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Conformal invariance and conserved quantities of non-conservative Lagrange systems by point transformations*

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This paper studies the conformal invariance by infinitesimal point transformations of non-conservative Lagrange systems. It gives the necessary and sufficient conditions of conformal invariance by the action of infinitesimal point transformations being Lie symmetric simultaneously. Then the Noether conserved quantities of conformal invariance are obtained. Finally an illustrative example is given to verify the results.

Keywords: non-conservative Lagrange systems, point transformations, conformal invariance, conserved quantities

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1. Introduction

The research on symmetries and conserved quantities of a mechanical system possesses important theoretical and practical significance.[1−26] Conserved quantities, also known as first integrals, can simplify the solutions of dynamical systems. Sophus Lie introduced the notion of continuous group, now known as Lie group. Lie showed that the order of an ordinary differential equation could be reduced by one, if it is invariant under a one-parameter Lie group of point transformations.[2−7] The symmetry method is a modern method for finding conserved quantities of dynamical systems. In 1979, Lutzky applied the Lie theory to mechanical systems, and obtained Lie symmetries and conserved quantities of dynamical systems.[27] The research on Lie symmetries has gained many important results.[7,10,28] In 1997, Galiullin studied the conformal invariance of Birkhoff equations and their relation with the Lie symmetry.[29] Recently Robert researched into conformal Hamiltonian systems by geometric methods, in which the author discussed the geometric structure of conformal invariance and pointed out the difference of conformal invariance from arbitrary symmetry of dynamical systems.[30] In Ref.[31] conformal invariance of Lagrange systems and Hojman conserved quantities was discussed. In this paper, the conformal invariance of non-conservative Lagrange systems by point transformations is first defined, then the conformal invariance and their relations with the Lie symmetry of non-conservative Lagrange systems is discussed. At the same time the Noether quantities of the non-conservative Lagrange systems by Noether equalities are obtained. Finally an illustrative example is given to verify the results.

2. The conformal invariance of non-conservative Lagrange systems

The differential equations of the non-conservative Lagrange systems with n degrees of freedom have the following forms:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} = Q_s \quad (s = 1, \ldots, n), \tag{1}$$

where $L = L(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is Lagrangian function, $Q_s = Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ are generalized non-potential forces.

From Eq.(1) we can get

$$F_s \equiv A_{sk}(t, q, \ddot{q}_k + B_s(t, q, \dot{q}) - Q_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}}) = 0, \tag{2}$$

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where

$$A_{sk}=\frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_k},$$
$$B_s=\frac{\partial^2 L}{\partial\dot{q}_s\partial q_k}\dot{q}_k+\frac{\partial^2 L}{\partial\dot{q}_s\partial t}-\frac{\partial L}{\partial q_s}.$$

Suppose systems (1) are non-singular, namely

$$\det\left(\frac{\partial^2 L}{\partial\dot{q}_s\partial\dot{q}_k}\right)\neq 0. \tag{3}$$

From Eq.(2), we can get

$$\ddot{q}_s=\alpha_s\left(t,\boldsymbol{q},\dot{\boldsymbol{q}}\right). \tag{4}$$

In order to discuss the conformal invariance of Eq.(2), we adopt the infinitesimal transformations of t and q_s as

$$t^*=t+\Delta t,$$
$$q_s^*(t^*)=q_s(t)+\Delta q_s, \tag{5}$$

or their expended forms

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol{q}),$$
$$q_s^*(t^*)=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q}), \tag{6}$$

where ε is the infinitesimal parameter, ξ_0, ξ are the generators of the one-parameter infinitesimal transformations group.

Choosing the infinitesimal generators vectors

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s}, \tag{7}$$

we find that its first prolonged vectors are

$$X^{(1)}=X^{(0)}+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial\dot{q}_s}, \tag{8}$$

and its second prolonged vectors are

$$X^{(2)}=X^{(1)}+(\ddot{\xi}_s-2\ddot{q}_s\dot{\xi}_0-\dot{q}_s\ddot{\xi}_0)\frac{\partial}{\partial\ddot{q}_s}. \tag{9}$$

Definition 1 By the action of infinitesimal point transformations of generators $\xi_0\,(t,q)$ and $\xi_s\,(t,\boldsymbol{q})$, if F_s satisfies the following determining expressions

$$X^{(2)}(F_s)=\ell_s^l(F_l), \tag{10}$$

then the infinitesimal transformations are conformal invariant for the corresponding second order differential equations. And we always call the Eq.(10) as determining equations of conformal invariance.

To obtain the general expressions of conformal invariance, we calculate the difference

$$X^{(2)}(F_s)-X^{(2)}(F_s)\mid_{F_s=0}. \tag{11}$$

Since

$$\dot{\xi}_s=\frac{\partial\xi_s}{\partial t}+\frac{\partial\xi_s}{\partial q_k}\dot{q}_k,\quad(s=0,1,\ldots,n), \tag{12}$$

$$\ddot{\xi}_s=\frac{\partial^2\xi_s}{\partial t^2}+2\frac{\partial^2\xi_s}{\partial q_k\partial t}\dot{q}_k+\frac{\partial^2\xi_s}{\partial q_k\partial q_j}\dot{q}_k\dot{q}_j+\frac{\partial\xi_s}{\partial q_k}\ddot{q}_k,$$
$$(s=0,1,\cdots,n;\quad k,j=1,\cdots,n), \tag{13}$$

therefore,

$$\begin{aligned}
X^{(2)}\left(F_s\right)&=A_{sk}\left(\ddot{\xi}_k-2\ddot{q}_k\dot{\xi}_0-\dot{q}_k\ddot{\xi}_0\right)+X^{(1)}\left(A_{sk}\right)\ddot{q}_k+X^{(1)}\left(B_s-Q_s\right)\\
&=A_{sk}\left(\ddot{\xi}_k-2\ddot{q}_k\dot{\xi}_0-\dot{q}_k\ddot{\xi}_0\right)+X^{(0)}\left(A_{sk}\right)\ddot{q}_k+X^{(0)}\left(B_s-Q_s\right)+\left(\dot{\xi}_k-\dot{q}_k\dot{\xi}_0\right)\frac{\partial\left(B_s-Q_s\right)}{\partial\dot{q}_k}\\
&=A_{sk}\left(\frac{\partial^2\xi_k}{\partial t^2}+2\frac{\partial^2\xi_k}{\partial q_r\partial t}\dot{q}_r+\frac{\partial^2\xi_k}{\partial q_r\partial q_j}\dot{q}_r\dot{q}_j+\frac{\partial\xi_k}{\partial q_r}\ddot{q}_r\right)-2A_{sk}\ddot{q}_k\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r\right)\\
&\quad+A_{sk}\dot{q}_k\left(\frac{\partial^2\xi_0}{\partial t^2}+2\frac{\partial^2\xi_0}{\partial q_r\partial t}\dot{q}_r+\frac{\partial^2\xi_0}{\partial q_r\partial q_j}\dot{q}_r\dot{q}_j+\frac{\partial\xi_0}{\partial q_r}\ddot{q}_r\right)+X^{(0)}\left(A_{sk}\right)\ddot{q}_k+X^{(0)}\left(B_s-Q_s\right)\\
&\quad+\left[\frac{\partial\xi_k}{\partial t}+\frac{\partial\xi_k}{\partial q_r}\dot{q}_r-\dot{q}_r\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r\right)\right]\frac{\partial\left(B_s-Q_s\right)}{\partial\dot{q}_k}\\
&=A_{sk}\left(\frac{\partial\xi_k}{\partial q_r}-\dot{q}_k\frac{\partial\xi_0}{\partial q_r}\right)\ddot{q}_r-2A_{sk}\ddot{q}_k\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r\right)+X^{(0)}\left(A_{sk}\right)\ddot{q}_k+C(t,q,\dot{q}),
\end{aligned} \tag{14}$$

where $C(t,q,\dot{q})$ is the summation of items in which $\ddot{q}_s$ are not contained. Equivalently, there exists

$$X^{(2)}(F_s)\mid_{F_s=0}=A_{sk}\left(\frac{\partial\xi_k}{\partial q_r}-\dot{q}_k\frac{\partial\xi_0}{\partial q_r}\right)\alpha_r-2A_{sk}\alpha_k\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r\right)+X^{(0)}(A_{sk})\alpha_k+C(t,q,\dot{q}), \tag{15}$$

where $\alpha_s = -A^{sk}B_k$. Let Eq.(14) subtract Eq.(15); we obtain the following expressions:

$$X^{(2)}(F_s) - X^{(2)}(F_s)|_{F_s=0} = A_{sk}\left(\frac{\partial \xi_k}{\partial q_r} - \dot{q}_k\frac{\partial \xi_0}{\partial q_r}\right)(\ddot{q}_r - \alpha_r) - 2A_{sk}(\ddot{q}_k - \alpha_k)\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right) + X^{(0)}(A_{sk})(\ddot{q}_k - \alpha_k) \quad (s,k,r = 1,\cdots,n). \tag{16}$$

On account of

$$\ddot{q}_k - \alpha_k = \ddot{q}_k + A^{kl}B_l = A^{kl}(A_{lm}\ddot{q}_m + B_l) = A^{kl}F_l, \tag{17}$$

we have

$$\begin{aligned} X^{(2)}(F_s) - X^{(2)}(F_s)|_{F_s=0} &= A_{sk}\left(\frac{\partial \xi_k}{\partial q_r} - \dot{q}_k\frac{\partial \xi_0}{\partial q_r}\right)A^{rl}F_l - 2A_{sk}A^{kl}F_l\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right) + X^{(0)}(A_{sk})A^{kl}F_l \\ &= \left[A_{sk}\left(\frac{\partial \xi_k}{\partial q_r} - \dot{q}_k\frac{\partial \xi_0}{\partial q_r}\right)A^{rl} - 2\delta_s^l\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right) + X^{(0)}(A_{sk})A^{kl}\right]F_l \\ &= \beta_s^l F_l \qquad (s,k,r,l = 1,\cdots,n). \end{aligned} \tag{18}$$

If Eq.(2) is Lie symmetric by the infinitesimal point transformations of the one-parameter group, then

$$X^{(2)}(F_s)\,|_{F_s=0} = 0. \tag{19}$$

From Eqs.(10), (18) and (19), we can obtain

$$(\ell_s^l - \beta_s^l)(F_l) = X^{(2)}(F_s)|_{F_s=0} \quad (s,l = 1,\ldots,n). \tag{20}$$

So the necessary and sufficient conditions of the conformal invariance by the action of the infinitesimal point transformations of the one-parameter group being Lie symmetric simultaneously are

$$\ell_s^l = \beta_s^l. \tag{21}$$

Thus, there is the following criterion.

Proposition 1 For the non-conservative Lagrange systems (1), the necessary and sufficient conditions of its conformal invariance being Lie symmetric simultaneously by the action of the infinitesimal point transformations of the one-parameter group are the infinitesimal generators fulfilling the following expressions:

$$\begin{aligned} \ell_s^l = {} & A_{sk}\left(\frac{\partial \xi_k}{\partial q_r} - \dot{q}_k\frac{\partial \xi_0}{\partial q_r}\right)A^{rl} \\ & - 2\delta_s^l\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right) \\ & + X^{(0)}(A_{sk})A^{kl} \\ & (s,k,r,l = 1,\ldots,n). \end{aligned} \tag{22}$$

From Proposition 1 we have the following deduction.

Corollary Suppose the differential equations (2) can be transformed into the standard forms

$$F_s \equiv \ddot{q}_s - \alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = 0, \quad (s = 1,\ldots,n), \tag{23}$$

by the action of the infinitesimal point transformations of one-parameter group, the necessary and sufficient conditions of conformal invariance being Lie symmetric simultaneously of Eq.(23) by the action of the infinitesimal point transformations of one-parameter group are

$$\ell_s^l = \frac{\partial \xi_s}{\partial q_l} - 2\dot{\xi}_0 - \dot{q}_s\frac{\partial \xi_0}{\partial q_l}. \tag{24}$$

In fact, Eq.(23) corresponds to

$$\begin{aligned} & A_{sk} = \delta_{sk}, \\ & B_s\,(t,\boldsymbol{q},\dot{\boldsymbol{q}}) - Q_s\,(t,\boldsymbol{q},\dot{\boldsymbol{q}}) = -\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}) \end{aligned}$$

in Eq.(2), so we can easily obtain Eq.(24) from Eq.(22).

3. The conserved quantities of conformal invariance

The conformal invariance can also lead to conserved quantities under some conditions; results are as follows.

Proposition 2 For the conformal invariance of non-conservative Lagrange systems, if the infinitesimal generators ξ_0, ξ_s meet the conformal facts Eq.(22)

and there exists a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ which satisfies the Noether equalities

$$L\dot{\xi}_0 + X^{(1)}(L) + Q_s\left(\xi_s - \dot{q}_s\xi_0\right) + \dot{G} = 0, \qquad (25)$$

then the conformal invariance of non-conservative Lagrange systems may induce Noether quantities:

$$I = L\xi_0 + \frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + G = \text{const.} \qquad (26)$$

Proof.

$$\begin{aligned}\frac{\mathrm{d}I}{\mathrm{d}t} &= \dot{L}\xi_0 + L\dot{\xi}_0 + \frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}(\xi_s - \dot{q}_s\xi_0) + \frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \ddot{q}_s\xi_0 - \dot{q}_s\dot{\xi}_0)\\ &\quad - \frac{\partial L}{\partial t}\xi_0 - \frac{\partial L}{\partial q_s}\xi_s - \frac{\partial L}{\partial \dot{q}_s}(\dot{\xi}_s - \dot{q}_s\dot{\xi}_0) - Q_s\left(\xi_s - \dot{q}_s\xi_0\right) - L\dot{\xi}_0\\ &= (\xi_s - \dot{q}_s\xi_0)\left(\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s} - \frac{\partial L}{\partial q_s} - Q_s\right) = 0.\end{aligned}$$

4. Illustrative example

The non-conservative Lagrange systems with one degree of freedom

$$L = \frac{1}{2}t^2\dot{q}^2, \quad Q = -t^2q^5, \qquad (27)$$

whose differential equation is

$$F = \ddot{q} + \frac{2}{t}\dot{q} + q^5 = 0.$$

When the infinitesimal generators are

$$\xi_0 = -2t, \qquad \xi = q, \qquad (28)$$

thus

$$\begin{aligned}X^{(2)} &= \xi_0\frac{\partial}{\partial t} + \xi\frac{\partial}{\partial q} + \left(\dot{\xi} - \dot{q}\dot{\xi}_0\right)\frac{\partial}{\partial \dot{q}}\\ &\quad + \left(\ddot{\xi} - 2\ddot{q}\dot{\xi}_0 - \dot{q}\ddot{\xi}_0\right)\frac{\partial}{\partial \ddot{q}}, \qquad (29)\end{aligned}$$

and

$$\begin{aligned}X^{(2)}F &= \xi\frac{\partial F}{\partial q} + \xi_0\frac{\partial F}{\partial t} + \left(\dot{\xi} - \dot{q}\dot{\xi}_0\right)\frac{\partial F}{\partial \dot{q}}\\ &\quad + \left(\ddot{\xi} - 2\ddot{q}\dot{\xi}_0 - \dot{q}\ddot{\xi}_0\right)\frac{\partial F}{\partial \ddot{q}}\\ &= 5q^5 + \frac{4}{t}\dot{q} + \frac{6}{t}\dot{q} + 5\ddot{q}\\ &= 5F.\end{aligned}$$

We obtain

$$\ell = 5. \qquad (30)$$

So the determining equation of the non-conservative systems is

$$X^{(2)}F = 5F. \qquad (31)$$

Utilizing Proposition 1, then we can also obtain

$$\begin{aligned}\ell_s^l &= A_{sk}\left(\frac{\partial \xi_k}{\partial q_r} - \dot{q}_k\frac{\partial \xi_0}{\partial q_r}\right)A^{rl}\\ &\quad - 2\delta_s^l\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r\right) + X^{(0)}(A_{sk})A^{kl}\\ &= \frac{\partial \xi}{\partial q} - 2\delta\frac{\partial \xi_0}{\partial t}\\ &= 1 + 4 = 5.\end{aligned}$$

So the result accords with Eq.(30). Here the systems are both conformal invariant and Lie symmetric. From Proposition 2, when

$$G = \frac{1}{3}t^3q^6,$$

we can obtain the Noether conserved quantity

$$I = t^3\dot{q}^2 + t^2q\dot{q} + \frac{1}{3}t^3q^6 = \text{const.} \qquad (32)$$

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