Vol 18 No 2, February 2009
1674-1056/2009/18(02)/0391-04

Chinese Physics B

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Direct perturbation method for perturbed complex Burgers equation*

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(Received 28 August 2008; revised manuscript received 16 October 2008)

So far, Lou's direct perturbation method has been applied successfully to solve the nonlinear Schrödinger equation(NLSE) hierarchy, such as the NLSE, the coupled NLSE, the critical NLSE, and the derivative NLSE. But to our knowledge, this method for other types of perturbed nonlinear evolution equations has still been lacking. In this paper, Lou's direct perturbation method is applied to the study of perturbed complex Burgers equation. By this method, we calculate not only the zero-order adiabatic solution, but also the first order modification.

Keywords: direct perturbation method, perturbed complex Burgers equation, asymptotical solutions

PACC: 0260, 0340K, 4735

1. Introduction

It has now been realized that the study of many different types of waves in numerous media can often be based on a few universal nonlinear equations, such as the KdV, the nonlinear Schrödinger equation, the sine-Gordon equation, the Boussinesq equation, the Burgers equation.

However, the standard nonlinear evolution equations are ideal models. In more real physical applications, correction terms, such as the effects of the external drive, the loss, the dissipative and dispersion, etc., are often present in the nonlinear equations. Generally, it is rather difficult to obtain exact analytical solutions of these equations. But fortunately, the correction terms are small enough in many important practical cases: i.e., they can be treated as the perturbations, thus the perturbed nonlinear equations can be taken into account by effective perturbation methods.

So far, many powerful perturbation methods have been proposed. One of the famous methods is the inverse scattering perturbation method.[1] This approach enables one to solve the most sophisticated dynamical problems. However, this method requires the unperturbed equations to be exactly solvable by the inverse scattering transformation (IST) approach, which is inconvenient to those who are not familiar with the IST. Another effective method is the direct perturbation approach.[2,3] In this method, the squared Jost solutions are employed as the basis for perturbation expansions. But actually, direct perturbation method can not completely get rid of the dependence on the IST because the squared Jost solutions appear as the particular solution in the process of the IST.

Later, Lou[4] proposed a new direct perturbation method. In the framework of this approach, no knowledge of IST is required and the algorithmic procedure is more simple and convenient to follow. Especially, different types of perturbation solutions, involving the periodic solutions, one-soliton solutions, multi-soliton solutions, etc., can be obtained at the same time. In addition, not only the zero-order adiabatic solutions but also the first-order modifications can be calculated by this method.

So far, Lou's direct perturbation method has been applied successfully to solve the NLSE,[5] the coupled NLSE,[6] the critical NLSE,[7] the derivative NLSE,[8] etc.. But to our knowledge, this method used for other types of perturbed nonlinear evolution equations has still been lacking.

*Project supported by the National Natural Science Foundation of China (Grant Nos 10575087 and 10875106).

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2. The direct perturbation method for perturbed complex Burgers equation

In this section, we are interested in applying Lou's direct perturbation method to the perturbed complex Burgers equation, which is written as

$$u_t + 2uu_x + \mathrm{i}u_{xx} = -\epsilon u, \tag{1}$$

where ϵ is a small positive constant for measuring the weakness of the linear damping ($0 < \epsilon \ll 1$). When $\epsilon = 0$, Eq.(1) reduces to the complex Burgers equation

$$u_t + 2uu_x + iu_{xx} = 0. \tag{2}$$

It is a standard example of a type II quadratic flux model.[9]

According to Lou's direct perturbation method, we expand u of Eq.(1) in an asymptotic series

$$u = \mathrm{e}^{\epsilon(a+\mathrm{i}b)}(u_0(\xi,\tau) + \epsilon u_1(\xi,\tau) + O(\epsilon^2)), \tag{3}$$

where

$$a \equiv a(x,t), \quad b \equiv b(x,t),$$
$$\xi \equiv \xi(x,t,\epsilon), \quad \tau \equiv \tau(x,t,\epsilon)$$

are real functions of indicated variables to be determined. Here ξ and τ should have the property as follows:

$$\{\xi,\tau\} \xrightarrow{\epsilon\to 0} \{x,t\}, \tag{4}$$

such that u_0 would be an exact solution of the complex Burgers equation as $\epsilon \to 0$.

Inserting expansion (3) into Eq.(1) yields

$$\begin{aligned}
&u_{0\tau}\tau_t + 2\mathrm{e}^{\epsilon(a+\mathrm{i}b)}u_0u_{0\xi}\xi_x + \mathrm{i}u_{0\xi\xi}\xi_x^2 + \epsilon\{u_{1\tau}\tau_t + 2\mathrm{e}^{\epsilon(a+\mathrm{i}b)}u_0u_{1\xi}\xi_x + 2\mathrm{e}^{\epsilon(a+\mathrm{i}b)}u_{0\xi}u_1\xi_x + \mathrm{i}u_{1\xi\xi}\xi_x^2\}\\
&+ \mathrm{i}u_{0\tau}\tau_{xx} + \mathrm{i}\epsilon a_{xx}u_0 - \epsilon b_{xx}u_0 + \epsilon a_t u_0 + \mathrm{i}\epsilon b_t u_0 + \epsilon u_0 + 2\mathrm{i}u_{0\xi\tau}\xi_x\tau_x + \mathrm{i}u_{0\xi}\xi_{xx} + u_{0\xi}\xi_t + 2i\epsilon a_x u_{0\xi}\xi_x - 2\epsilon b_x u_{0\xi}\xi_x\\
&+ 2\epsilon\mathrm{e}^{\epsilon(a+\mathrm{i}b)}a_x u_0^2 + 2i\epsilon\mathrm{e}^{\epsilon(a+\mathrm{i}b)}b_x u_0^2 + 2\mathrm{e}^{\epsilon(a+\mathrm{i}b)}u_0u_{0\tau}\tau_x = 0.
\end{aligned} \tag{5}$$

Here we have considered that ξ_t, τ_x and τ_{xx} are all in ϵ order because of Eq.(4). Vanishing the coefficients of ϵ^0 and ϵ^1 orders in Eq.(5), we have

$$u_{0\tau}\tau_t + 2\mathrm{e}^{\epsilon(a+\mathrm{i}b)}u_0u_{0\xi}\xi_x + \mathrm{i}u_{0\xi\xi}\xi_x^2 = 0, \tag{6}$$

$$\begin{aligned}
&\epsilon\{u_{1\tau}\tau_t + 2\mathrm{e}^{\epsilon(a+\mathrm{i}b)}u_0u_{1\xi}\xi_x + 2\mathrm{e}^{\epsilon(a+\mathrm{i}b)}u_{0\xi}u_1\xi_x + \mathrm{i}u_{1\xi\xi}\xi_x^2\} + \mathrm{i}u_{0\tau}\tau_{xx} + \mathrm{i}\epsilon a_{xx}u_0 - \epsilon b_{xx}u_0 + \epsilon a_t u_0 + \mathrm{i}\epsilon b_t u_0 + \epsilon u_0\\
&+ 2\mathrm{i}u_{0\xi\tau}\xi_x\tau_x + \mathrm{i}u_{0\xi}\xi_{xx} + u_{0\xi}\xi_t + 2\mathrm{i}\epsilon a_x u_{0\xi}\xi_x - 2\epsilon b_x u_{0\xi}\xi_x\\
&+ 2\epsilon\mathrm{e}^{\epsilon(a+\mathrm{i}b)}a_x u_0^2 + 2\mathrm{i}\epsilon\mathrm{e}^{\epsilon(a+\mathrm{i}b)}b_x u_0^2 + 2\mathrm{e}^{\epsilon(a+\mathrm{i}b)}u_0u_{0\tau}\tau_x = 0.
\end{aligned} \tag{7}$$

As we know, u_0 in Eq.(6) is not explicitly ϵ dependent and ξ, τ have the limiting property of Eq.(4), so ξ, τ satisfy

$$\tau_t = \mathrm{e}^{\epsilon(a+\mathrm{i}b)}\xi_x = \xi_x^2, \tag{8}$$

and the function u_0 should be an exact solution of complex Burgers equation with adiabatic independent variables ξ, τ, i.e.,

$$u_{0\tau} + 2u_0u_{0\xi} + \mathrm{i}u_{0\xi\xi} = 0. \tag{9}$$

From the expansion, we know that u_1 is just the symmetry definition of Eq.(9), i.e., the first-order correction u_1 satisfies the linearized equation of Eq.(9)

$$u_{1\tau} + 2u_0u_{1\xi} + 2u_{0\xi}u_1 + \mathrm{i}u_{1\xi\xi} = 0, \tag{10}$$

whose easiest solutions can be given by

$$u_1 = u_{0\xi} \quad \text{and} \quad u_1 = u_{0\tau}. \tag{11}$$

Equations (7), (10) with Eq.(8) are consistent at any time t and position x for a given nontrivial solution u_0, which leads to that the last terms of Eq.(7) are all zeros, i.e.,

$$\begin{aligned}
&\mathrm{i}\epsilon a_{xx} - \epsilon b_{xx} + \epsilon a_t + \mathrm{i}\epsilon b_t + \epsilon = 0,\\
&\mathrm{i}\xi_{xx} + \xi_t + 2\mathrm{i}\epsilon a_x\xi_x - 2\epsilon b_x\xi_x = 0,\\
&a_x + \mathrm{i}b_x = 0, \quad \tau_x = 0.
\end{aligned} \tag{12}$$

Solving Eqs.(8) and (12) we have

$$\begin{aligned}
&a = -t, \quad b = C, \quad \xi = x\mathrm{e}^{\epsilon(-t+\mathrm{i}C)},\\
&\tau = \frac{\mathrm{e}^{2\epsilon(-t+\mathrm{i}C)} - 1}{2\epsilon}.
\end{aligned} \tag{13}$$

where C is an arbitrary real constant.

3. Effects of perturbation on solutions

As can be seen from the above, the zero-order solution u_0 in Eq.(3) can be any one of the solutions of the complex Burgers equation (9), and the first-order modification u_1 can be any one of the symmetries of Eq.(9). So compared to other perturbation theories, the asymptotical solutions obtained by Lou's direct perturbation method are more abundant, and different types of perturbation solutions can be obtained at the same time.

In this section, to demonstrate the effects of perturbation, we take into account the periodic travelling wave solution of the complex Burgers equation. Setting $\phi = \xi - \omega\tau$, we obtain

$$u_0 = \frac{1}{2}\frac{\omega e^{\varphi} - \sqrt{\omega^2 - 4C_0}e^{\varphi} - \omega - \sqrt{\omega^2 - 4C_0}}{e^{\varphi} - 1} \quad (\omega^2 > 4C_0), \tag{14}$$

where $\varphi = \sqrt{\omega^2 - 4C_0}(\mathrm{i}(\xi - \omega\tau) + \chi_0))$, C_0 is an arbitrary real constant under the condition $\omega^2 > 4C_0$, and ω and χ_0 describe the velocity and initial position of travelling wave, respectively. As soon as the first-order solution u_1 in Eq.(3) are determined in terms of Eq.(11), the corresponding approximate solution of perturbed complex Burgers equation (1) can then be written as

$$u = -\frac{(-\omega + \sqrt{\omega^2 - 4C_0})e^{2\phi} + (2\omega + 8\mathrm{i}\epsilon C_0 - 2\mathrm{i}\epsilon\omega^2)e^{\phi} - \omega - \sqrt{\omega^2 - 4C_0}}{2(e^{\varphi} - 1)^2} e^{\epsilon(-t + \mathrm{i}C)},$$

where $\varphi = \sqrt{\omega^2 - 4C_0}(i(\xi - \omega\tau) + \chi_0))$ with ξ, τ given by Eq.(13).

Figure 1 plots the evolution of intensity profile $|u_0|^2$ of the periodic travelling wave solution (14) of complex Burgers equation, the parameters being fixed at $\omega = 2,\ C_0 = 0,\ \xi_0 = 1$. In the absence of perturbation, general behaviours of periodic travelling wave solution can all be found in Fig.1. For example, the propagating direction of periodic travelling wave are subject to not only the transverse coordinate ξ but also the propagating coordinate τ. From Figs.2 and 3, it is not difficult to note that the perturbation does not introduce any change in the periodicity of the travelling wave, but the velocity, the amplitude and initial position of the travelling wave have all been changed. Under the action of the small linear damping term, the travelling wave amplitude decays exponentially with time, leading to the broadening of travelling wave. These features can be seen distinctly from Fig.3. For larger ϵ value, the amplitude will decay faster. Moreover, with the increase of transverse coordinate x, the velocity of travelling wave increases as well. And we believe that the effects of perturbation on other types of solutions for the complex Burgers equation can also be analysed in a similar manner.

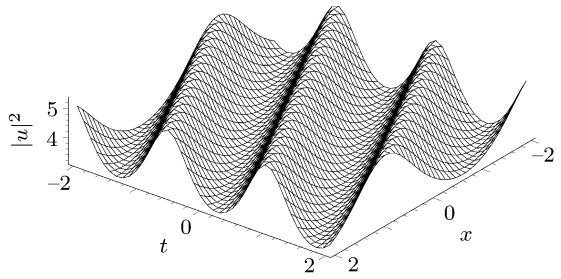


Fig.1. Evolution of intensity profile $|u_0|^2$ of periodic travelling solution (14) of complex Burgers equation with $\omega = 2,\ C_0 = 0,\ \xi_0 = 1$.

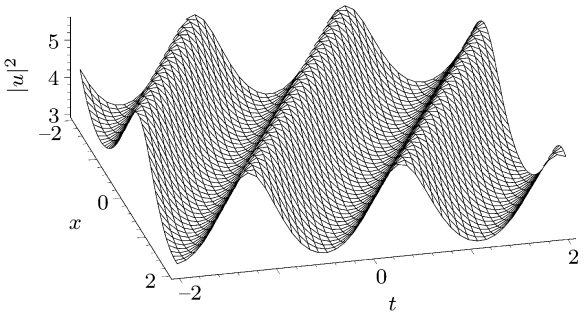


Fig.2. Evolution of intensity profile $|u|^2$ of solution of perturbed complex Burgers equation (1) with $\epsilon = 0.01$ and $C = 0$, other parameters are chosen the same as those in Fig.1.

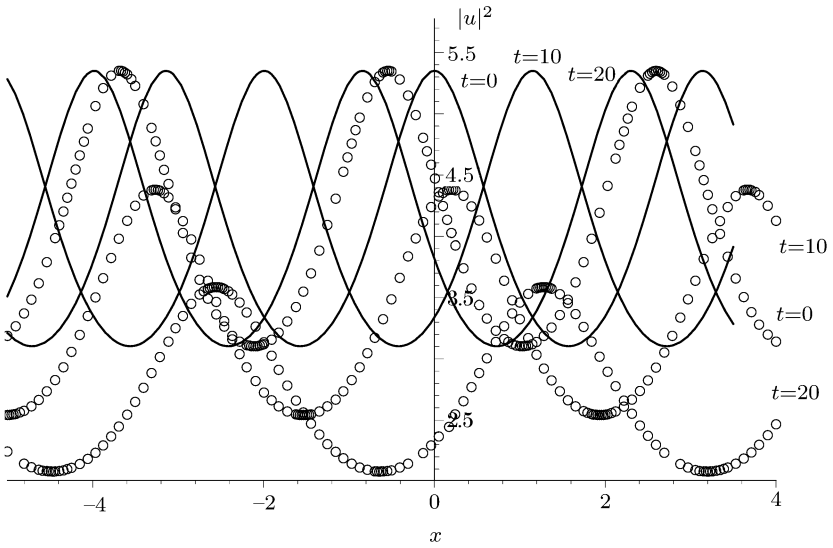


Fig.3. The evolution of $|u|^2$ with amplitudes x of solutions at $t = 0,\ 10,\ 20$. The solid lines are from solution (14) and the circle ones are from Eq.(15).

4. Conclusion

Using Lou's direct perturbation method, we have obtained the asymptotical solutions for the complex Burgers equation with perturbation of $-\epsilon u$, which include not only the zero-order solutions but also the first-order ones. In the asymptotical solutions (3), u_0 can be any one of the solutions for the complex Burgers equation, and the first modification u_1 can be any one of its symmetries. For further study, we take the periodic travelling wave solution (14) for complex Burgers equation as a typical example to analyse the effects of perturbation on solution. It is shown that, in the presence of perturbation of linear damping, the amplitude, the velocity and the initial position of periodic travelling wave are changed, but the perturbation does not bring any change in the periodicity of the travelling wave. In addition, we believe that Lou's direct perturbation method can also be extended to other more perturbed nonlinear equations, such as the KdV, the sine-Gordon equation, the Boussinesq equation, etc.. And how to apply the direct perturbation method to systems with other different perturbations are also worthy of further study.

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