3}{2}t\right)}{\cosh\left[\frac{\lambda_3}{2}(b_0-t)\right]},$$
$$\varphi_2 = \frac{\lambda_3}{8}\tanh\left[\frac{\lambda_3}{2}(b_0-t)\right] - \frac{\lambda_3}{8}, \tag{52}$$

where $b_0, b_2$ and $b_4$ are constants and

$$b_1(t) = \frac{2b_2}{\lambda_3}\coth\left[\frac{\lambda_3}{2}(b_0-t)\right] + \frac{\lambda_3}{4}\int \frac{\coth\left[\frac{\lambda_3}{2}(b_0-t)\right]}{\exp(\lambda_3 t)}\mathrm{d}t + \left(\frac{1}{4}-\frac{\lambda_2}{\lambda_3}\right)\exp\left(-\lambda_3 t\right),$$

and

$$b_3(t) = \frac{-2b_4}{\lambda_3}\tanh\left[\frac{\lambda_3}{2}(b_0-t)\right] + \frac{\lambda_3}{4}\int \frac{\tanh\left[\frac{\lambda_3}{2}(b_0-t)\right]}{\exp(\lambda_3 t)}\mathrm{d}t + \left(\frac{1}{4}-\frac{\lambda_2}{\lambda_3}\right)\exp(-\lambda_3 t).$$

From expressions (47), (51) and (52), we obtain the solutions of Eq.(3) as

$$u = b_1(t)\exp(\lambda_3 t) + \frac{b_2\exp\left(\frac{\lambda_3}{2}t\right)}{\sinh\left[\frac{\lambda_3}{2}(b_0-t)\right]}x + \left\{\frac{\lambda_3}{8}\coth\left[\frac{\lambda_3}{2}(b_0-t)\right] - \frac{\lambda_3}{8}\right\}x^2, \tag{53}$$

and

$$u = b_3(t)\exp(\lambda_3 t) + \frac{b_4\exp\left(\frac{\lambda_3}{2}t\right)}{\cosh\left[\frac{\lambda_3}{2}(b_0-t)\right]}x + \left\{\frac{\lambda_3}{8}\tanh\left[\frac{\lambda_3}{2}(b_0-t)\right] - \frac{\lambda_3}{8}\right\}x^2. \tag{54}$$

**Remark** Using the symbolic computation Maple and separately substituting solutions (38), (41), (42), (45), (46), (53) and (54) into Eq.(3), we find that these solutions entirely satisfy Eq.(3). It implies that the results obtained by using the additional generating technique to investigate Eq.(3) are valid.

## 4. Conclusion

In this paper, the additional generating condition method based on the analysis of a fixed nonlinear partial differential equation together with an additional condition in the form of a linear ordinary differential equation is employed to study a reaction diffusion Eq.(3) which describes a real process in physics and in chemistry. The exact solutions are obtained under several circumstances. The constructive method may be applied to finding the exact solutions for other nonlinear diffusion equations.

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