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Bifurcation control and chaos in a linear impulsive system*

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Bifurcation control and the existence of chaos in a class of linear impulsive systems are discussed by means of both theoretical and numerical ways. Chaotic behaviour in the sense of Marotto's definition is rigorously proven. A linear impulsive controller, which does not result in any change in one period-1 solution of the original system, is proposed to control and anti-control chaos. The numerical results for chaotic attractor, route leading to chaos, chaos control, and chaos anti-control, which are illustrated with two examples, are in good agreement with the theoretical analysis.

Keywords: periodic solution, bifurcation control, chaos, controller
PACC: 0547, 4610

1. Introduction

Many systems in physics, chemistry, biology, and information science have impulsive dynamical behaviours due to abrupt jumps at certain instants during the evolving processes. These complex dynamical behaviours can be modelled by impulsive differential equations (IDEs).[1,2] Recent years have seen wide-scope potential applications of IDEs in various scientific fields, such as pulse vaccination[3] and impulsive synchronization,[4] etc.

For a dynamical system to be classified as chaotic, it must have the following properties: it must be sensitive to initial conditions, it must be topologically mixing, and its periodic orbits must be dense. Many papers about the chaos theory for continuous and discrete systems appeared during the last decades, (see Refs.[5] and [6]), however, little is known about the existence of chaos in the impulsive differential system. Many authors, such as Wang *et al* [7] and Georgescu and Morosanu[8] obtained the results of the existence of chaos in the impulsive differential system by numerical way. Thus, how to prove that an impulsive differential system is indeed chaotic in a rigorous mathematical sense, is an important problem for study. Jiang *et al* [9] discussed the existence of chaos in an impulsive differential system. Unlike the nonlinear autonomous impulsive system considered in Ref.[9], a linear impulsive system with impulses at periodic fixed time is discussed in this paper. As we know, chaos does not exist in linear continuous and discrete systems, but the situation in linear impulsive system is different. In the following we give the rigorous proof of the existence of chaos in the sense of Marotto's definition in this paper.

Chaos is useful in some cases while harmful in other cases. In recent years, research on chaos control and chaotification (chaos anticontrol) has seen in rapid evolution and extension toward engineering applications. Many methods for chaos control and chaotification were proposed (see Refs.[10]–[12] and references cited there for example). Bifurcation control, is one of these methods and is usually used in continuous and discrete systems. Wang and Abed[13] controlled chaos by controlling associated bifurcations. Ding *et al* [14] discussed bifurcation control in a one-dimensional discrete dynamical system.

In this paper, bifurcation control is introduced in an impulsive system and is used to control and anticontrol chaos; while bifurcation control and the existence of chaos of a linear impulsive system are discussed. The rest of this paper is organized as follows: A linear impulsive system is introduced in Section 2. In Section 3, rigorous proof of existence of chaos is given. In Section 4, a linear impulsive controller is presented and chaos is controlled and anticontrolled

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through bifurcation control. The conclusion is finally presented in Section 5.

2. The linear impulsive system

It is easy to see that to analyse the vibration of a mass m on a spring with a dashpot, a second-order linear differential equation $m\ddot{x} + c\dot{x} + kx = F(t)$ is usually used, where c is the damping constant of the dashpot, k is the spring constant, and $F(t)$ is a given external force. This equation transforms via substituting $\dot{x} = y$ into the two-dimensional linear system

$$\begin{cases} \dot{x} = y, \\ \dot{y} = -\dfrac{k}{m}x - \dfrac{c}{m}y + F(t). \end{cases}$$

In view of the external force $F(t)$, the continuous control is used here. However, in many cases impulsive control is more efficient than continuous control. By using impulsive control of inputs, the velocity of mass m changes impulsively. It follows from the above linear system that

$$\left.\begin{array}{l} \dot{x} = y, \\ \dot{y} = -\dfrac{k}{m}x - \dfrac{c}{m}y, \end{array}\right\} \quad t \neq nT,$$

$$\left.\begin{array}{l} \Delta x = 0, \\ \Delta y = h(x,\ y), \end{array}\right\} \quad t = nT.$$

Moreover, reference [15] has investigated the complex dynamics of the following linear impulsive system

$$\left.\begin{array}{l} \dot{x} = ax + by, \\ \dot{y} = cx - ay, \\ \Delta x = -(1-p)x + h, \\ \Delta y = qy^2 - y, \end{array}\right\} \begin{array}{l} t \neq nT, \\ \\ t = nT, \end{array} \tag{1}$$

where $a > 0$, $b = 0$, $c < 0$, $q > 0$, $0 < p < 1$, $n \in \mathbb{N}_+$, and T is the time between two consecutive pulses; $\Delta x(t)$ and $\Delta y(t)$ are the impulsive terms, with

$$\Delta x(t) = x(t^+) - x(t)\,, \qquad \Delta y(t) = y(t^+) - y(t)\,.$$

The existence and stability of the period-one solution are discussed by using a discrete map and the conditions of existence for flip bifurcation are derived by using centre manifold theorem and bifurcation theorem.

In this paper, we further discuss the chaotic dynamics and bifurcation control of system (1). In the case of a mechanical system (1) subject to an impulsive force, the external impulsive force will make the displacement $x(nT)$ and velocity $y(nT)$ change to $x(nT^+)$ and $y(nT^+)$ instantaneously at $t = nT$, where $x(nT^+) = x(nT) + \triangle x(nT) = px(nT) + h$ and $y(nT^+) = qy^2(nT)$. For more details about the impulsive system and its application see Ref.[2]. It follows from Ref.[15] that the impulsive system (1) is complex in dynamics, although the corresponding linear system without impulses is simple. Impulse plays an important role in the change of dynamics. It generally occurs at the discrete moments τ_1, τ_2, ..., τ_n, ... To reveal the essence of the dynamics, we study a simple and important case: impulses are assumed to occur at periodic fixed moments T, $2T$, ..., nT, ... Hence, it is easy to investigate the effect of impulses on the dynamics of linear systems by viewing T as a parameter in system (1).

In order to discuss chaos, the results of Ref.[15] needed in this paper are as follows. The trajectory of system (1) originating from the initial point $A_k(x_k,\ y_k)$ reaches the point $B_k(\bar{x}_k,\ \bar{y}_k)$, and then jumps to point $A_{k+1}(x_{k+1},\ y_{k+1})$ by the effect of impulse. It follows from system (1) that

$$\begin{cases} x_{k+1} = p\exp(aT)x_k + h, \\ y_{k+1} = q\left(\dfrac{c}{2a}J(T)x_k + \exp(-aT)y_k\right)^2, \end{cases} \tag{2}$$

where $J(T) = \exp(aT) - \exp(-aT)$. For each fixed point of map (2) there is an associated periodic solution of system (1), and *vice versa*. Map (2) has two fixed points

$$\begin{aligned} &B\left(\frac{h}{1-p\exp(aT)},\ \frac{\left(1+\sqrt{H(T)}\right)^2}{4q\exp(-2aT)}\right), \\ &C\left(\frac{h}{1-p\exp(aT)},\ \frac{\left(1-\sqrt{H(T)}\right)^2}{4q\exp(-2aT)}\right), \end{aligned} \tag{3}$$

where

$$H(T) = 1 - \frac{2q\mathrm{ch}(1-\exp(-2aT))}{a(1-p\exp(aT))}. \tag{4}$$

There exists a unique

$$T_0 \in \left(0,\ \frac{1}{a}\ln\frac{1}{p}\right)$$

such that $H(T_0) = 4$. For $0 < T < \frac{1}{a}\ln\frac{1}{p}$, the period-1 solution corresponding to the fixed point B is unstable. The period-1 solution corresponding to the fixed point C is stable for $0 < T < T_0$ while unstable for

$$T_0 < T < \frac{1}{a}\ln\frac{1}{p}.$$

Lemma 1 Assume that the following condition holds:

$$A_4\left(A_7-\frac{A_1apx_0\exp(aT_0)}{1+p\ \exp(aT_0)}\right)+\left(A_3apx_0\exp(aT_0)+A_6+\frac{2A_1apx_0\exp(aT_0)}{1+p\exp(aT_0)}\right)\neq 0.$$

Then a flip bifurcation occurs at $T=T_0$ in system (1). For some $\epsilon>0$, system (1) has a positive stable period-two solution for $T\in(T_0,T_0+\epsilon)$.

For more details about x_0, y_0, A_1, A_2, A_3, A_4, A_5, A_6, and A_7 see Ref.[15].

3. Existence of chaos

Both theoretical and experimental investigations have revealed that three main routes to chaos are the route via torus bifurcation, period-doubling route, and intermittency. Lemma 1 implies that period-doubling bifurcation can exist in system (1). Does chaos exist in our case? Now we search a snap-back repeller[6] to prove that system (1) is indeed chaotic in the sense of Marotto's definition.

Definition 1 Suppose z is a fixed point of f with all eigenvalues of $Df(z)$ exceeding 1 in magnitude and suppose there exists a point $x_0\neq z$ in a repelling neighbourhood $B_r(Z)$ of z such that $x_M=z$ and $|Df(x_k)|\neq 0$ for $1<k<M$, where $x_k=f^k(x_0)$. Then z is called a snap-back repeller of f.

Lemma 2 (Marotto Theorem[6]) If f possesses a snap-back repeller, then the map f is chaotic.

Theorem 1 Assume the following conditions hold:

(F_1) $T_0<T<\frac{1}{a}\ln\frac{1}{p}$,

(F_2) $-1+H(T)-2\sqrt{H(T)+2\sqrt{H(T)}-3}>\exp(2a(T_0-T))$, where $H(T)$ is defined in Eq.(4) and $H(T_0)=4$. Then system (1) is chaotic in the sense of Marotto's definition.

Proof From the above, the fixed points B and C of map (2) are unstable under condition (F_1). Note that the first map $x_{k+1}=p\exp(aT)x_k+h$ of map (2) is independent of y_k; it may be rewritten as $x_{k+1}=(p\exp(aT))^k x_1+\dfrac{1-(p\exp(aT))^k}{1-p\exp(aT)}h$, which possesses the fixed point $\bar{x}=\dfrac{h}{1-p\exp(aT)}$. Substituting this fixed point into the second map of map (2) gives the sub-map $y_{k+1}=G(y_k,\ T)$, where

$$G(y_k,\ T)=q\left(\frac{(1-H(T))\exp(aT)}{4q}+\exp(-aT)y_k\right)^2. \tag{5}$$

Under condition (F_1), one of the fixed points of map (5) is

$$y^*(T)=\frac{\left(1-\sqrt{H(T)}\right)^2}{4q\exp(-2aT)} \tag{6}$$

and

$$\frac{\partial G}{\partial y_k}(y^*(T),\ T)=1-\sqrt{H(T)}<-1. \tag{7}$$

Now suppose $G^2(y_0,\ T)=y^*(T)$, then there exists Y such that

$$q\left(\frac{(1-H(T))\exp(aT)}{4q}+\exp(-aT)y_0\right)^2=Y \tag{8}$$

and

$$q\left(\frac{(1-H(T))\exp(aT)}{4q}+\exp(-aT)Y\right)^2=\frac{\left(1-\sqrt{H(T)}\right)^2}{4q\exp(-2aT)}. \tag{9}$$

It follows from $H(T)>4$ for $T\in\left(T_0,\ \frac{1}{a}\ln\frac{1}{p}\right)$ that equation (9) has two positive roots

$$Y_1=\frac{\left(1-\sqrt{H(T)}\right)^2}{4q\exp(-2aT)},$$
$$Y_2=\frac{H(T)+2\sqrt{H(T)}-3}{4q\exp(-2aT)}. \tag{10}$$

Substituting Y_2 into Eq.(8) yields

$$y_{01}=\frac{-1+H(T)+2\sqrt{H(T)+2\sqrt{H(T)}-3}}{4q\exp(-2aT)},$$
$$y_{02}=\frac{-1+H(T)-2\sqrt{H(T)+2\sqrt{H(T)}-3}}{4q\exp(-2aT)},$$

which means that

$$G^2(y_{02},\ T)=y^*(T). \tag{11}$$

For $T=T_0$, $H(T_0)=4$ and then $y^*(T_0)=\frac{1}{4q\exp(-2aT_0)}$. In view of Eq.(6), $y^{*\prime}(T)>0$ and then $y^*(T)>y^*(T_0)$ for $T\in\left(T_0,\ \frac{1}{a}\ln\frac{1}{p}\right)$.

It follows from condition (F_2) that

$$\frac{-1+H(T)-2\sqrt{H(T)+2\sqrt{H(T)}-3}}{4q\exp(-2aT)} > \frac{1}{4q\exp(-2aT_0)},$$

that is, $y_{02} > y^*(T_0)$. Taking account of $y^*(T) > y_{02}$, now set $r = y^*(T) - y_{02} + \frac{1}{2}(y_{02} - y^*(T_0))$, then in view of Eq.(7),

$$\frac{\partial G}{\partial y_k}(y,\ T) < -1, \quad y \in (y^*(T) - r,\ y^*(T) + r)\,. \tag{12}$$

From sub-map (5),

$$\begin{aligned}\frac{\partial^2 G}{\partial y_k}(y_{02},\ T) &= \frac{\partial G}{\partial y_k}(Y_2,\ T) \times \frac{\partial G}{\partial y_k}(y_{02},\ T)\\ &= 2q\exp(-aT)\left(\frac{1}{4q}(1-H(T))\exp(aT) + \exp(-aT)Y_2\right)\\ &\quad \times 2q\exp(-aT)\left(\frac{1}{4q}(1-H(T))\exp(aT) + \exp(-aT)y_{02}\right)\\ &= -\left(\sqrt{H(T)}-1\right)\sqrt{H(T)+2\sqrt{H(T)}-3}\\ &\neq 0.\end{aligned} \tag{13}$$

In view of Eqs.(6), (11), (13), and inequality (12), $y^*(T)$ is a snap-back repeller. From the Marotto Theorem, sub-map (5) is chaotic, and then system (1) is chaotic under conditions (F_1) and (F_2). The proof is then completed.

Now consider the following system

$$\left\{\begin{array}{l}\left.\begin{array}{l}\dot{x} = x,\\ \dot{y} = -2x - y,\end{array}\right\} t \neq nT,\\ \left.\begin{array}{l}\Delta x = -(1-0.5)x + 0.5,\\ \Delta y = 1.5y^2 - y,\end{array}\right\} t = nT.\end{array}\right. \tag{14}$$

In our case, $a = 1$, $b = 0$, $c = -2$, $p = 0.5$, $q = 1.5$, $h = 0.5$, $\frac{1}{a}\ln\frac{1}{p} = \ln 2$. $H(T_0) = 4$ yields $T_0 = \frac{1}{3}\ln 2$.

Set $T = 0.44$, then condition (F_1) holds obviously. On the other hand,

$$H(T) = 1 - \frac{2q\mathrm{ch}(1-\exp(-2aT))}{a(1-p\exp(aT))} \approx 8.8501,$$

$$-1 + H(T) - 2\sqrt{H(T)+2\sqrt{H(T)}-3} \approx 0.9799,$$

and

$$\exp(2a(T_0 - T)) \approx 0.6584,$$

then condition (F_2) holds. It follows from Theorem 1 that system (14) is chaotic for $T = 0.44$. For the initial point (0.5, 0.1) and $t \in (1555,\ 1580)$, a chaotic solution of system (14) with $T = 0.44$ is given in Fig.1 and the time-series of y is given in Fig.2.

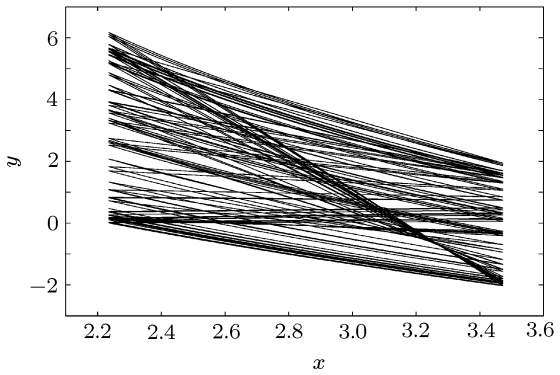


Fig.1. A chaotic solution of system (14) with $T = 0.44$.

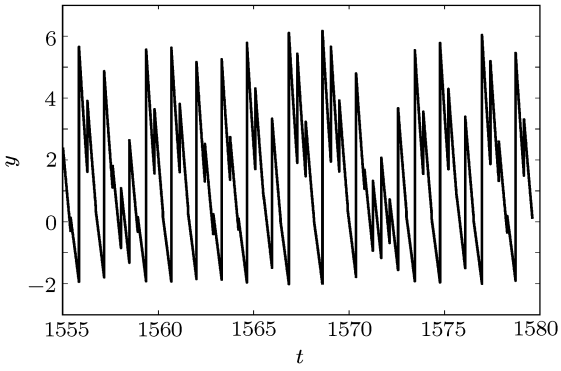


Fig.2. The time-series of y of system (14) with $T = 0.44$.

Figure 3 shows the bifurcation diagram of stable periodic solutions of system (14) and figure 4 shows

the variation of Lyapunov exponents with respect to the parameter T. In our case, the route to chaos is period-doubling bifurcation, and the positive Lyapunov exponents further claim the existence of chaos in system (14).

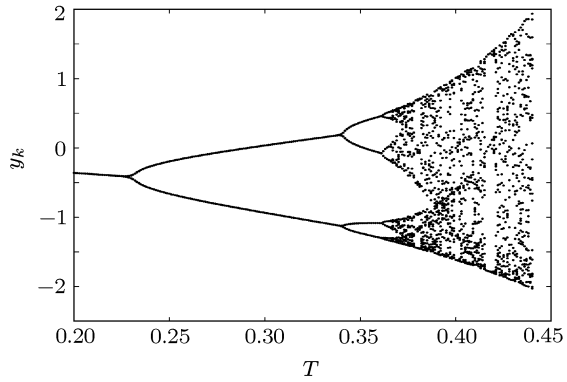


Fig.3. The bifurcation diagram of system (14) with respect to T.

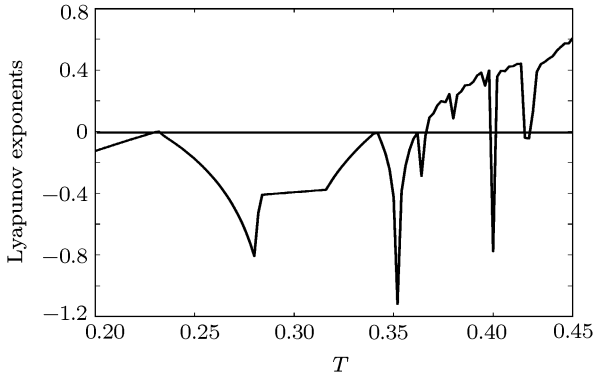


Fig.4. Variation of the Lyapunov exponents with respect to T in map (2).

4. Bifurcation control

Chaos is needed to be controlled or anti-controlled for practical reasons in some cases. To control and anti-control chaos in system (1), we first control the bifurcation in it by using a linear impulsive controller $u = k_1 + k_2 y(t)$ for $t = nT$, that is,

$$\left\{\begin{array}{l} \dot{x} = ax + by, \\ \dot{y} = cx - ay, \\ \Delta x = -(1-p)x + h, \\ \Delta y = qy^2 - y + k_1 + k_2 y, \end{array}\right\} \begin{array}{l} t \neq nT, \\ \\ t = nT. \end{array} \tag{15}$$

From condition (F_1), system (15) is discussed under condition $0 < T < \frac{1}{a}\ln\frac{1}{p}$ in this section. In view of y, k_1, and k_2 in the controller, a hybrid control strategy using both state feedback and parameter perturbation is employed impulsively to control the bifurcation in our case. The trajectory of system (15) originating from the initial point $\bar{A}_k(x_k,\ y_k)$ reaches point $\bar{B}_k(\bar{x}_k,\ \bar{y}_k)$, and next jumps to the point $\bar{A}_{k+1}(x_{k+1},\ y_{k+1})$ due to the effect of impulse. From the first and second equations in system (15), it follows that

$$\left\{\begin{array}{l} \bar{x}_k = x_k \exp(aT), \\ \bar{y}_k = \dfrac{cx_k}{2a}\exp(aT) + \dfrac{y_k - cx_k}{2a}\exp(-aT). \end{array}\right. \tag{16}$$

Taking account of the effect of impulse in system (15), one obtains

$$\left\{\begin{array}{l} x_{k+1} = p\bar{x}_k + h, \\ y_{k+1} = q\bar{y}_k^2 + k_1 + k_2 y. \end{array}\right. \tag{17}$$

From Eqs.(16) and (17), we have the following map

$$\left\{\begin{array}{l} x_{k+1} = p\exp(aT)x_k + h, \\ y_{k+1} = q\left(\dfrac{c}{2a}J(T)x_k + \exp(-aT)y_k\right)^2 \\ \qquad + k_1 + k_2\left(\dfrac{c}{2a}J(T)x_k + \exp(-aT)y_k\right). \end{array}\right. \tag{18}$$

Set

$$k_1 = -k_2\left(\frac{c}{2a}J(T)\frac{h}{1-p\exp(aT)} + \exp(-aT)\frac{\left(1-\sqrt{H(T)}\right)^2}{4q\exp(-2aT)}\right). \tag{19}$$

It is easy to calculate that one of the fixed points of map (18) is one of the fixed points of map (2), that is,

$$C\left(\frac{h}{1-p\exp(aT)},\ \frac{\left(1-\sqrt{H(T)}\right)^2}{4q\exp(-2aT)}\right),$$

and the eigenvalues of the fixed point C are

$$\lambda_{1C} = p\exp(aT),$$
$$\lambda_{2C} = 1 - \sqrt{H(T)} + k_2\exp(-aT). \tag{20}$$

Thus the period-1 solution corresponding to the fixed point C in the system keeps the same under the linear impulsive controller $u = k_1 + k_2 y(t)$ for $t = nT$, where k_1 and k_2 satisfy the condition (19).

It follows from expression (3) that $\lambda_{2C} = -1$ for $T = T_0$. Flip bifurcation occurs at $T = T_0$ in system (1) according to Lemma 1. Under the linear impulsive controller $u = k_1 + k_2 y(t)$ for $t = nT$, the bifurcation value is different from $T = T_0$.

In the case of $k_2 > 0$, there exists some $T^* > T_0$ such that

$$\lambda_{2C} = 1 - \sqrt{H(T^*)} + k_2 \exp(-aT^*) = -1.$$

The stable range of the period-1 solution is extended and the bifurcation is suppressed to occur at $T = T^* > T_0$. Then some chaos in system (1) is controlled.

In the case of $k_2 < 0$, there exists some $T^{**} < T_0$ such that

$$\lambda_{2C} = 1 - \sqrt{H(T^{**})} + k_2 \exp(-aT^{**}) = -1.$$

The stable range of the period-1 solution is compressed and the bifurcation occurs at $T = T^{**} < T_0$. Hence some chaos in system (1) is anti-controlled.

Consider the controlled system

$$\left\{ \begin{array}{l} \dot{x} = x, \\ \dot{y} = -2x - y, \\ \Delta x = -(1-0.5)x + 0.5, \\ \Delta y = 1.5y^2 - y + k_1 + k_2 y, \end{array} \right\} \begin{array}{l} t \neq nT, \\ \\ t = nT. \end{array} \tag{21}$$

Now set $k_2 = 0.3$ in Eq.(19), then $|\lambda_{2C}| = -1$ for $T \approx 0.28$. Flip bifurcation occurs at $T = T_0 = \frac{1}{3}\ln 2 \approx 0.2310$ in system (14), and is suppressed to occur at $T \approx 0.28$ in system (21). The bifurcation diagram of stable periodic solutions of system (21) is given in Fig.5.

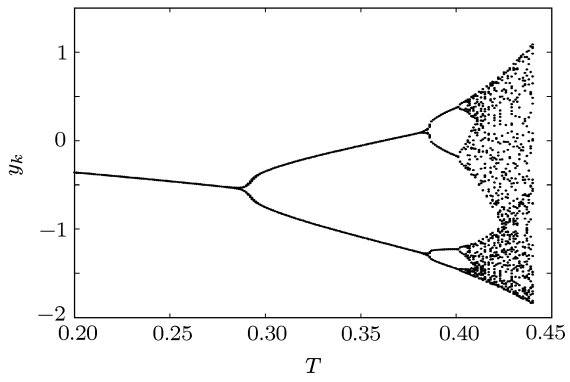


Fig.5. The bifurcation diagram of system (21) with $k_2 = 0.3$.

To show chaos control clearly, set $k_2 = 0.85$ in Eq.(19). Figure 6 shows the bifurcation diagram of stable periodic solutions of system (21) with $k_2 = 0.85$. In Fig.3 we show that system (14) is chaotic for $T = 0.37$. However, figure 6 indicates that the controlled system (21) with $k_2 = 0.85$ has a stable period-1 solution for $T = 0.37$. Figure 7 shows the result of chaos control. The linear impulsive controller $u = k_1 + k_2 y(t)$ is used to control system (14) at $t = nT$, where $n \in \mathbb{N}_+$, $n \geq 160$, and $T = 0.37$. It shows that system (14) is chaotic for $t < 160 \times 0.37 = 59.2$ and is stabilized to the period-1 solution rapidly after applying the linear impulsive controller.

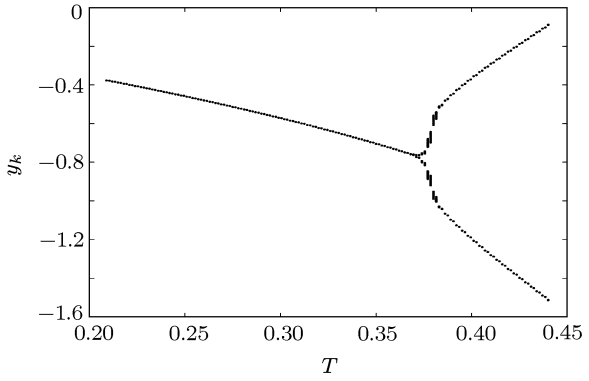


Fig.6. The bifurcation diagram of system (21) with $k_2 = 0.85$.

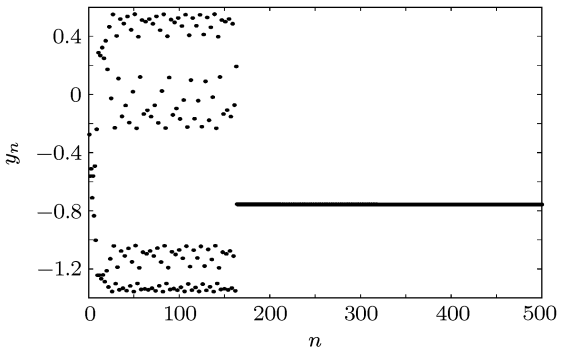


Fig.7. Time response of y (without controlling ($n < 160$), after controlling ($n \geq 160$)), with parameters $k_2 = 0.85$, $T = 0.37$, $n \in \mathbb{N}_+$.

Set $k_2 = -0.3$ in Eq.(19). The bifurcation diagram of stable periodic solutions of system (21) is shown in Fig.8. Flip bifurcation occurs at $T \approx 0.155 < 0.2310$ in advance and the controlled system is chaotic for $T = 0.34$. However, it is shown in Fig.3 that a stable period-2 solution exists in system (14) for $T = 0.34$. The result of chaotifacation (chaos anticontrol) is shown in Fig.9. The linear impulsive controller $u = k_1 + k_2 y(t)$ is used to anticontrol system (14) at $t = nT$, where $n \in \mathbb{N}_+$, $n \geq 120$, and $T = 0.34$. It shows that the chaotic behaviour occurs rapidly after applying this linear impulsive controller.

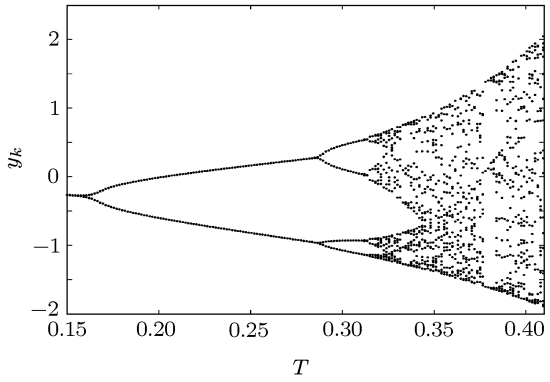


Fig.8. The bifurcation diagram of system (21) with $k_2 = -0.3$.

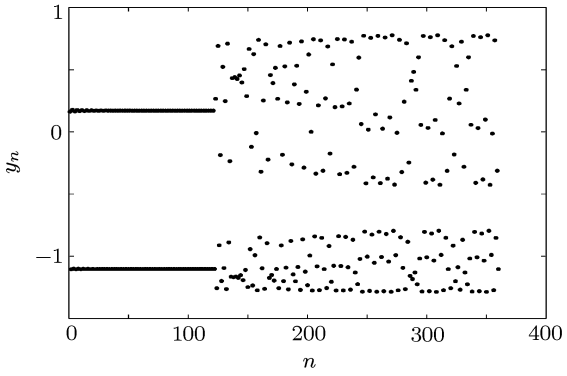


Fig.9. Time response of y (without controlling ($n < 120$), after controlling ($n \geq 120$)), with parameters $k_2 = -0.3$, $T = 0.34$, $n \in \mathbb{N}_+$.

5. Conclusion

The existence of chaos and the bifurcation control of a class of linear impulsive systems are discussed. Unlike Refs.[7] and [8], not only numerical results but also the theoretical analysis of the existence of chaos are given. A linear impulsive controller is proposed to control bifurcation. An important feature of this controller is that it does not result in any change in the period-1 solution of the original system. Numerical simulation results have demonstrated that chaos is controlled and anticontrolled effectively. These results about linear impulsive systems are of significance in the dynamics analysis of nonlinear impulsive systems. We will adopt more real economics, ecology and environment models for further study.

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