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The (ω/g)-expansion method and its application to Vakhnenko equation

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This paper presents a new function expansion method for finding travelling wave solutions of a nonlinear evolution equation and calls it the (ω/g)-expansion method, which can be thought of as the generalization of (G'/G)-expansion given by Wang *et al* recently. As an application of this new method, we study the well-known Vakhnenko equation which describes the propagation of high-frequency waves in a relaxing medium. With two new expansions, general types of soliton solutions and periodic solutions for Vakhnenko equation are obtained.

Keywords: (ω/g)-expansion method, Vakhnenko equation, travelling wave solutions
PACC: 0340K, 0290

1. Introduction

The investigation of the exact solutions for nonlinear evolution equations plays an important role in the study of soliton theory. In the past decade, a number of powerful methods have been proposed, such as the tanh function expansion method,[1,2] Jacobi elliptic function method,[3,4] Exp-function method,[5] the hyperbolic tangent function expansion method,[2,6,7] the F-expansion.[8−10] A great number of nonlinear equations can be solved analytically by the above methods.[11−18] However, although many efforts have been devoted to find various methods to solve nonlinear equation, there is not a unified method. Recently (G'/G)-expansion method[19] has been proposed which can be applied to many nonlinear equations and result in a few new kinds of solution. Then Zhang *et al*[20] generalized this method to solve nonlinear equations with variable coefficients. Motivated by this method, we introduce the (ω/g)-expansion which actually is a family of expansion methods. When the ω and g are taken as special choices, some familiar expansion methods can be obtained, such as tanh-expansion, (G'/G)-expansion. Based on these interesting results, we further give two new forms of expansion. In order to illustrate well the effectiveness of our method, it is applied to Vakhnenko equation which is an important equation describing the propagation of high-frequency waves in a relaxing medium. We will show that several new types of solution can be derived by using our method.

This paper is organized as follows. The next section is devoted to the description of our method. In Section 3, we apply it to Vakhnenko equation and discuss briefly its solutions. At Last, a brief summary is given in Section 4.

2. Description of the (ω/g)-expansion method

A general nonlinear wave equation can be written as

$$P(u, u_t, u_x, u_{tt}, u_{xt}, u_{xx}, ...) = 0. \tag{1}$$

We seek its travelling wave solution $u(\varepsilon)$ by letting

$$\varepsilon = x - Vt, \tag{2}$$

where V is a parameter to be determined later. Now we briefly illustrate the (ω/g)-expansion method.

Step 1 Uniting the independent variables x and t into one variable ε as usual, then from Eq.(1) we obtain

$$P(u, -Vu', u', V^2u'', -Vu'', u'', ...) = 0. \tag{3}$$

Step 2 Suppose the solution of Eq.(3) can be expressed by a polynomial in (ω/g), and ω, g satisfy the following relation:

$$\left(\frac{\omega}{g}\right)' = a + b\left(\frac{\omega}{g}\right) + c\left(\frac{\omega}{g}\right)^2,$$

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namely,

$$\omega' g - \omega g' = ag^2 + b\omega g + c\omega^2, \tag{4}$$

where $a,\ b,\ c$ are arbitrary constants. Let us examine Eq.(4) carefully. If we make the choice $\omega = g',\ a = -\mu,\ b = -\lambda,\ c = -1$, then $u(\varepsilon)$ can be expressed as

$$u(\varepsilon) = \sum_{m=0} a_m \left(\frac{g'}{g}\right)^m, \tag{5}$$

where g satisfies relation $g'' + \lambda g' + \mu g = 0$. This is just the (G'/G)-expansion method that Wang *et al*[19] have proposed recently. Furthermore, if we put $\omega = \tanh\varepsilon,\ g = 1,\ a = 1,\ b = 0,\ c = -1$, then $u(\varepsilon)$ becomes

$$u(\varepsilon) = \sum_{m=0} a_m (\tanh\varepsilon)^m, \tag{6}$$

which is the tanh-function expansion method.

In the present paper, we propose another two new kinds of expansion from which new solutions of the nonlinear wave equation can be obtained. For the first one, let $\omega = g'/g$, $b = 0$, thus

$$u(\varepsilon) = \sum_{m=0} a_m \left(\frac{g'}{g^2}\right)^m, \tag{7}$$

where g satisfies

$$g'' g^2 - 2gg'^2 = ag^4 + cg'^2. \tag{8}$$

For another, let $\omega = gg'$, then

$$u(\varepsilon) = \sum_{m=0} a_m (g')^m. \tag{9}$$

Now the differential equation about g becomes

$$g'' = a + bg' + cg'^2. \tag{10}$$

Step 3 Substituting Eq.(7) or Eq.(9) into Eq.(3), making use of Eq.(8) or Eq.(10), and setting the coefficients of all powers of $(\omega/g)^m$ to zeros, we will get a system of algebraic equations, from which V and a_m can be found explicitly.

Step 4 Substituting the values a_m obtained in Step 3 into Eq.(7) or Eq.(9), we may get all possible solutions.

3. New solutions of Vakhnenko equation

Vakhnenko equation,[21−23] is a nonlinear equation with loop soliton solutions describing the propagation of high-frequency waves in a relaxing medium, which can be written as

$$u_{tx} + u_x^2 + uu_{xx} + u = 0. \tag{11}$$

Following Vakhnenko *et al*,[21] we introduce new independent variables $X,\ T$, defined by

$$x = T + W(X,T) + x_0, \quad t = X, \tag{12}$$

where $u(x,t) = W_X(X,T),\ x_0$ is an arbitrary constant.

From Eq.(12) we have

$$\frac{\partial}{\partial X} = \frac{\partial}{\partial t} + u\frac{\partial}{\partial x}, \quad \frac{\partial}{\partial T} = (1 + W_T)\frac{\partial}{\partial x}. \tag{13}$$

Thus equation (11) can be rewritten as

$$W_{XXT} + W_X W_T + W_X = 0. \tag{14}$$

Now we look for the travelling solution of W by putting

$$W = \phi(\varepsilon), \quad \varepsilon = X - VT. \tag{15}$$

Substituting Eq.(15) into Eq.(14), we have

$$-V\phi''' - V(\phi')^2 + \phi' = 0. \tag{16}$$

Considering the homogeneous balance ϕ''' and $(\phi')^2$ in Eq.(16) and noticing Eq.(4), we require that the highest order of the polynomial in (ω/g) is 1.

3.1. (g'/g^2)-expansion

Suppose

$$\phi(\varepsilon) = a_0 + a_1\left(\frac{g'}{g^2}\right). \tag{17}$$

Noting $\left(\frac{g'}{g^2}\right)' = a + c\left(\frac{g'}{g^2}\right)^2$, we have the concrete form of ϕ', ϕ'', ϕ''' and $(\phi')^2$, then substitute them into Eq.(16), collect all terms with same order of $\left(\frac{g'}{g^2}\right)$, and set the coefficients of all powers of $\left(\frac{\omega}{g}\right)^m$ to zeros. We will get a system of algebraic equations for $a_0,\ a_1$ and V as follows:

$$\begin{aligned} &-2Va_1a^2c - Va_1^2a^2 + a_1a = 0,\\ &-8Va_1ac^2 - 2Va_1^2ac + a_1c = 0,\\ &-6Va_1c^3 - Va_1^2c^2 = 0. \end{aligned} \tag{18}$$

After some algebraic calculations we have

$$a_1 = -6c, \quad V = -\frac{1}{4ac}. \tag{19}$$

Substituting Eq.(19) and the general solution of Eq.(8) (see Eqs.(A5), (A7) and Eq.(A9) in the Appendix) into Eq.(17), we therefore have two types of solutions as follows.

For $ac > 0$,

$$\phi_1(\varepsilon) = a_0 - 6\sqrt{ac}\frac{C_1\cos(\sqrt{ac}\varepsilon) + C_2\sin(\sqrt{ac}\varepsilon)}{C_1\sin(\sqrt{ac}\varepsilon) - C_2\cos(\sqrt{ac}\varepsilon)}, \quad (20)$$

where $\varepsilon = X + \dfrac{1}{4ac}T$ and C_1, C_2 are arbitrary constants.

Thus the solution of Vakhnenko equation is

$$u_1(x,t) = \frac{6ac\left(C_1^2 + C_2^2\right)}{\left(C_1\sin(\sqrt{ac}\varepsilon) - C_2\cos(\sqrt{ac}\varepsilon)\right)^2},$$

$$x + 4act = 4ac\varepsilon + x_0 + a_0 - 6\sqrt{ac} \times \frac{C_1\cos(\sqrt{ac}\varepsilon) + C_2\sin(\sqrt{ac}\varepsilon)}{C_1\sin(\sqrt{ac}\varepsilon) - C_2\cos(\sqrt{ac}\varepsilon)}, \quad (21)$$

where u_1 here had not been given in Refs.[21–23]. It is a general form of periodic solution.

For $ac < 0$,

$$\phi_2(\varepsilon) = a_0 + 6\sqrt{|ac|}\left(\frac{C_1 e^{2\varepsilon\sqrt{|ac|}} + C_2}{C_1 e^{2\varepsilon\sqrt{|ac|}} - C_2}\right), \quad (22)$$

where $\varepsilon = X + \dfrac{1}{4ac}T$ and C_1, C_2 are arbitrary constants. In particular, if C_1 and C_2 take the special value, for example, $C_2/C_1 = -e^{2\varepsilon_0}$, then

$$\begin{aligned}\phi_2(\varepsilon) =& a_0 + 6\sqrt{|ac|}\tanh\left(\sqrt{|ac|}\varepsilon + \varepsilon_0\right)\\ =& a_0 + \frac{3}{\sqrt{V}}\tanh\left(\frac{\varepsilon}{2\sqrt{V}} + \varepsilon_0\right),\end{aligned}$$

which has the same form as that in Ref.[21].

In general, the soliton solution is

$$u_2(x,t) = -6ac + 6ac\left(\frac{C_1 e^{2\varepsilon\sqrt{|ac|}} + C_2}{C_1 e^{2\varepsilon\sqrt{|ac|}} - C_2}\right)^2,$$

$$x + 4act = 4ac\varepsilon + x_0 + a_0 + 6\sqrt{|ac|} \times \left(\frac{C_1 e^{2\varepsilon\sqrt{|ac|}} + C_2}{C_1 e^{2\varepsilon\sqrt{|ac|}} - C_2}\right). \quad (23)$$

3.2. g'-expansion

Let

$$\phi(\varepsilon) = b_0 + b_1 g'. \quad (24)$$

Similarly, noting $(g')' = a + bg' + cg'^2$, one substitutes the new form of ϕ', ϕ'', ϕ''' and $(\phi')^2$ into Eq.(16), and gets

$$\begin{aligned}&-V\left(ab^2b_1 + 2a^2b_1c\right) - Vb_1^2a^2 + b_1a = 0,\\ &-V\left(b_1b^3 + 8b_1abc\right) - 2Vb_1^2ab + b_1b = 0,\\ &-V\left(7b_1b^2c + 8b_1ac^2\right) - V\left(b_1^2b^2 + 2b_1^2ac\right) + b_1c = 0,\\ &-12Vb_1bc^2 - 2Vb_1^2bc = 0,\\ &-6Vb_1c^3 - Vb_1^2c^2 = 0 \end{aligned} \quad (25)$$

Then we have

$$b_1 = -6c, \quad V = -\frac{1}{4ac - b^2}. \quad (26)$$

Substituting Eq.(26) and the general solution of Eq.(10) (see Eqs.(A18) – (A20) in the Appendix) into Eq.(24), we have two types of travelling wave solutions of the Vakhnenko equation as follows:

Case 1 When $\Delta \equiv 4ac - b^2 < 0$,

$$\begin{aligned}\phi_3(\varepsilon) =& b_0 - 3\left[\sqrt{-\Delta}\tanh\left(-\frac{\sqrt{-\Delta}}{2}\varepsilon\right) - b\right]\\ =& (b_0 + 3b) + \frac{3}{\sqrt{V}}\tanh\left(\frac{\varepsilon}{2\sqrt{V}}\right),\end{aligned} \quad (27)$$

Then, we have

$$u_3(x,t) = -\frac{3\Delta}{2}\mathrm{sech}^2\left(-\frac{\sqrt{-\Delta}}{2}\varepsilon\right),$$

$$x + \Delta t = \Delta\varepsilon + b_0 - 3\left[\sqrt{-\Delta}\tanh\left(-\frac{\sqrt{-\Delta}}{2}\varepsilon\right) - b\right] + x_0. \quad (28)$$

which is the one-loop soliton solution.[21–24]

Case 2 When $\Delta > 0$,

$$\phi_4(\varepsilon) = b_0 - 3\left(\sqrt{\Delta}\tan\left(\frac{\sqrt{\Delta}}{2}\varepsilon\right) - b\right), \quad (29)$$

$$u_4(x,t) = -\frac{3\Delta}{2}\sec^2\left(\frac{\sqrt{\Delta}}{2}\varepsilon\right),$$

$$x + \Delta t = \Delta\varepsilon + b_0 - 3\left(\sqrt{\Delta}\tan\left(\frac{\sqrt{\Delta}}{2}\varepsilon\right) - b\right) + x_0. \quad (30)$$

where u_4 had not been given in Refs.[21–24]. Obviously, u_3 and u_4 are the special cases of u_1 and u_4. Compared with g'-expansion, the (g'/g^2)-expansion is a more powerful tool to explore the solutions for nonlinear evolution equations.

4. Summary

In this work, the (ω/g)-expansion method has been proposed, which is a generalization of the (G'/G)-expansion method. With two new expansions, several types of travelling solutions of the Vakhnenko equation are obtained, such as periodic solution and loop soliton solution. As far as we know, some of these solutions are found for the first time. It is also proved that $\left(g'/g^2\right)$-expansion is more effective than the g'-expansion because the former can give a general form of periodic solutions and soliton solutions while the latter can not. Though this new method only represents the unification of several expansion methods, we believe that it may contribute to finding a method which can solve most nonlinear equations and obtain many new types of solutions.

Appendix

In this section, general solutions of Eqs.(8) and (10) will be given.

(i) $g''g^2 - 2gg'^2 = ag^4 + cg'^2$

Let $g = 1/y$, then the above equation becomes

$$y'' + cy'^2 + a = 0, \tag{A1}$$

which has a general solutions as follows,

when $ac > 0$,

$$y(\varepsilon) = \frac{1}{2c}\ln\left[\frac{c}{a}\Big(C_1\sin\left(\sqrt{ac}\varepsilon\right) - C_2\cos\left(\sqrt{ac}\varepsilon\right)\Big)^2\right], \tag{A2}$$

when $ac < 0$,

$$y(\varepsilon) = -\frac{1}{2c}\left(2\sqrt{|ac|}\varepsilon - \ln\left[\frac{c}{4a}\left(C_1\mathrm{e}^{2\varepsilon\sqrt{|ac|}} - C_2\right)^2\right]\right), \tag{A3}$$

when $a = 0$, $c \neq 0$,

$$y(\varepsilon) = \frac{1}{c}\ln\left(C_1\varepsilon c + C_2 c\right). \tag{A4}$$

Thus, we have

when $ac > 0$,

$$g(\varepsilon) = \frac{2c}{\ln\left[\frac{c}{a}\left(C_1\sin\left(\sqrt{ac}\varepsilon\right) - C_2\cos\left(\sqrt{ac}\varepsilon\right)\right)^2\right]}, \tag{A5}$$

$$\frac{g'}{g^2} = \sqrt{\frac{a}{c}}\frac{C_1\cos\left(\sqrt{ac}\varepsilon\right) + C_2\sin\left(\sqrt{ac}\varepsilon\right)}{C_1\sin\left(\sqrt{ac}\varepsilon\right) - C_2\cos\left(\sqrt{ac}\varepsilon\right)}, \tag{A6}$$

when $ac < 0$,

$$g(\varepsilon) = -\frac{2c}{2\sqrt{|ac|}\varepsilon - \ln\left[\frac{c}{4a}\left(C_1\mathrm{e}^{2\varepsilon\sqrt{|ac|}} - C_2\right)^2\right]}, \tag{A7}$$

$$\frac{g'}{g^2} = \frac{1}{2c}\left(2\sqrt{|ac|} - \frac{4\sqrt{|ac|}C_1\mathrm{e}^{2\varepsilon\sqrt{|ac|}}}{C_1\mathrm{e}^{2\varepsilon\sqrt{|ac|}} - C_2}\right), \tag{A8}$$

when $a = 0$, $c \neq 0$,

$$g(\varepsilon) = \frac{c}{\ln\left(C_1\varepsilon c + C_2 c\right)}, \tag{A9}$$

$$\frac{g'}{g^2} = -\frac{C_1}{C_1\varepsilon c + C_2 c}. \tag{A10}$$

(ii) $g'' = a + bg' + cg'^2$

By putting $y = g'$, $\Delta = 4ac - b^2$, the above equation becomes

$$y' = a + by + cy^2,$$

or

$$\frac{\mathrm{d}y}{a + by + cy^2} = \mathrm{d}\varepsilon. \tag{A11}$$

Integrating both sides of (A11), one has

$$\varepsilon = \int\frac{\mathrm{d}y}{a + by + cy^2} = \frac{1}{\sqrt{-\Delta}}\ln\frac{b + 2cy - \sqrt{-\Delta}}{b + 2cy + \sqrt{-\Delta}} = \frac{-2}{\sqrt{-\Delta}}\mathrm{Arcth}\frac{b + 2cy}{\sqrt{-\Delta}} \quad \text{for} \quad \Delta < 0, \tag{A12}$$

$$= \frac{-2}{b + 2cy} \quad \text{for} \quad \Delta = 0, \tag{A13}$$

$$= \frac{2}{\sqrt{\Delta}}\arctan\frac{b + 2cy}{\sqrt{\Delta}} \quad \text{for} \quad \Delta > 0, \tag{A14}$$

where we have set the integration constant to zero. Therefore,

$$y = \frac{1}{2c}\left[\sqrt{-\Delta}\tanh\left(-\frac{\sqrt{-\Delta}}{2}\varepsilon\right) - b\right], \quad \text{for} \quad \Delta < 0, \tag{A15}$$

$$y = -\frac{1}{c\varepsilon} - \frac{b}{2c}, \quad \text{for} \quad \Delta = 0, \tag{A16}$$

$$y = \frac{1}{2c}\left[\sqrt{\Delta}\tan\left(\frac{\sqrt{\Delta}}{2}\varepsilon\right) - b\right], \quad \text{for} \quad \Delta > 0, \tag{A17}$$

$$g = \frac{1}{2c}\left[\ln\left(\tanh^2\left(\frac{\sqrt{-\Delta}}{2}\varepsilon\right) - 1\right) - b\varepsilon\right], \quad \text{for} \quad \Delta < 0, \tag{A18}$$

$$g = -\frac{1}{c}\left[\ln\left(\varepsilon\right) + \frac{b}{2}\varepsilon\right], \quad \text{for} \quad \Delta = 0, \tag{A19}$$

$$g = \frac{1}{2c}\left[\ln\left(1 + \tan^2\left(\frac{\sqrt{\Delta}}{2}\varepsilon\right)\right) - b\varepsilon\right], \quad \text{for} \quad \Delta > 0. \tag{A20}$$

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