Vol 18 No 12, December 2009
1674-1056/2009/18(12)/5575-08

Chinese Physics B

© 2009 Chin. Phys. Soc.
and IOP Publishing Ltd

Neutrino oscillation interference phase in Kerr space–time

Ren Jun(任 军)†, Jia Meng-Wen(贾孟文), and Yuan Chang-Qing(袁常青)
School of Science, Hebei University of Technology, Tianjin 300130, *China*

(Received 8 April 2009; revised manuscript received 6 May 2009)

The mass neutrino interference phases along the null trajectory and the geodesic line in Kerr space–time are studied on the plane $\theta = \pi/2$. Because of the rotation object in Kerr space–time, a particle travelling along the radial geodesic must have a dragging effect produced by the angular momentum of the central object. We give the correction of the phase due to the rotation of the space–time. We find that the type-I interference phase along the geodesic remains the double of that along the null on the condition that the rotating quantity parameter a^2 is preserved and the higher order terms are negligible (e.g. a^4). In addition, we calculate the proper oscillation length in Kerr space–time. All of our results can return to those in Schwarzschild space–time as the rotating parameter a approaches zero.

Keywords: gravitational field, general relativity, neutrino interference phase, Kerr space–time, geodesic line

PACC: 9760L, 0460K, 0470D

1. Introduction

Since the mass neutrino was confirmed by the Super-Kamiokande atmospheric neutrino experiment,[1] consideration of mass neutrino oscillations[2,3] has been a hot topic. In a theoretical study, neutrino oscillations in flat space–time were extended to cases in gravitational field space–time.[4] Cardall and Fuller have developed a simple formalism for treating neutrino oscillations in curved space–time.[5] In 2001, Zhang and Beesham also studied the propagation of mass neutrinos in Schwarzschild space–time with a detailed scheme. They calculated the interference phase along the geodesic line, which produces a factor 2, compared with the value along the null line.[6] In 2003, they analysed in detail the mass neutrino phase along the geodesic line and the null line in flat and curved space–time, separately.[7] This issue of the factor 2 can be solved if the two neutrino arrival time difference is taken into account.[6−8] As an extended study on the mass neutrino, gravitational effects on neutrino oscillation have attracted much attention recently.[9−11] Furthermore, some alternative mechanisms have been proposed to account for the gravitational effect on neutrino oscillation.[12,13] As a further theoretical exploration, neutrino oscillations in space–time with both curvature and torsion have been studied.[14−16]

Recently, researchers have made new progress on the neutrino oscillation. Cuesta and Lambiase studied the neutrino mass spectrum.[17] They introduced a model-independent novel nonpareil method to overcome difficulties of the issue of determining the ν masses from solar or atmosphere ν experiments concerning the ability of ν detectors to be sensitive to the species mass-square difference instead of to the ν mass itself. Akhmedov, Maltoni and Smirnov presented neutrino oscillograms for different oscillation channels and discussed the effects of non-vanishing 1–2 mixing and CP-violation.[18]

In this paper, we extend the work from Schwarzschild space–time to Kerr space–time and calculate the neutrino oscillation phase along the null and radial geodesic on the plane, $\theta = \pi/2$, respectively. Although this work was studied in Ref.[10], they only calculated the phase in the radial direction of $\theta = 0$. It is more meaningful to study the interference phase on the plane, $\theta = \pi/2$. It is noticed that in the rotational Kerr space–time, a particle travelling along the radial geodesic must have angular momentum, which is different from the case in the static Schwarzschild space–time. The effects of the rotation of the space–time on the phase are given. The relation Φ (geod) $= 2\Phi$ (null) is still true on the condition that the rotating quantity a^2 is retained and the higher order quantity, e.g. a^4, is omitted, where a is the angular momentum per unit mass. In addition, we discuss the proper oscillation length in Kerr space–time,

†Corresponding author. E-mail: thebest2@163.com

which increases due to the correction of the rotation of the gravitational field, compared with the length in Schwarzschild space–time.

The paper is organized as follows. In Sections 2 and 3, we calculate the neutrino phase along the null and along the geodesic on the plane $\theta = \pi/2$, respectively. We discuss the proper oscillation length in Kerr space–time in Section 4. Finally, the conclusion and discussion are given. Throughout the paper, the units $G = c = \hbar = 1$ and $\eta_{\mu\nu} = \mathrm{diag}(+1, -1, -1, -1)$ are used.

2. The interference phase along the light-ray trajectory on the plane $\theta = \pi/2$

The line element of Kerr space–time is described by

$$\begin{aligned} \mathrm{d}s^2 &= g_{00}\mathrm{d}t^2 + g_{11}\mathrm{d}r^2 + g_{22}\mathrm{d}\theta^2 \\ &\quad + g_{33}\mathrm{d}\varphi^2 + 2g_{03}\mathrm{d}t\mathrm{d}\varphi. \end{aligned} \tag{1}$$

The canonical conjugate momenta to coordinates x^μ which are the motion constants along the geodesic can be defined as[4]

$$p_\mu = m g_{\mu\nu}\frac{\mathrm{d}x^\nu}{\mathrm{d}s}, \tag{2}$$

where m is the mass of the particle. The relevant components are

$$p_t = p_0 = m g_{00}\frac{\mathrm{d}t}{\mathrm{d}s} + m g_{03}\frac{\mathrm{d}\varphi}{\mathrm{d}s}, \tag{3}$$

$$p_r = m g_{11}\frac{\mathrm{d}r}{\mathrm{d}s}, \tag{4}$$

$$p_\varphi = m g_{33}\frac{\mathrm{d}\varphi}{\mathrm{d}s} + m g_{03}\frac{\mathrm{d}t}{\mathrm{d}s}. \tag{5}$$

For neutrinos propagating in a radial direction, we have $\mathrm{d}\varphi = 0$ and

$$\begin{aligned} p_0 &= m g_{00}\frac{\mathrm{d}t}{\mathrm{d}s}, \\ p_\varphi &= m g_{03}\frac{\mathrm{d}t}{\mathrm{d}s} = \frac{g_{03}}{g_{00}}p_0. \end{aligned} \tag{6}$$

The mass shell condition is[4]

$$m^2 = g_{\mu\nu}p^\mu p^\nu. \tag{7}$$

The phase along the radial direction from point A to point B is given by[6]

$$\begin{aligned} \Phi(\mathrm{null}) &= \int_A^B g_{\mu\nu}p^\mu \mathrm{d}x^\nu \\ &= \int_A^B (g_{00}p^0\mathrm{d}t + g_{11}p^r\mathrm{d}r) \\ &= \int_A^B \left(g_{00}p^0\frac{\mathrm{d}t}{\mathrm{d}r} + g_{11}p^r\right)\mathrm{d}r. \end{aligned} \tag{8}$$

We calculate the interference phase on the plane $\theta = \pi/2$. On this plane, the components of the Kerr metric can be written as

$$\begin{aligned} g_{00} &= 1 - \frac{2M}{r}, \quad g_{11} = -\frac{r^2}{\Delta}, \quad g_{03} = \frac{2Ma}{r}, \\ g^{00} &= \frac{1}{r^2}\left[\frac{(r^2+a^2)^2}{\Delta} - a^2\right], \quad g^{11} = -\frac{\Delta}{r^2}, \\ g_{33} &= -\left(r^2 + a^2 + \frac{2Ma^2}{r}\right), \\ g^{03} &= \frac{2Ma}{r\Delta}, \quad g^{33} = -\frac{1}{r^2} + \frac{a^2}{r^2\Delta}. \end{aligned} \tag{9}$$

Because of $g_{03} \neq 0$, the angular momentum, $m g_{03}(\mathrm{d}t/\mathrm{d}s)$, does not vanish. So, the mass shell condition becomes

$$\begin{aligned} m^2 &= g_{00}(p^0)^2 + g_{11}(p^r)^2 + 2g_{03}p^0p^\varphi \\ &\quad + g_{33}(p^\varphi)^2. \end{aligned} \tag{10}$$

Substituting $p^0 = g^{00}p_0 + g^{03}p_\varphi$, $p^\varphi = g^{33}p_\varphi + g^{30}p_0$ and $p_\varphi = (g_{03}/g_{00})p_0$ into Eq.(10), we have

$$\begin{aligned} & g_{11}m^2 \\ &= g_{11}g_{00}(p^0)^2 + g_{11}^2(p^r)^2 + 2g_{03}g_{11}p^0p^\varphi + g_{11}g_{33}(p^\varphi)^2 \\ &= g_{11}g_{00}\left(g^{00}p_0 + \frac{g^{03}g_{03}p_0}{g_{00}}\right)^2 + g_{11}^2(p^r)^2 \\ &\quad + 2g_{03}g_{11}\left(g^{00}p_0 + \frac{g_{03}g^{03}p_0}{g_{00}}\right)\left(\frac{g^{33}g_{03}p_0}{g_{00}} + g^{03}p_0\right) \\ &\quad + g_{11}g_{33}\left(\frac{g^{33}g_{03}}{g_{00}} + g^{03}\right)^2 p_0^2 \\ &= A(r)p_0^2 + g_{11}^2(p^r)^2, \end{aligned} \tag{11}$$

where

$$\begin{aligned} A(r) &= g_{11}g_{00}(g^{00})^2 + 2g_{11}g^{03}g_{03}g^{00} \\ &\quad + \frac{g_{11}(g^{03})^2g_{03}^2}{g_{00}} + \frac{2(g_{03})^2g_{11}g^{00}g^{33}}{g_{00}} \\ &\quad + 2g^{00}g^{03}g_{03}g_{11} + \frac{2(g^{03})^2g_{11}g^{03}g^{33}}{g_{00}^2} \\ &\quad + \frac{2g_{03}^2g_{11}(g^{03})^2}{g_{00}} + \frac{g_{11}g_{33}(g^{33})^2g_{03}^2}{g_{00}^2} \\ &\quad + g_{11}g_{33}(g^{03})^2 + \frac{2g_{11}g_{33}g^{33}g_{03}g^{03}}{g_{00}} \\ &= g_{11}g_{00}(g^{00})^2 + B(r). \end{aligned} \tag{12}$$

It is noticed that $B(r)$ contains terms with the order of a^2 or more than a^2. If a is a small quantity,

we have $B(r) \ll g_{11}g_{00}(g^{00})^2$, which is useful in the calculation.

Although the mass of the neutrino is confirmed, it is a very small quantity and close to zero, in fact. Moreover, in the high energy limit, the neutrino is a relativistic particle with a high speed close to light. Therefore, it is reasonable to calculate the neutrino phase along the null line. The radial null geodesic is

$$0 = g_{00}\mathrm{d}t^2 + g_{11}\mathrm{d}r^2. \tag{13}$$

The interference phase can be written as

$$\begin{aligned}&\Phi(\text{null})\\ &= \int_A^B \left(g_{00}p^0\frac{\mathrm{d}t}{\mathrm{d}r} + g_{11}p^r\right)\mathrm{d}r\\ &= \int_A^B \left\{g_{00}p^0\sqrt{\frac{-g_{11}}{g_{00}}} - \sqrt{g_{11}m^2 - A(r)p_0^2}\right\}\mathrm{d}r.\end{aligned}$$

Substituting $p^0 = g^{00}p_0 + g^{03}p_\varphi$ and relativistic condition $p_0 \gg m$, $\sqrt{p_0^2 - \frac{g_{11}m^2}{A(r)}} \simeq p_0 - \frac{g_{11}}{A(r)}\frac{m^2}{2p_0}$ into Eq.(14), we have

$$\Phi(\text{null}) = \int_A^B \left[\sqrt{-g_{11}g_{00}}g^{00}p_0 + \sqrt{\frac{-g_{11}}{g_{00}}}g_{03}g^{03}p_0 - \sqrt{-A(r)}p_0 - \frac{g_{11}}{\sqrt{-A(r)}}\frac{m^2}{2p_0}\right]\mathrm{d}r. \tag{14}$$

Under the condition that a is rather small, we can expand

$$\begin{aligned}\sqrt{-A(r)} &\simeq \sqrt{-g_{11}g_{00}}g^{00} - \frac{1}{2}\frac{B(r)}{\sqrt{-g_{11}g_{00}}g^{00}},\\ \frac{1}{\sqrt{-A(r)}} &\simeq \frac{1}{\sqrt{-g_{11}g_{00}}g^{00}} + \frac{1}{2}\frac{B(r)}{(-g_{11}g_{00})^{3/2}(g^{00})^3}.\end{aligned} \tag{15}$$

Adopting the relation $1/g^{00} = g_{00} - g_{03}^2/g_{33}$, equation (15) can be written as

$$\begin{aligned}\Phi(\text{null}) = \int_A^B \Bigg[&\sqrt{\frac{-g_{11}}{g_{00}}}g_{03}g^{03}p_0 + \frac{1}{2}\frac{B(r)p_0}{\sqrt{-g_{11}g_{00}}g^{00}} + \sqrt{-g_{11}g_{00}}\frac{m^2}{2p_0}\\ &- \frac{1}{2}\frac{g_{11}B(r)}{(-g_{11}g_{00})^{3/2}(g^{00})^3}\frac{m^2}{2p_0} - \sqrt{\frac{-g_{11}}{g_{00}}}\frac{g_{03}^2m^2}{g_{33}2p_0}\Bigg]\mathrm{d}r.\end{aligned} \tag{16}$$

We will do the integral carefully in Section 3.

3. The interference phase along the geodesic line on the plane $\theta = \pi/2$

For more general situations, we will calculate the neutrino phase along the radial geodesic. The classical orbit can be confined to a plane, $\theta = \pi/2$, $\mathrm{d}\theta = 0$, and in a radial direction, $\mathrm{d}\varphi = 0$. So, the geodesic equation can be written as

$$g_{00}\dot{t}^2 + g_{11}\dot{r}^2 = 1, \tag{17}$$

where $\dot{t} = \mathrm{d}t/\mathrm{d}s$, $\dot{r} = \mathrm{d}r/\mathrm{d}s$. Substituting Eq.(6) into Eq.(18), we have

$$\frac{\mathrm{d}t}{\mathrm{d}r} = \sqrt{\frac{-g_{11}}{g_{00}}}\frac{1}{\sqrt{1 - \frac{m^2}{p_0^2}g_{00}}} \simeq \sqrt{\frac{-g_{11}}{g_{00}}}\left[1 + \frac{g_{00}}{2}\frac{m^2}{p_0^2}\right], \tag{18}$$

where the high energy limit, $p_0 \gg m$, is used. The phase can be calculated as follows:[6]

$$\begin{aligned}\Phi(\text{geod}) &= \int_A^B \left(g_{00}p^0\frac{\mathrm{d}t}{\mathrm{d}r} + g_{11}p^r\right)\mathrm{d}r\\ &= \int_A^B \left[g_{00}p^0\sqrt{\frac{-g_{11}}{g_{00}}} + g_{00}^2p^0\sqrt{\frac{-g_{11}}{g_{00}}}\frac{m^2}{2p_0^2} - \sqrt{g_{11}m^2 - A(r)p_0^2}\right].\end{aligned} \tag{19}$$

As we compare Eq.(20) with Eq.(14), the difference is the term

$$\begin{aligned} g_{00}^2 p^0 \sqrt{\frac{-g_{11}}{g_{00}}}\frac{m^2}{2p_0^2} &= g_{00}^2\left(g^{00}p_0+\frac{g_{03}g^{03}p_0}{g_{00}}\right)\sqrt{\frac{-g_{11}}{g_{00}}}\frac{m^2}{2p_0^2} \\ &\simeq \sqrt{-g_{11}g_{00}}\left[g_{00}\left(\frac{1}{g_{00}}+\frac{g_{03}^2}{g_{00}^2g_{33}}\right)\frac{m^2}{2p_0}+g_{03}g^{03}\frac{m^2}{2p_0}\right] \\ &= \sqrt{-g_{11}g_{00}}\left(\frac{m^2}{2p_0}+g_{03}g^{03}\frac{m^2}{2p_0}\right)+\sqrt{\frac{-g_{11}}{g_{00}}}\frac{g_{03}^2}{g_{33}}\frac{m^2}{2p_0}, \end{aligned}$$

where $g^{00}=1/(g_{00}+g_{03}^2/g_{33})\simeq 1/g_{00}+g_{03}^2/(g_{00}^2g_{33})$. Hence, we have

$$\begin{aligned} \Phi(\text{geod}) = \int_A^B &\left[\sqrt{\frac{-g_{11}}{g_{00}}}g_{03}g^{03}p_0+\frac{1}{2}\frac{B(r)p_0}{\sqrt{-g_{11}g_{00}}g^{00}}+\sqrt{-g_{11}g_{00}}\frac{m^2}{p_0}\right. \\ &\left.-\frac{1}{2}\frac{g_{11}B(r)}{(-g_{11}g_{00})^{3/2}(g^{00})^3}\frac{m^2}{2p_0}+\sqrt{-g_{11}g_{00}}g_{03}g^{03}\frac{m^2}{2p_0}\right]\mathrm{d}r. \end{aligned} \tag{20}$$

Now we start to do the integral in Eq.(22). Because of the complication of the calculation in Kerr space–time, we have to adopt an approximation. We will omit the terms with the order of a^4. Except for the third term $\sqrt{-g_{11}g_{00}}m^2/p_0$ on the right-hand side of Eq.(22), the first and second terms are important in the integral. Therefore, we do the integral of these two terms at first. We only retain the terms with the order of a^2 in $B(r)$:

$$\begin{aligned} B(r) \simeq\ & 2g_{11}g^{03}g_{03}g^{00}+\frac{2g_{03}^2g_{11}g^{00}g^{33}}{g_{00}}+2g^{00}g_{11}g^{03}g_{03}+\frac{g_{11}g_{33}(g^{33})^2g_{03}^2}{g_{00}^2} \\ &+g_{11}g_{33}(g^{03})^2+\frac{2g_{11}g_{33}g^{33}g_{03}g^{03}}{g_{00}}. \end{aligned} \tag{21}$$

Equation (23) can be simplified further as $1/g^{00}=g_{00}-g_{03}^2/g_{33}\simeq g_{00}$, $g_{33}=-(r^2+a^2+2Mra^2/r^2)\simeq -r^2$ and $g^{33}=-1/r^2+a^2/(r^2\Delta)\simeq -1/r^2$. Because g_{03}^2, $(g^{03})^2$ and $g_{03}g^{03}$ are the terms with the order of a^2, the product of the terms with a^2 in $1/g^{00}$, g_{33}, g^{33} and a^2 is the order of a^4 which can be omitted. Thus the integral of the first and second terms in Eq.(22) is

$$\begin{aligned} &\int\left[\sqrt{\frac{-g_{11}}{g_{00}}}g_{03}g^{03}p_0+\frac{1}{2}\frac{B(r)}{\sqrt{-g_{11}g_{00}}g^{00}}p_0\right]\mathrm{d}r \\ \simeq &\int\left[-2\sqrt{\frac{-g_{11}}{g_{00}}}g_{03}g^{03}-\frac{3}{2}\frac{\sqrt{-g_{11}}}{g_{00}^{3/2}}g^{33}g_{03}^2-\frac{1}{2}\sqrt{-g_{11}g_{00}}g_{33}(g^{03})^2\right]p_0\mathrm{d}r \\ = &\ 4M^2a^2\int\left[-2\sqrt{\frac{-g_{11}}{g_{00}}}\frac{1}{r^2\Delta}+\frac{3}{2}\sqrt{\frac{-g_{11}}{g_{00}}}\frac{1}{g_{00}r^4}+\frac{1}{2}\sqrt{-g_{11}g_{00}}\frac{1}{\Delta^2}\right]p_0\mathrm{d}r. \end{aligned} \tag{22}$$

Omitting $O(a^2)$, we have $1/\Delta\simeq 1/r(r-2M)$ and $1/\Delta^2\simeq 1/r^2(r-2M)^2$. So, the integral is

$$4M^2a^2\int\left[-2\sqrt{\frac{-g_{11}}{g_{00}}}\frac{1}{r^3(r-2M)}+\frac{3}{2}\sqrt{\frac{-g_{11}}{g_{00}}}\frac{1}{r^3(r-2M)}+\frac{1}{2}\frac{\sqrt{-g_{11}g_{00}}}{r^2(r-2m)^2}\right]\mathrm{d}r=0. \tag{23}$$

Therefore the terms with p_0 do not contribute to the integral as the terms with the higher order of a^4 are negligible. Now we have to calculate the fourth and fifth terms on the right-hand side of Eq.(22). The fourth term is

$$\begin{aligned} &\int-\frac{1}{2}\frac{g_{11}B(r)}{(-g_{11}g_{00})^{3/2}(g^{00})^3}\frac{m^2}{2p_0}\mathrm{d}r \\ =&\int\frac{1}{2}\frac{B(r)}{\sqrt{-g_{11}g_{00}}}g_{00}^2\frac{m^2}{2p_0}\mathrm{d}r \\ =&\int\left[-3\sqrt{-g_{11}g_{00}}g_{03}g^{03}-\frac{3}{2}\sqrt{\frac{-g_{11}}{g_{00}}}g^{33}g_{03}^2-\frac{1}{2}\sqrt{-g_{11}}(g_{00})^{3/2}g_{33}(g^{03})^2\right]\frac{m^2}{2p_0}\mathrm{d}r \\ \simeq&\int\left[-3\sqrt{-g_{11}g_{00}}\frac{4M^2a^2}{r^2\Delta}+\frac{3}{2}\sqrt{-g_{11}g_{00}}\frac{4M^2a^2}{g_{00}r^4}+\frac{1}{2}\sqrt{-g_{11}g_{00}}g_{00}\frac{4M^2a^2}{\Delta^2}\right]\frac{m^2}{2p_0}\mathrm{d}r \end{aligned}$$

$$\simeq \int \left[-\frac{4M^2a^2\sqrt{-g_{11}g_{00}}}{r^3(r-2M)}\right]\frac{m^2}{2p_0}\mathrm{d}r. \tag{24}$$

In the calculation, we use the approximation: $1/\Delta \simeq 1/r(r-2M)$ and $1/\Delta^2 \simeq 1/r^2(r-2M)^2$. The fifth term in Eq.(22) is

$$\int \sqrt{-g_{11}g_{00}}g_{03}g^{03}\frac{m^2}{2p_0}\mathrm{d}r = \int \sqrt{-g_{11}g_{00}}\frac{4M^2a^2}{r^2\Delta}\frac{m^2}{2p_0}\,\mathrm{d}r \simeq \int \sqrt{-g_{11}g_{00}}\frac{4M^2a^2}{r^3(r-2M)}\frac{m^2}{2p_0}\mathrm{d}r. \tag{25}$$

It is surprising that the sum of the fourth and fifth terms on the right-hand side of Eq.(22) vanishes too when we omit the order of a^4. Finally, let us do the integral of the fifth term on the right-hand side of Eq.(17):

$$\int \left[-\sqrt{\frac{-g_{11}}{g_{00}}}\frac{g_{03}^2}{g_{33}}\right]\frac{m^2}{2p_0}\mathrm{d}r = \int \sqrt{\frac{-g_{11}}{g_{00}}}\frac{4M^2a^2}{r^4}\frac{m^2}{2p_0}\mathrm{d}r = \int \sqrt{-g_{11}g_{00}}\frac{4M^2a^2}{r^3(r-2M)}\frac{m^2}{2p_0}\mathrm{d}r. \tag{26}$$

Consequently, the phase along the null is

$$\Phi(\mathrm{null}) = \int_A^B \sqrt{-g_{11}g_{00}}\frac{m^2}{2p_0}\mathrm{d}r, \tag{27}$$

and the phase along the radial geodesic is

$$\begin{aligned}
\Phi(\mathrm{geod}) &= \int_A^B \sqrt{-g_{11}g_{00}}\frac{m^2}{p_0}\mathrm{d}r \\
&= \int_A^B \sqrt{\frac{r(r-2M)}{\Delta}}\frac{m^2}{p_0}\mathrm{d}r \\
&\simeq \int_A^B \left[1-\frac{a^2}{2(r^2-2Mr)}\right]\frac{m^2}{p_0}\mathrm{d}r \\
&= \frac{m^2}{p_0}(r_B-r_A) - \frac{a^2}{4M}\left[\ln\left(\frac{r_B-2M}{r_B}\frac{r_A}{r_A-2M}\right)\right]\frac{m^2}{p_0} \\
&= \frac{m^2}{p_0}(r_B-r_A) - \frac{a^2}{4M}\left[\ln\left(1-\frac{2M}{r_B}\right) - \ln\left(1-\frac{2M}{r_A}\right)\right] \\
&= \frac{m^2}{p_0}(r_B-r_A) - \frac{a^2}{4M}\ln\frac{g_{00}(r_B)}{g_{00}(r_A)}.
\end{aligned} \tag{28}$$

It is noticed that $g_{00}(r_B) = 1-2M/r_B$ and $g_{00}(r_A) = 1-2M/r_A$ are the square of the gravitational red-shift factors at point B and point A, respectively, on the plane $\theta = \pi/2$. If the condition of $\Delta r = r_B - r_A \ll r_A$ is considered, we can obtain a simpler form of the interference phase

$$\begin{aligned}
\Phi(\mathrm{geod}) &= \frac{m^2}{p_0}\left[r_B - r_A - \frac{a^2}{4M}\ln\left(1-\frac{2M}{r_B}\right)\left(\frac{1}{1-\frac{2M}{r_A}}\right)\right] \\
&\simeq \frac{m^2}{p_0}\left[r_B - r_A - \frac{a^2}{4M}\ln\left(1-\frac{2M}{r_A+\Delta r}\right)\left(\frac{1}{1-\frac{2M}{r_A}}\right)\right] \\
&\simeq \frac{m^2}{p_0}\left[\Delta r - \frac{a^2}{4M}\ln\left(1+\frac{2M}{r_A(r_A-2M)}\Delta r\right)\right] \\
&\simeq \frac{m^2}{p_0}\left(1-\frac{a^2}{2r_A(r_A-2M)}\right)\Delta r.
\end{aligned} \tag{29}$$

We can conclude that the relation

$$\Phi(\mathrm{null}) = \int_A^B \sqrt{-g_{11}g_{00}}\frac{m^2}{2p_0} = \frac{1}{2}\Phi(\mathrm{geod}) \tag{30}$$

is also right on the condition that we only retain the terms with the order of a^2. The above result is obtained in the high energy limit, $p_0 \gg m$. The first term on the right-hand side of Eq.(30) appears to be identical to that of the Schwarzschild space–time oscillation case. However, the rotational effects are presented in the first term implicitly. The propagation of a neutrino is over its proper distance, but $(r_B - r_A)$ in Eq.(30) is not. The influence of the rotation of the gravitational field on the phase in the first term is embodied in the proper distance which is analysed in the next section in detail. The second one on the right-hand side of Eq.(30) shows the effects of the rotation of the gravitational field on the oscillation of radially propagating neutrinos explicitly. So, equation (30) gives the standard result of the neutrino oscillation phase in Kerr space–time. Because the contribution of the second term to the phase is much less than the first term on the right-hand of Eq.(30), Kerr rotation has just a little contribution to the neutrino oscillation phase. Also, if we let $a = 0$, equation (30) will be reduced to the Schwarzschild space–time case.

4. Proper oscillation length in Kerr space–time

It is noticed that $r_B - r_A$ is not the proper physical distance in Kerr space–time, which is given by

$$\begin{aligned} & L_B - L_A \\ &= \int_A^B \sqrt{-g_{11}}\mathrm{d}r \\ &= \int_A^B \frac{r\mathrm{d}r}{\sqrt{\Delta}} \\ &= [\sqrt{\Delta_B} + M\ln(2\sqrt{\Delta_B} + 2r_B - 2M)] \\ &\quad - \left[\sqrt{\Delta_A} + M\ln(2\sqrt{\Delta_A} + 2r_A - 2M)\right], \end{aligned} \tag{31}$$

where $\Delta_A = r_A^2 - 2Mr_A + a^2$ and $\Delta_B = r_B^2 - 2Mr_B + a^2$. If $a = 0$, the result can be reduced to the proper distance in Schwarzschild space–time[6]

$$\begin{aligned} L_B - L_A = {} & r_B\sqrt{1 - \frac{2M}{r_B}} - r_A\sqrt{1 - \frac{2M}{r_A}} \\ & + 2M\Big[\ln(\sqrt{r_B - 2M} + \sqrt{r_B}) \\ & - \ln(\sqrt{r_A - 2M} + \sqrt{r_A})\Big]. \end{aligned} \tag{32}$$

It is generally not possible to extract a separate "gravitational phase" from expression (33).

At first, let us discuss the oscillation length in the Schwarzschild space–time. We consider the case of a weak field and the condition $\Delta r = r_B - r_A \ll r_A$; the proper distance is approximated to

$$\begin{aligned} \Delta L_{\mathrm{Sch}} &= L_B - L_A \\ &\simeq r_B - r_A + M\ln\frac{r_B}{r_A} \\ &= \Delta r + M\ln\frac{r_A + \Delta r}{r_A} \\ &= \Delta r + M\ln\left(1 + \frac{\Delta r}{r_A}\right) \\ &\simeq \left(1 + \frac{M}{r_A}\right)\Delta r, \end{aligned} \tag{33}$$

where $r_B > r_A$. So, the relative phase of the mass neutrino is

$$\Delta\Phi = \frac{\Delta m^2}{2p_0}\left(\frac{\Delta L_{\mathrm{Sch}}}{1 + \dfrac{M}{r_A}}\right), \tag{34}$$

where $\Delta m^2 = m_\mu^2 - m_{\mathrm{e}}^2$ is the mass square difference. The proper oscillation length can be given by

$$L_{\mathrm{OSC}}(r_A) = \frac{4\pi p_0}{\Delta m^2}\left(1 + \frac{M}{r_A}\right). \tag{35}$$

The term on the right-hand side of Eq.(37), $4\pi P_0/\Delta m^2$, is analogous to the flat space–time oscillation length. Thus the proper oscillation length increases in the gravitational field. If $r_A \to \infty$, $M/r_A \to 0$, the result can be reduced to the oscillation length in flat space–time.

Assuming $\Delta r = r_B - r_A \ll r_A$, the proper physical length in Kerr space–time Eq.(33) can be written as

$$\begin{aligned} \Delta L_{\mathrm{Kerr}} &= L_B - L_A \\ &= [(r_A + \Delta r)^2 - 2M(r_A + \Delta r) + a^2]^{1/2} \\ &\quad - \Delta_A^{1/2} + M\ln\{[(r_A + \Delta r)^2 \\ &\quad - 2M(r_A + \Delta r) + a^2]^{1/2} + r_A - M + \Delta r\} \\ &\quad - M\ln[\sqrt{\Delta} + r_A - M] \\ &\simeq r_A\Delta r\Delta_A^{-1/2}. \end{aligned} \tag{36}$$

Under the condition of the weak field, the above equation can be rewritten as

$$\Delta L(\mathrm{Kerr}) \simeq \Delta r + \frac{M}{r_A}\Delta r - \frac{a^2}{2}\frac{\Delta r}{r_A^2}. \tag{37}$$

It is obvious that the proper distance in Kerr space–time is shorter than ΔL in Schwarzschild space–time. Substituting Eq.(38) into Eq.(32), we can obtain the relative phase in Kerr space–time

$$\Delta\Phi = \frac{\Delta m^2}{2p_0}\left[1 - \frac{a^2}{2(r_A^2 - 2Mr_A)}\right]\frac{\sqrt{\Delta_A}}{r_A}\Delta L(\text{Kerr}). \tag{38}$$

Therefore, the oscillation length in Kerr space–time is

$$\begin{aligned} & L_{\text{OSC}}(r_A) \\ \simeq\ & \frac{4\pi p_0}{\Delta m^2}\left[\frac{r_A}{\sqrt{r_A^2 - 2Mr_A}} + \frac{a^4 r_A}{4(r_A^2 - 2Mr_A)^{5/2}}\right] \\ \simeq\ & \frac{4\pi p_0}{\Delta m^2}\left(1 + \frac{M}{r_A}\right) + \frac{4\pi p_0}{\Delta m^2}\frac{a^4 r_A}{4(r_A^2 - 2Mr_A)^{5/2}}. \end{aligned} \tag{39}$$

The first term on the right-hand side of Eq.(41) is analogous to the oscillation length in Schwarzschild space–time. The second term represents the correction due to the rotation of the gravitational field. It is obvious that the second term is positive. Consequently, the oscillation length increases due to the rotation of the gravitational field compared with that in Schwarzschild space–time. In addition, the oscillation length in Kerr space–time equals approximately the case in Schwarzschild space–time if the higher order quantity a^4 can be omitted. Because the quantity in the second term is rather small, the effect of the rotation of space–time on the proper oscillation length is much smaller than the effect of the gravitational field.

5. Conclusion and discussion

In this paper, we have given the interference phase of a mass neutrino propagating in the radial direction along the geodesic line and the null line in the gravitational field of a rotating symmetric object, which is described by a Kerr metric. Most astrophysical bodies in the universe generally have angular momentum and rotation. Thus work on neutrino oscillation in rotating symmetric space–time is important and meaningful.

Although the calculation is very complicated, we work out the phase on the plane $\theta = \pi/2$ with the universality by an approximation. We divide the integrand into terms with different orders of a: a^0, a^2 and a^4. We retain the terms with a^0 and a^2, and omit those with a^4. The type-I interference phase along the geodesic remains the double of that along the null in the high energy limit. We obtain a correction due to the rotation of space–time. Because a is a small quantity, the contribution of the rotation of the gravitational field is small. If a^2 vanishes, the rotating effect does not contribute to the phase Φ and all of our results can be reduced to the cases in Schwarzschild space–time. Assuming that the source and the detector are at a sufficiently large distance from the rotating object, the phase is reduced to a simpler form as given in Eq.(31). In addition, the proper oscillation length in Kerr space–time is studied. Because of the correction of the rotation of the gravitational field, the oscillation length is longer than that in Schwarzschild space–time. However, the difference can be only shown on the condition that the higher order a^4 is taken into account.

Our result has a universal meaning and can be used to study neutrino oscillation near compact stars such as neutron stars and in a rotationally gravitational field. Moreover, it is necessary to study in depth the properties of the mass neutrino in particle physics and cosmology. The establishment of the results of neutrino oscillation will show their great influence on the origin and evolution of the universe.

References

[1] Fukuda Y, Hayakawa T, Ichihara E, Inoue K, Ishihara K, Ishino H, Itow Y, Kajita T, Kameda J, Kasuga S, Kobayashi K, Kobayashi Y, Koshio Y, Miura M, Nakahata M, Nakayama S, Okada A, Okumura K, Sakurai N, Shiozawa M, Suzuki Y, Takeuchi Y, Totsuka Y, Yamada S, Earl M, Habig A, Kearns E, Messier M D, Scholberg K, Stone J L, Sulak L R, Walter C W, Goldhaber M, Barszczak T, Casper D, Gajewski W, Halverson P G, Hsu J, Kropp W R, Price L R, Reines F, Smy M, Sobel H W, Vagins M R, Ganezer K S, Keig W E, Ellsworth R W, Tasaka S, Flanagan J W, Kibayashi A, Learned J G, Matsuno S, Stenger V J, Takemori D, Ishii T, Kanzaki J, Kobayashi T, Mine S, Nakamura K and Nishikawa K 1999 *Phys. Rev. Lett.* **82** 1810

[2] Wolfenstein L 1978 *Phys. Rev.* D **17** 2369

[3] Mikheyev S P and Smirnov A Y 1986 Il *Nuovo Cimento* C **9** 17

[4] Fornengo N, Giunti C, Kim C W and Song J 1997 *Phys. Rev.* D **56** 1895

[5] Cardall C and Fuller G 1997 *Phys. Rev.* D **55** 7960

[6] Zhang C M and Beesham A 2001 *Gen. Rel. Grav.* **33** 1011

[7] Zhang C M and Beesham A 2003 *Int. J. Mod. Phys.* D **12** 727

[8] Bhattacharya T, Habib S and Mottola E 1999 *Phys. Rev.* D **59** 067301

[9] Pereira J G and Zhang C M 2000 *Gen. Rel. Grav.* **32** 1633
Pereira J G and Zhang C M 2001 *Gen. Rel. Grav.* **33** 2297

[10] Huang X J and Wang Y J 2003 *Commun. Theor. Phys.* **40** 742

[11] Huang X J and Wang Y J 2004 *Chin. Phys.* **13** 1588
Huang X J and Wang Y J 2006 *Chin. Phys.* **15** 229

[12] Gasperini M 1988 *Phys. Rev.* D **38** 2635
[13] Mureika J and Mann R B 1996 *Phys. Lett.* B **368** 112
Mureika J and Mann R B 1996 *Phys. Rev.* D **54** 2761
[14] Alimohammadi M and Shariati A 1999 *Mod. Phys. Lett.* A **14** 267
[15] Capozziello S, Iovane G, Lambiase G and Stomaiolo C 1999 *Europhys. Lett.* **46** 710
[16] Capozziello S, Lambiase G and Stornaiolo C 2001 *Annalen Phys.* **10** 713
[17] Cuesta H J M and Lambiase G 2008 *Astrophysical Journal* **689** 371
[18] Akhmedov E K, Maltoni M and Smirnov A Y 2008 *Journal of High Energy Physics* **0806** 072