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Band rules for the frequency spectra of three kinds of aperiodic photonic crystals with negative refractive index materials*

Quan Xiao-Lin(全小林) and Yang Xiang-Bo(杨湘波)†

MOE Key Laboratory of Laser Life Science & Institute of Laser Life Science, South China Normal University, Guangzhou 510631, *China*

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By means of the theory of electromagnetic wave propagation and transfer matrix method, this paper investigates the band rules for the frequency spectra of three kinds of one-dimensional (1D) aperiodic photonic crystals (PCs), generalized Fibonacci GF(p, 1), GF(1,2), and Thue–Morse (TM) PCs, with negative refractive index (NRI) materials. It is found that all of these PCs can open a broad zero-$\bar{n}$ gap, TM PC possesses the largest zero-$\bar{n}$ gap, and with the increase of p, the width of the zero-$\bar{n}$ gap for GF(p, 1) PC becomes smaller. This characteristic is caused by the symmetry of the system and the open position of the zero-$\bar{n}$ gap. It is found that for GF(p, 1) PCs, the possible limit zero-$\bar{n}$ gaps open at lower frequencies with the increase of p, but for GF(1,2) and TM PCs, their limit zero-$\bar{n}$ gaps open at the same frequency. Additionally, for the three bottom-bands, we find the interesting perfect self-similarities of the evolution structures with the increase of generation, and obtain the corresponding subband-number formulae. Based on 11 types of evolving manners Q_i ($i = 1, 2, \ldots, 11$) one can plot out the detailed evolution structures of the three kinds of aperiodic PCs for any generation.

Keywords: negative refractive index, photonic crystals, generalized Fibonacci photonic crystal, Thue–Morse photonic crystal

PACC: 4225B, 7820P, 6140M

1. Introduction

Photonic crystal (PC) is a kind of well-known periodic structure being composed of materials with different permittivities.[1,2] It influences the nature of the photonic modes significantly, which leads to some very unique physical phenomena, such as the photonic band gap and localization of electromagnetic waves. In recent years, people have paid great attention to PC due to their remarkable features.[3−25] Now, whether in theory or in experiment, great progress has been made in PC, and the studies on PC have been one of the focuses in condensed matter physics, optics, materials, and many other fields.

At the beginning of the investigations of PC the ordinary materials with positive refractive index (RI) were mainly used. However, more and more so-called double-negative[26,27] ($\varepsilon < 0$ and $\mu < 0$) and single-negative[28,29] ($\varepsilon < 0$ or $\mu < 0$) materials have received renewed attention. It is found that PCs with negative refractive index (NRI) materials result in two novel photonic band gaps, the zero-$\bar{n}$ and zero-ϕ_{eff} gaps.[30,31] Distinct from band gaps induced by Bragg scattering, the zero-$\bar{n}$ gap is invariant under a change of scale length and is insensitive to disorder which is symmetric in the random variable. Later, the zero-$\bar{n}$ gap is shown with omnidirectional properties, which means that the edge of the gap is insensitive to the incident angle and polarization, even when an impurity is introduced, a defect mode appears inside the gap with a very weak dependence on incident angle and invariant with scaling,[32,33] but subsequent studies demonstrated that behaviour is not universally true.[34] Some of these properties of the zero-$\bar{n}$ gap have been confirmed by experiment.[35] The another gap, zero-ϕ_{eff} gap, also possesses many novel properties. The properties of such an omnidirectional gap is insensitive to the incident angles and the light polarizations, and is invariant under the change of scale length, a defect mode appears inside the omnidirectional gap, the spectral position of the defect mode is almost independent of incident angles and nearly invariant with the scaling.[36] Moreover, zero-ϕ_{eff} gap is insensitive to thickness fluctuation. In contrast to

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†Corresponding author. E-mail: xbyang@scnu.edu.cn

a photonic gap corresponding to zero-$\bar{n}$, the zero-$\phi_{\rm eff}$ gap can be made very wide by varying the ratio of the thicknesses of two media. A photonic quantum-well structure based on zero-$\phi_{\rm eff}$ gaps is proposed as a multiple channeled filter that is compact and robust against disorder.[31]

Additionally, PCs ordered in quasiperiodic and non-quasiperiodic sequences have been also widely studied.[37−40] People have paid much more attention to PCs ordered in aperiodic sequences and composed of NRI materials. It is found that, the Fibonacci PC stacking of positive and NRI materials possesses a very stable zero-$\bar{n}$ gap, and some of its wave functions in the edge of this zero-$\bar{n}$ gap are self-similar.[41] For the symmetrical Fibonacci-system PC made up of both positive and NRI materials, there is also a zero-$\bar{n}$ gap, and when the NRI material is not dispersive, self-similar behaviour and perfect transmission peaks are displayed in its transmission spectrum.[42] Recently, it was reported that PCs containing both positive and NRI materials, and arranged following Fibonacci, Thue–Morse (TM) and double-period sequences, possess zero-$\bar{n}$ gap and their photonic bandwidth to satisfy different power-law.[43]

In this paper we investigate the band rules for the frequency spectra of three kinds of one-dimensional (1D) aperiodic PCs with NRI, whose unit cells are composed of generalized Fibonacci GF$(p,1)$, GF(1,2) and TM PCs, respectively. The positions and widths of the possible limit zero-$\bar{n}$ gaps are obtained and they are tightly related with the symmetry of the aperiodic systems. On the other hand, we find perfect self-similar evolution structure with the increase of generation for the three bottom-bands of the aforementioned three kinds of aperiodic PCs, and obtain the corresponding subband-number formulae. It would have potential applications for the designing of PCs with NRI materials.

2. Constructions of the photonic crystals

2.1. Structure

The structure of three kinds of 1D aperiodic PCs with NRI materials which we studied in this paper is shown in Fig.1, where j denotes the j-th unit cell, the thickness d_A and RI n_A of thin film A are 8.0 mm and 2.3 nm, respectively. The thickness of thin film B is 8.0 mm and its RI satisfies the following equation:[44,45,33]

$$\begin{aligned} n_B(F) &= \sqrt{\varepsilon(F)}\sqrt{\mu(F)}, \\ \varepsilon(F) &= \varepsilon_1 - \frac{\alpha}{(2\pi F)^2}, \\ \mu(F) &= \mu_1 - \frac{\beta}{(2\pi F)^2}, \end{aligned} \tag{1}$$

where F is the frequency measured in GHz, $\varepsilon_1 = 1.21$, $\mu_1 = 1.0$, $\alpha = 80$, and $\beta = 160$. The dispersion curves of material B is given in Fig.2.

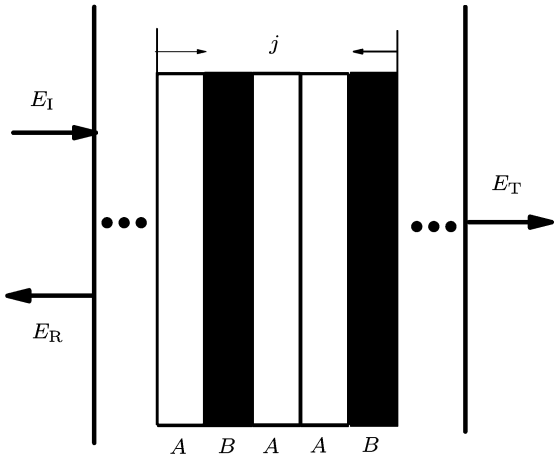


Fig.1. Propagation of light through the reflective and the transmissive surfaces of GF(1,1) PC, where $E_{\rm I}$, $E_{\rm R}$, and $E_{\rm T}$ are the input, reflective, and output electric fields, respectively. A and B represent two types of thin films, j denotes the j-th unit cell.

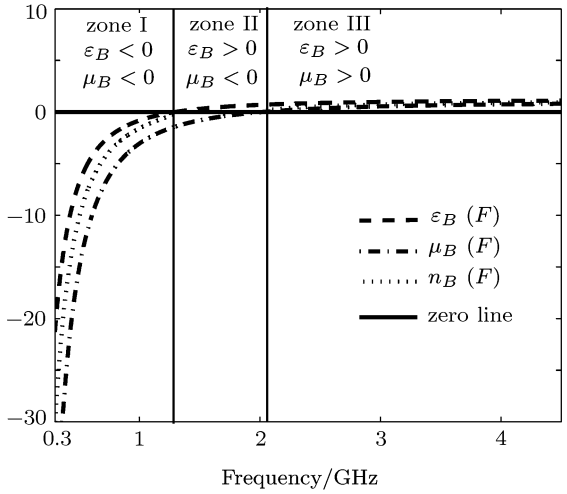


Fig.2. The dispersion curves of material B, which satisfies Eq.(1).

2.2. GF(p, q) sequence

The unit cell which we studied in Fig.1 can be one of the three kinds of aperiodic systems, whose order follows GF$(p,1)$, GF(1,2) and TM sequences. It

is well known that GF(p,q) model is a general extension of Fibonacci sequence and can be generated by the following substitution rules:[46]

$$B \to A, \quad A \to A^p B^q, \tag{2}$$

where p and q are all positive integers. Starting with an A, we can express the first-three generations of GF(p,q) as follows:

$$\begin{aligned} &\mathrm{GF}(p,q)_{G_0} = A, \\ &\mathrm{GF}(p,q)_{G_1} = A^p B^q, \\ &\mathrm{GF}(p,q)_{G_2} = (A^p B^q)^p A^q, \end{aligned} \tag{3}$$

which shows the following recursion relation:

$$\mathrm{GF}(p,q)_{G_l} = \mathrm{GF}(p,q)^p_{G_{l-1}} \mathrm{GF}(p,q)^q_{G_{l-2}}, \ (l \geq 2). \tag{4}$$

By means of Eqs.(3) and (4) one can obtain the number of total elements, N, the number of elements A, N_A, and the number of elements B, N_B as follows:

$$\begin{aligned} &N_{G_0} = 1, \\ &N_{AG_0} = 1, \\ &N_{BG_0} = 0, \end{aligned} \tag{5}$$

$$\begin{aligned} &N_{G_1} = p + q, \\ &N_{AG_1} = p, \\ &N_{BG_1} = q, \end{aligned} \tag{6}$$

$$\begin{aligned} &N_{G_2} = p^2 + pq + q, \\ &N_{AG_2} = p^2 + q, \\ &N_{BG_2} = pq, \end{aligned} \tag{7}$$

$$\begin{aligned} &N_{G_l} = pN_{G_{l-1}} + qN_{G_{l-2}}, \\ &N_{AG_l} = pN_{AG_{l-1}} + qN_{AG_{l-2}}, \\ &N_{BG_l} = pN_{BG_{l-1}} + qN_{BG_{l-2}}, \end{aligned} \tag{8}$$

where $l \geq 2$. Generally, GF$(p,1)$ $(p \geq 1)$ is defined as family A of GF(p,q) while GF$(1,q)$ $(q \geq 2)$ is defined as family B of GF(p,q), and their associated substitution matrices are as follows:[46]

$$M_A = \begin{pmatrix} p & 1 \\ 1 & 0 \end{pmatrix}, \tag{9}$$

$$M_B = \begin{pmatrix} 1 & q \\ 1 & 0 \end{pmatrix}. \tag{10}$$

The corresponding eigenvalues of M_A and M_B are

$$\tau_{A\pm} = \frac{m \pm \sqrt{m^2+4}}{2} \tag{11}$$

and

$$\tau_{B\pm} = \frac{2 \pm \sqrt{1+4n}}{2}, \tag{12}$$

respectively. It shows that M_A satisfies Pisot condition and possesses the unit determinant requirement,[47] then GF$(p,1)$ $(p \geq 1)$ is a kind of real quasiperiodic model and has a volume-preserving trace map and a Cantor-like energy spectrum with critical electronic states.[48] However, GF$(1,q)$ $(q \geq 2)$ has a volume-nonpreserving trace map and Bloch-like part as well as the singular part of the energy spectrum such that extended electronic states are allowed, it is a kind of non-quasiperiodic model. The GF(1,2) is a critical aperiodic model between the quasiperiodic GF$(p,1)$ $(p \geq 1)$ sequence and the non-quasiperiodic GF$(1,q)$ $(q \geq 2)$ chain.

2.3. TM sequence

TM model is a typical aperiodic system and has been widely studied. Its substitution rules are as follows:[49,50]

$$B \to BA, \quad A \to AB. \tag{13}$$

Starting with an A, the first-two generations of TM are:

$$\begin{aligned} &\mathrm{TM}_{G_0} = A, \quad \overline{\mathrm{TM}_{G_0}} = B, \\ &\mathrm{TM}_{G_1} = AB, \quad \overline{\mathrm{TM}_{G_1}} = BA, \end{aligned} \tag{14}$$

which shows the following recursion relation:

$$\begin{aligned} &\mathrm{TM}_{G_l} = \mathrm{TM}_{G_{l-1}} \overline{\mathrm{TM}_{G_{l-1}}}, \\ &\overline{\mathrm{TM}_{G_l}} = \overline{\mathrm{TM}_{G_{l-1}}} \mathrm{TM}_{G_{l-1}}, \end{aligned} \tag{15}$$

where $l \geq 1$ and the $\overline{\mathrm{TM}_{G_l}}$ sequence is complementary to the TM_{G_l} one. Obviously, the even generations of TM sequence are symmetric while the odd ones are antisymmetric. In addition, according to Eqs.(13)–(15), the numbers of elements for TM sequence can be expressed as follows:

$$\begin{aligned} &N_{G_l} = 2^l, \\ &N_{AG_l} = N_{BG_l} = \frac{N_{G_l}}{2} = 2^{l-1}, \end{aligned} \tag{16}$$

where $l \geq 1$. The substitution matrix of TM is[50]

$$M_{\mathrm{TM}} = \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}. \tag{17}$$

One can see that the determinant of this substitution matrix equals zero and is the same as that of periodic systems. Meanwhile, the ratio of the element numbers A to B for TM sequence is rational. It means that TM sequence is a middle model between quasiperiodic models and periodic ones.

3. Theory

The isotropic electromagnetic medium can be generally described by permeability ε and permittivity μ, the electromagnetic wave for transverse electric mode satisfies the following equation:

$$-\frac{Z}{n}\frac{\mathrm{d}}{\mathrm{d}x}\left[\frac{1}{Zn}\frac{\mathrm{d}E(x)}{\mathrm{d}x}\right]=\left(\frac{\omega}{C}\right)^2 E(x), \tag{18}$$

where $Z=\sqrt{\varepsilon}/\sqrt{\mu}$ and $n=\sqrt{\varepsilon}\sqrt{\mu}$ are the impedance and RI at a given frequency, respectively, C is the light velocity in vacuum. For our studied 1D PC system, the above differential equation can be casted in terms of a transfer matrix which relates the incident electromagnetic wave at one side x to the transmitted electromagnetic wave at other side $x+d_i$:

$$\begin{pmatrix} E(x) \\ H(x) \end{pmatrix} = T(d_i)\begin{pmatrix} E(x+d_i) \\ H(x+d_i) \end{pmatrix}, \tag{19}$$

where $E(x)$ and $H(x)$ are the electric and magnetic fields, respectively. The transfer matrix $T(d_i)$ for TE wave can be expressed as follows:

$$T(d_i)=\begin{pmatrix} \cos(k_i d_i\cos\theta_i) & \mathrm{i}\sin(k_i d_i\cos\theta_i)(Z_i\cos\theta_i) \\ \dfrac{\mathrm{i}\sin(k_i d_i\cos\theta_i)}{Z_i\cos\theta_i} & \cos(k_i d_i\cos\theta_i) \end{pmatrix}, \tag{20}$$

where the wave number $k_i=n_i\omega/C$ and θ_i is the refracted angle in the i-th layer. We choose normal incidence for simplicity, then $\theta_i=0$ and $\cos\theta_i=1$. On the other hand, we define that

$$\begin{pmatrix} E(x_j) \\ H(x_j) \end{pmatrix} = \prod_{i=1}^{N} T(d_i)\begin{pmatrix} E(x_{j+1}) \\ H(x_{j+1}) \end{pmatrix} = T\begin{pmatrix} E(x_{j+1}) \\ H(x_{j+1}) \end{pmatrix}, \tag{21}$$

where $E(x_j)$ and $H(x_j)$ are the electric and magnetic fields at the left side of the j-th unit cell. It is generally known that Bloch' boundary condition gives

$$\begin{pmatrix} E(x_j) \\ H(x_j) \end{pmatrix} = \mathrm{e}^{-\mathrm{i}ka}\begin{pmatrix} E(x_{j+1}) \\ H(x_{j+1}) \end{pmatrix}, \tag{22}$$

where a is the lattice constant. It means that

$$T\begin{pmatrix} E(x_{j+1}) \\ H(x_{j+1}) \end{pmatrix} = \mathrm{e}^{-\mathrm{i}ka}\begin{pmatrix} E(x_{j+1}) \\ H(x_{j+1}) \end{pmatrix}. \tag{23}$$

Obviously, the photonic spectra can be obtained by calculating the eigenvalue of the T matrix. When $|(1/2)\mathrm{Tr}(T)|\leq 1$, the wave propagation is allowed, indicating a passband, while $|(1/2)\mathrm{Tr}(T)|>1$, the wave propagation is prohibited, corresponding to a frequency spectra gap.

Based on the aforementioned transfer matrix method we calculate the frequency spectra of GF$(p,1)$, GF$(1,2)$ and TM aperiodic PCs, study the characteristics of the zero-$\bar{n}$ gaps, and investigate the self-similarities of the three bottom-bands.

4. Results and discussions

4.1. Zero-$\bar{n}$ gap

The zero-$\bar{n}$ gap denotes the gap where the system's average RI $\bar{n}$ is equal to zero. For the l-th generation, $\bar{n}$ is defined as

$$\overline{n}_{G_l}=\frac{n_A d_A N_{AG_l}+n_{BG_l}d_B N_{BG_l}}{d_A N_{AG_l}+d_B N_{BG_l}}. \tag{24}$$

For the zero-$\bar{n}$ gap, the RI of material B is

$$n_{BG_l}=-\frac{N_{AG_l}}{N_{BG_l}}\bullet\frac{n_A d_A}{d_B}=-R\bullet\frac{n_A d_A}{d_B}, \tag{25}$$

where R is the ratio of the numbers of elements A to B. Obviously, when $n_A d_A/d_B$ is constant, n_B will be proportional to R. Consequently, from Fig.2 one can see that the zero-$\bar{n}$ gap will open at lower frequencies for bigger R. For GF(p,q) sequence, R possesses the following limit:[51]

$$\lim_{l\to\infty} R=\frac{p+\sqrt{p^2+4q}}{2q}. \tag{26}$$

For TM sequence, from the first generation on, R will be constant:

$$R = 1. \tag{27}$$

Substituting Eqs.(26) and (27) into Eq.(25), one can obtain the RI of material B for the zero-$\bar{n}$ gap as follows:

$$\mathrm{GF}(p,q) : n_{BG_l} \overset{l\to\infty}{\longrightarrow} -\frac{p+\sqrt{p^2+4q}}{2q} \bullet \frac{n_A d_A}{d_B},$$
$$\mathrm{TM} : n_{BG_l} = -\frac{n_A d_A}{d_B}. \tag{28}$$

By means of Eqs.(1) and (28) one can obtain the limited positions of the possible zero-$\bar{n}$ gaps. For GF$(p,1)$ PCs, R becomes larger with the increase of p, this makes the possible limit zero-$\bar{n}$ gaps open at lower frequencies. In Fig.3 we plot the frequency spectra of GF(1,1), GF(2,1), GF(3,1), GF(4,1), GF(1,2), and TM aperiodic PCs. The limit zero-$\bar{n}$ gaps for GF$(1,1)$, GF$(2,1)$, GF$(3,1)$, and GF$(4,1)$ PCs open at frequency $F = 0.7598$, 0.6499, 0.5699, and 0.5112 GHz, respectively. For GF(1,2) PC, the limit of R equals 1. For TM PC, R keeps to be 1 from the first generation on. It makes the limit zero-$\bar{n}$ gaps for both GF(1,2) and TM PCs open at the same frequency $F = 0.8968$ GHz.

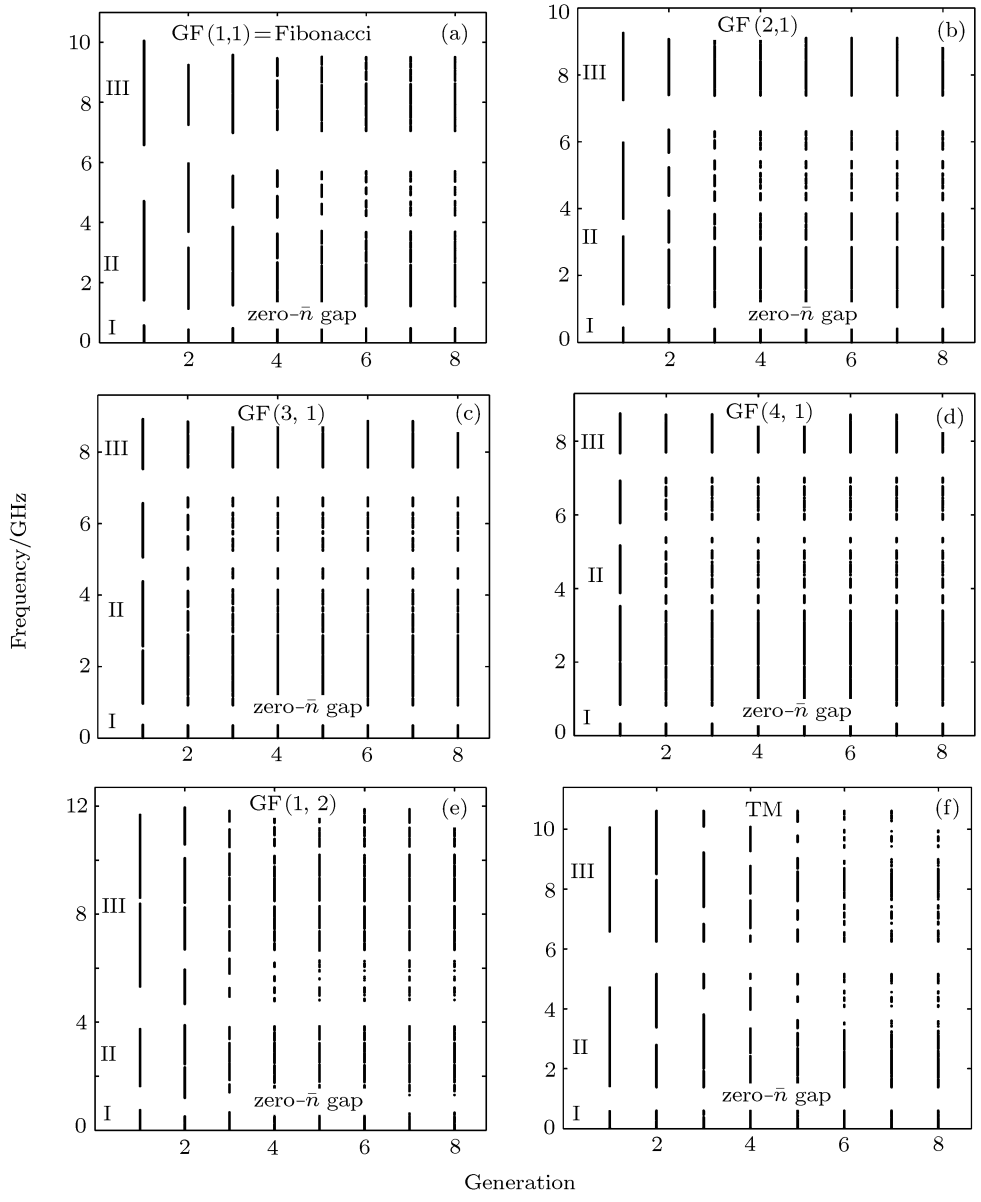


Fig.3. The frequency spectra of GF(1,1), GF(2,1), GF(3,1), GF(4,1), GF(1,2), and TM aperiodic PCs.

From Fig.3 one can also see that, the widths of the zero-$\bar{n}$ gaps tend to be stable with the increase of generations. The stable widths for GF(1,1), GF(2,1), GF(3,1), GF(4,1), GF(1,2), and TM PCs are ΔF = 0.7434, 0.6510, 0.5698, 0.5026, 0.6608, and 0.7911 GHz, respectively. Obviously, TM PC possesses the largest limit zero-$\bar{n}$ gap, the width of GF(1,1) PC is the second, that of GF(1,2) is in the middle, and those of GF(2,1), GF(3,1) and GF(4,1) PCs are the smallest. Furthermore, the width of zero-$\bar{n}$ gap of GF$(p,1)$ PC becomes smaller with the increase of p. It can be explained by two factors: the system symmetry and the open position of the zero-$\bar{n}$ gap. (i) Generally, the higher the symmetry of a system is, the larger the width of its zero-$\bar{n}$ gap will be. It is well known that the symmetry of TM sequence is between periodic and quasiperiodic ones. GF$(p,1)$ ($p\geq$1) model is quasiperiodic system while GF(1,2) sequence is a non-quasiperiodic system. So TM PC possesses the highest symmetry while GF(1,2) sequence possesses the lowest one. (ii) On the other hand, based on the calculations we find that, the lower the open position of zero-$\bar{n}$ gap is, the smaller the width of the limit zero-$\bar{n}$ gap will be. From the aforementioned values one can see that, the limit zero-$\bar{n}$ gaps for TM and GF(1,2) PCs open at the highest frequency, those of GF$(p,1)$ PCs open at lower frequencies. Consequently, the symmetry of TM PC is highest and its limit zero-$\bar{n}$ gap opens at the highest frequency, it makes the width of its limit zero-$\bar{n}$ gap be the largest. The symmetry of GF$(p,1)$ PCs is the second but their limit zero-$\bar{n}$ gaps open at the lowest frequency, it makes the widths of their limit zero-$\bar{n}$ gaps be the second ($p = 1$) or the smallest ($p \geq 2$). The symmetry of GF(1,2) is the lowest but its limit zero-$\bar{n}$ gap opens at the highest frequency, it makes the width of its limit zero-$\bar{n}$ gap be the third. In conclusion, the width of the limit zero-$\bar{n}$ gap of a system is finally dependent on its symmetry and the open position of its limit zero-$\bar{n}$ gap all together.

4.2. Self-similarities of three bottom-bands

Comparing Fig.3 with Fig.2 one can see that the RI of material B, n_B, for band I locates in the dispersion zone I, where B is NRI material. The n_B for band III locates in dispersion zone III and B is positive RI material. The dispersion zone II denotes an intermediate dispersive state, where B is single-negative RI material. It is interesting that the n_B for band II covers dispersion zones I, II, and III. So the three bottom-bands I, II, and III represent all types of frequency spectra of material B and then in this section we investigate the self-similarity of these three bottom-bands but not the total frequency spectra.

4.2.1. GF$(p,1)$ PCs

First, we study the properties of GF$(p,1)$ ($p \geq 1$) PCs. In order to see the self-similarity of the three bottom-bands of GF(1,1) (i.e., Fibonacci) PC clearly we enlarge Fig.3(a), the frequency spectra of GF(1,1) PC, in Fig.4. From Fig.4(a), band III of GF(1,1) PC, one can see that with the increase of generation the subband-number increases following 1, 1, 2, 3, 5, ..., which is equal to the number of elements B of GF(1,1) sequence, and the band-structure is perfectly self-similar. We draw the evolution structure of band III of GF(1,1) PC in Fig.5. Obviously, for band III there are three types of evolving manner, Q_1, Q_2, and Q_3. The Q_1 manner denotes the way for the band in the l-th generation evolving into two subbands in the $(l+1)$-th generation, i.e., 1:2 pattern. The Q_2 manner means 2:3 pattern. Q_3 manner indicates 3:5 pattern. Band II (see Fig.4(b)) has the similar evolving structure as that of band III, the only difference between them is that the beginning of the former is one generation earlier than that of the latter, i.e., the structure of band II for the l-th generation is equal to that of band III for the $(l+1)$-th generation. It means that the subband-number of band II of GF(1,1) is equal to the number of elements A of GF(1,1) sequence. For band I, the frequency tends to be zero infinitely and the spectra can be divided into infinite subband-zones, I-1, I-2, ... For example, we plot the enlarged drawings of the frequency spectra of subband-zones I-1 and I-2 in Figs.4(c) and 4(d). From Figs.4(c) and 4(d) one can see that each subband-zone of band I has the same evolution structure as that of band III.

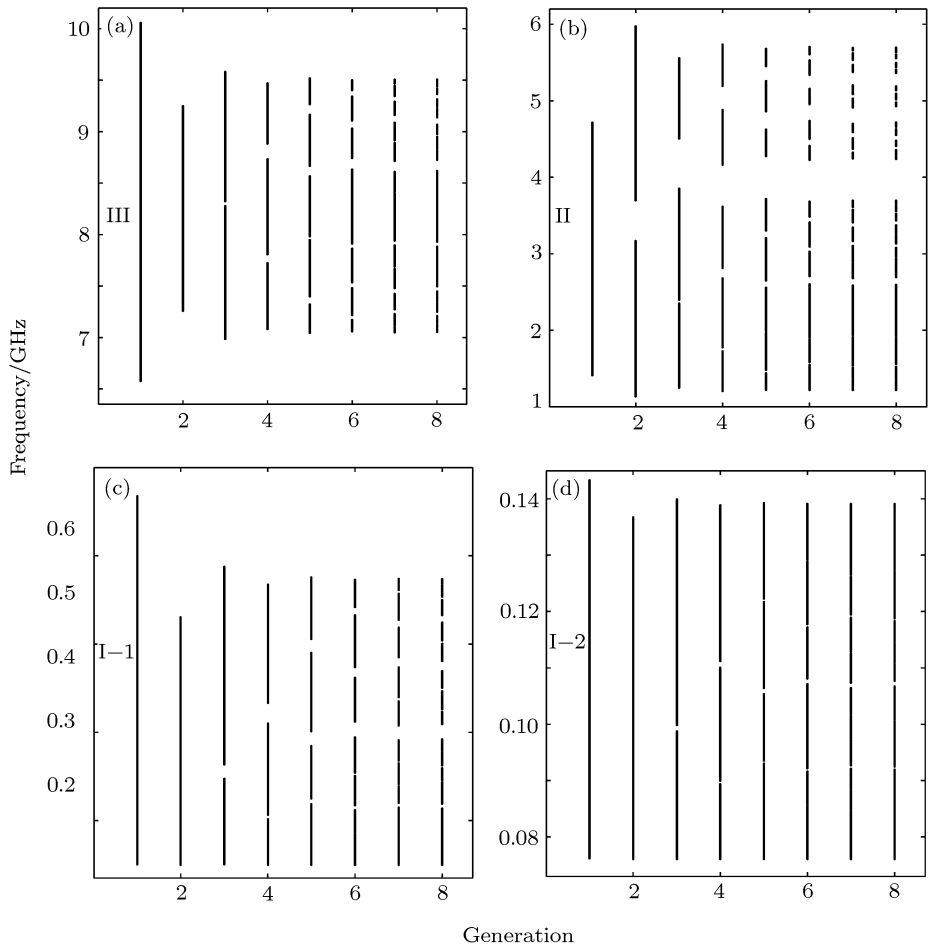


Fig.4. The enlarged drawing of the frequency spectra of GF(1,1) (i.e., Fibonacci) PC.

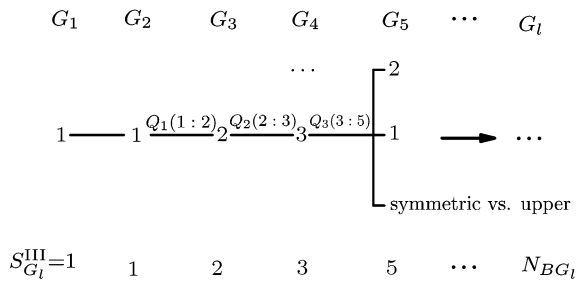


Fig.5. The evolution structure of band III for GF(1,1) PC.

Consequently, the subband-numbers for GF(1,1) (i.e., Fibonacci) PC can be written as follows:

$$\begin{aligned} S^{\rm III}_{G_l}({\rm GF}(1,1)) &= S^{\rm II}_{G_{l-1}}({\rm GF}(1,1) \\ &= S^{\rm I-i}_{G_l}({\rm GF}(1,1)) \\ &= N_{BG_l}({\rm GF}(1,1)) \\ &= N_{AG_{l-1}}({\rm GF}(1,1)), \end{aligned} \tag{29}$$

where S is the subband-number, $l \geq 2$, $i = 1, 2, \ldots$. $N_{AG_l}({\rm GF}(1,1))$ and $N_{BG_l}({\rm GF}(1,1))$ are the numbers of elements A and B of the l-th generation for GF(1,1) sequence, respectively, which can be obtained from Eqs.(5)–(8). When $l = 1$,

$$S^{\rm III}_{G_1}({\rm GF}(1,1)) = S^{\rm I-i}_{G_1}({\rm GF}(1,1)) = 1. \tag{30}$$

From Fig.3 one can also see that when $p \geq 2$, for the self-similarity of the three bands of GF$(p,1)$ PCs, the case of even p is different from that of odd p. As an example of the cases of even p, we enlarge Fig.3(b), the frequency spectra of GF(2,1), in Fig.6. Similarly to GF(1,1) PC, when p is even we plot the evolving structure of band III (see Fig.6(a)) for GF(p,1) PCs in Fig.7. One can see that for band III there are two evolving manners, Q_4 and Q_5. The Q_4 manner indicates that the band in the l-th generation will evolve into p subbands in the $(l+1)$-th generation, i.e., 1:p pattern.

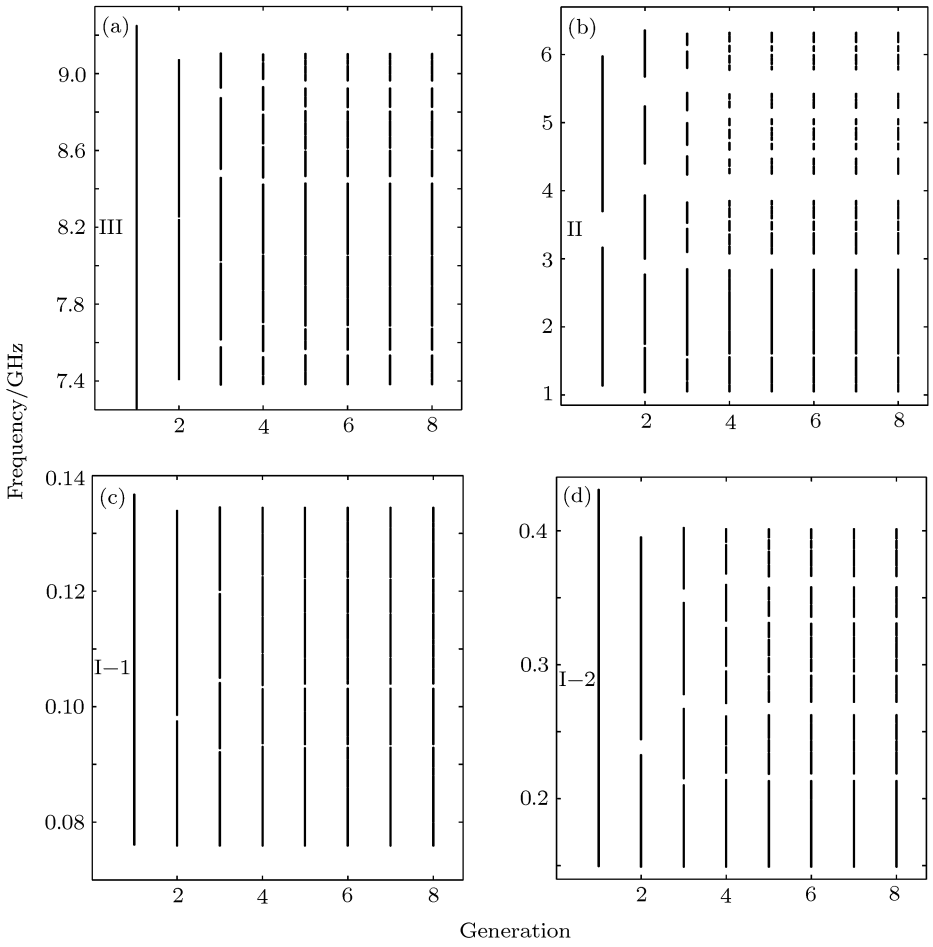


Fig.6. The enlarged drawing of the frequency spectra of GF(2,1) PC.

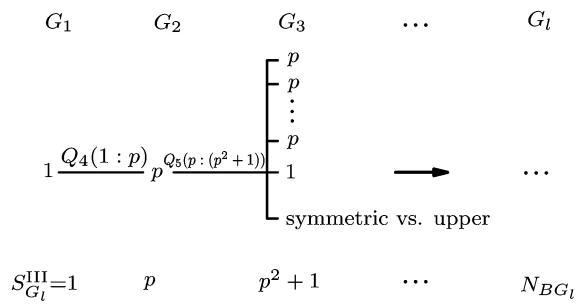


Fig.7. The evolution structure of band III for GF$(p,1)$ PC, where p is even.

The Q_5 manner denotes $p:(p^2+1)$ pattern and the p^2+1 subbands in the $(l+1)$-th generation belong to $p+1$ groups, where the middle group is non-degenerate and the other p groups distribute symmetrically up and down and are all p degenerate. Band II (see Fig.6(b)) has the similar evolving structure as that of band III, the subband-number of band II of GF(p,1) (where p is even) is equal to the number of element A of GF(p,1) (where p is even) sequence. From Figs.6(c) and 6(d) one can see that each subband-zone of band I has the same evolution structure as that of band III.

When p is odd and $p \geq 3$, as an example, we enlarge Fig.3(c), the frequency spectra of GF(3,1), in Fig.8. The evolving structure of band III (see Fig.8(a)) is shown in Fig.9. For band III there are three types of evolving manner, Q_4, Q_6 and Q_7. The Q_6 manner indicates that the p subbands in the l-th generation will evolve into p^2+1 subbands in the $(l+1)$-th generation (i.e., $p:(p^2+1)$ pattern) but this pattern is different from Q_5 pattern. The p^2+1 subbands of Q_6 pattern belong to p groups, where the middle group is $p+1$ degenerate and the other $p-1$ groups distribute symmetrically up and down and are all p degenerate. The Q_7 manner denotes $(p+1):(p^2+p+1)$ pattern and the p^2+p+1 subbands belong to $p+2$ groups, where the middle group is non-degenerate and the other $p+1$ groups distribute symmetrically up and down and are all p degenerate. Band II (see Fig.8(b)) has the similar evolving

structure as that of band III, the subband-number of band II of GF(p,1) (where p is odd) is equal to the number of elements A of GF(p,1) (where p is odd) sequence. From Figs.8(c) and 8(d) one can see that each subband-zone of band I has the same evolution structure as that of band III.

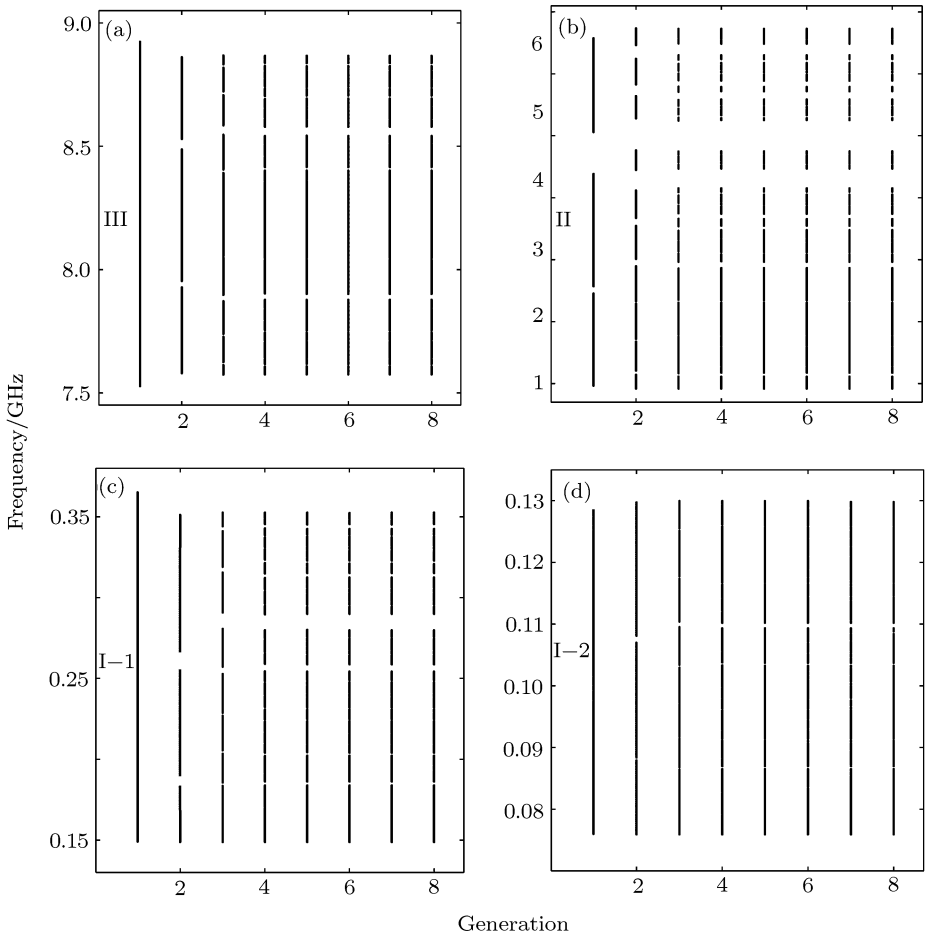


Fig.8. The enlarged drawing of the frequency spectra of GF(3,1) PC.

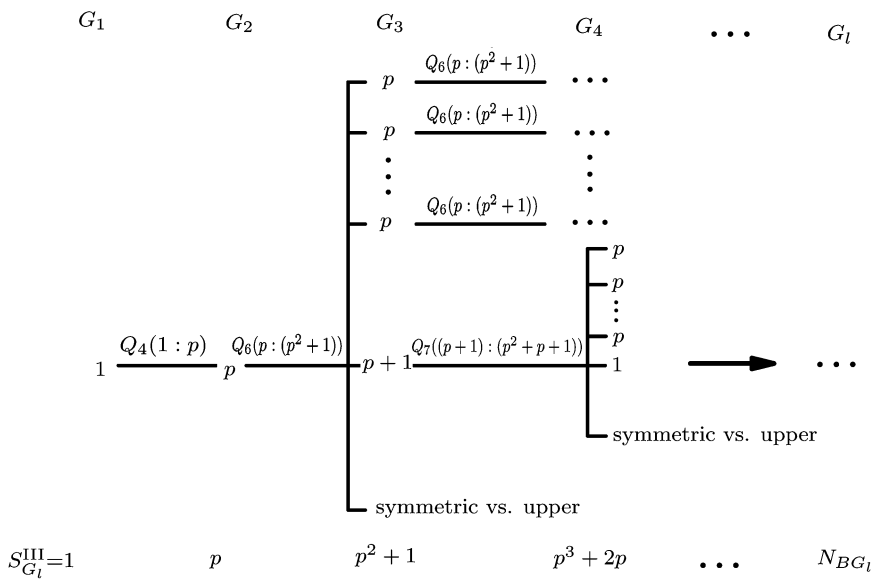


Fig.9. The evolution structure of band III of GF(p, 1) PC, where p is odd.

Obviously, based on the three evolution structures, Figs.5, 7, and 9, one can obtain the subband-numbers for GF(p,1) PCs generally as follows:

$$\begin{aligned} S^{\mathrm{III}}_{G_l}(\mathrm{GF}(p,1)) &= S^{\mathrm{II}}_{G_{l-1}}(\mathrm{GF}(p,1)) \\ &= S^{\mathrm{I}-i}_{G_l}(\mathrm{GF}(1,1)) \\ &= N_{BG_l}(\mathrm{GF}(p,1)) \\ &= N_{AG_{l-1}}(\mathrm{GF}(p,1)), \end{aligned} \tag{31}$$

where $l \geq 2$ and $i = 1, 2, \ldots$. When $l = 1$,

$$S^{\mathrm{III}}_{G_1}(\mathrm{GF}(1,1)) = S^{\mathrm{I}-i}_{G_1}(\mathrm{GF}(1,1)) = 1. \tag{32}$$

4.2.2. GF(1, 2) PC

In order to investigate the self-similarity of GF(1,2), we enlarge Fig.3(e), the frequency spectra of GF(1,2) PC, in Fig.10. From Fig.10 one can see that, the evolving structure of band III (see Fig.10(a)) is similar to that of band II (see Fig.10(b)), but the beginning of the former is one generation earlier than that of the latter. The evolving structure of Band I (see Figs.10(c) and (d)) is different from that of band II and band III. We present the evolving structures of band III and I-i ($i = 1, 2, 3, \ldots$) of GF(1,2) PC in Figs.11 and 12, respectively. Consequently, the formula for the subband-numbers of GF(1,2) PC can be obtained as follows:

$$\begin{aligned} &S^{\mathrm{III}}_{G_l}(\mathrm{GF}(1,2)) = 2^l, \\ &S^{\mathrm{II}}_{G_l}(\mathrm{GF}(1,2)) = 2^{l-1}, \\ &S^{\mathrm{I}-i}_{G_l}(\mathrm{GF}(1,2)) = N_{AG_{l-1}}(\mathrm{GF}(1,2)), \end{aligned} \tag{33}$$

where $l \geq 1$ and $i = 1, 2, \ldots$.

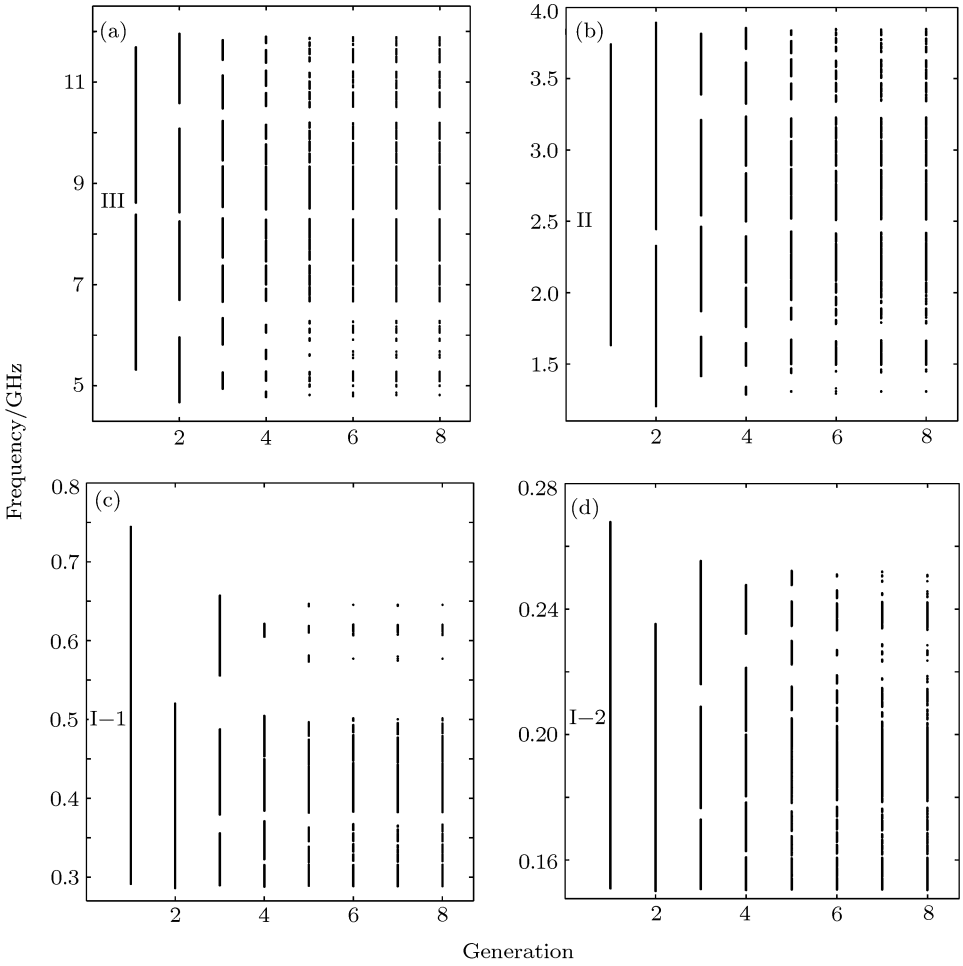


Fig.10. The enlarged drawing of the frequency spectra of GF(1,2) PC.

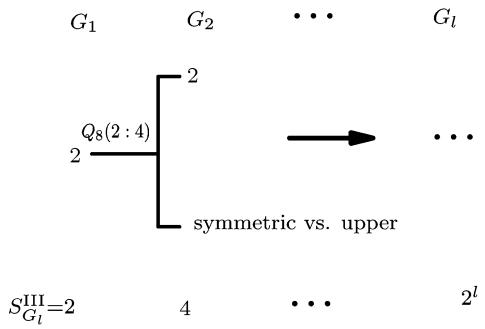


Fig.11. The evolution structure of band III of GF(1,2) PC.

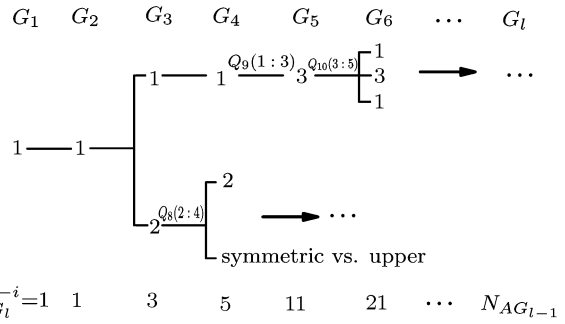


Fig.12. The evolution structure of band I-i ($i = 1, 2, \ldots$) of GF(1,2) PC.

4.2.3. TM PC

For TM PC, we show the enlarged drawing of the frequency spectra of Fig.3(f) in Fig.13. From Fig.13 one can deduce the evolution structure as Fig.14 and obtain the following formula of the subband-number of TM PC:

$$S^{\mathrm{III}}_{G_l}(\mathrm{TM}) = S^{\mathrm{II}}_{G_l}(\mathrm{TM}) = S^{\mathrm{I}-i}_{G_l}(\mathrm{TM}) = 2^{l-2} + 1, \tag{34}$$

where $l \geq 2$ and $i = 1, 2, \ldots$. When $l = 1$, all of the three kinds of S are equal to 1.

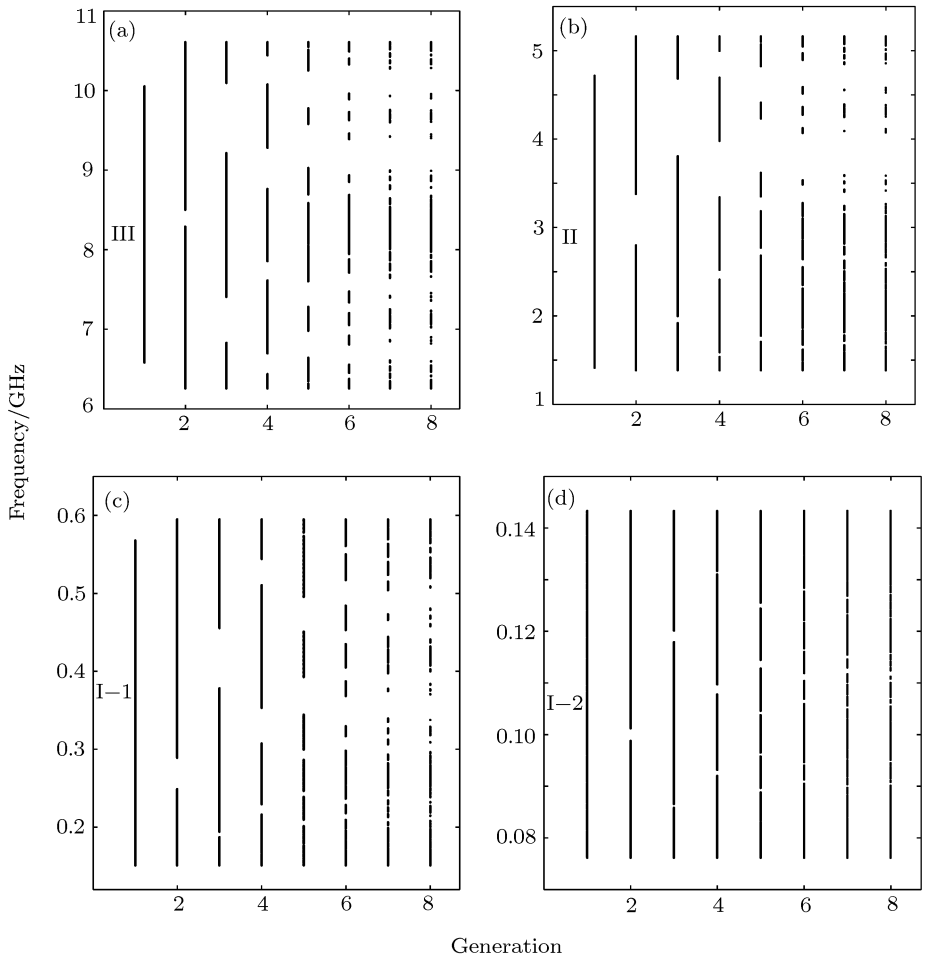


Fig.13. The enlarged drawing of frequency spectra of TM PC.

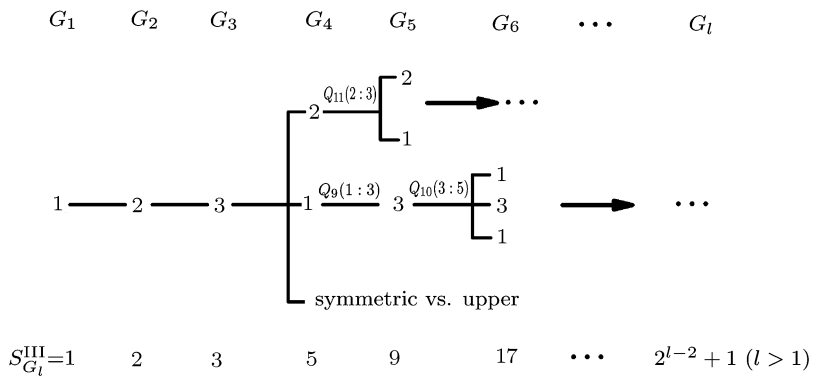


Fig.14. The evolution structure of band III of TM PC.

5. Conclusion

In conclusion, by means of the theory of electromagnetic wave propagation and transfer matrix method we investigated the band rules of the frequency spectra for three kinds of 1D aperiodic PCs, GF$(p,1)$, GF(1,2) and TM PCs, with NRI materials.

First, the limited positions of the possible zero-$\bar{n}$ gaps are studied. It is found that for GF$(p,1)$ PCs, the ratio of the numbers of elements A to B becomes larger with the increase of p, which makes the possible limit zero-$\bar{n}$ gaps open at lower frequencies. For GF(1,2) PC, the limit of the ratio equals 1. For TM PC, the ratio keeps to be 1 from the first generation on. It makes the limit zero-$\bar{n}$ gaps for both GF(1,2) and TM PCs open at the same frequency.

On the other hand, the widths of the zero-$\bar{n}$ gaps have been also calculated. The calculations show that the widths of the zero-$\bar{n}$ gaps tend to be stable with the increase of generations. The TM PC possesses the largest limit zero-$\bar{n}$ gap, the width of GF(1,1) PC is the second, that of GF(1,2) is in the middle, and those of other GF$(p,1)$ PCs are the smallest. With the increase of p, the width for GF$(p,1)$ PC becomes smaller. This characteristic is caused by the system symmetry and the open position of the zero-$\bar{n}$ gap.

Finally, we investigated the self-similarity of the three bottom-bands, found a perfectly self-similar evolution structure with the increase of generation, and obtained the corresponding subband-number formulae. Although the formulae for the subband-numbers of GF$(p,1)$ PCs are uniform, which are all equal to the number of elements B of GF$(p,1)$ sequences, the evolution structures for their spectra are different from each other. There are three types of evolving manners Q_1, Q_2, and Q_3 for GF(1,1) PC, but when p is odd and $p \geq 3$, the evolving structure is composed of Q_4, Q_6 and Q_7. When p is even, there are only two types of evolving manners, Q_4 and Q_5. For GF(1,2) PC, the subband-numbers of bands III and II are similar, which are equal to 2^l and 2^{l-1}, respectively, while that for band I equals the number of elements A of GF(1,2) sequence. Of course, the corresponding evolution structures for the three bottom-bands are different from each other. There is only one kind of evolving manner Q_8 for bands III and II, but for band I-i $(i = 1, 2, \dots)$, the evolving manners are Q_8, Q_9 and Q_{10}. For TM PC, the rule is very simple. Its subband-numbers for all of the three bottom-bands are equal to $2^{l-2}+1$ and there are three types of evolving manners Q_9, Q_{10} and Q_{11}. It would have potential applications for the designing of PCs with NRI materials.

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