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Disturbance rejection and H_∞ pinning control of linear complex dynamical networks*

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This paper concerns the disturbance rejection problem of a linear complex dynamical network subject to external disturbances. A dynamical network is said to be robust to disturbance, if the H_∞ norm of its transfer function matrix from the disturbance to the performance variable is satisfactorily small. It is shown that the disturbance rejection problem of a dynamical network can be solved by analysing the H_∞ control problem of a set of independent systems whose dimensions are equal to that of a single node. A counter-intuitive result is that the disturbance rejection level of the whole network with a diffusive coupling will never be better than that of an isolated node. To improve this, local feedback injections are applied to a small fraction of the nodes in the network. Some criteria for possible performance improvement are derived in terms of linear matrix inequalities. It is further demonstrated via a simulation example that one can indeed improve the disturbance rejection level of the network by pinning the nodes with higher degrees than pinning those with lower degrees.

Keywords: complex dynamical network, pinning control, H_∞ control, robustness
PACC: 0545

1. Introduction

The notion of complex dynamical networks has received compelling attention from various fields of scientific communities in recent years (see Refs.[1]–[6] and references therein). The main motivation is that many physical systems can be represented and characterized by various complex network models, in which nodes are the elements of the system and edges are the interactions among them. Typical examples of complex networks include the Internet, the World Wide Web, metabolic networks, scientific citation webs, and so forth.[7]

Traditionally, complex dynamics have been studied in the framework of random graph theory, introduced by Erdös and Rényi.[8] This model dominates the research of complex networks for nearly half a century. The past few years have witnessed dramatic advances in this direction. Watts and Strogatz introduced the so-called small-world network model to represent the transition from a regular network to a random one.[5] Such networks exhibit a high degree of clustering along with a short average distance between nodes. Another even more important discovery is the scale-free network model, introduced by Barabási and Albert, of which the connectivity distribution has a power-law form.[9] A distinctive feature of scale-free networks is that most nodes have very few connections while a small number of nodes have many connections, leading to the interesting "robust yet fragile" characteristics.[10]

Recently, a large body of literatures have focused on the research of dynamics and control problems of complex dynamical networks. The synchronization phenomenon in both small-world networks and scale-free networks is investigated in Refs.[11]–[17], and the synchronization of general complex dynamical network with coupling delays is considered in Refs.[18]–[20]. The control problem of scale-free networks is investigated by applying local feedback injections only to a small fraction of nodes in Ref.[21], and the pinning control strategy is further used to study the stability of general complex networks with or without coupling delays in Refs.[1], [22]–[26]. The controllability of complex networks via pinning is addressed in Ref.[27]. The synchronizability of complex dynamical networks is completely analysed through a graph and control theoretic viewpoint in Refs.[28]–[30]. Moreover, reference [31] concerns the synchronization problem of dynamical networks from a state-observer ap-

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proach, where only a scalar coupling signal is required to achieve network synchronization, in contrast to the common diagonally coupled networks where full state coupling is typically needed.

This paper studies the disturbance rejection problem of a linear complex dynamical network subject to external disturbances. A dynamical network is said to be robust to disturbance, if the H_∞ norm of its transfer function matrix from the disturbance to the performance variable is satisfactorily small. The disturbance rejection problem of a dynamical network is converted to the H_∞ control problem of a set of independent systems whose dimensions are equal to that of a single node. A counter-intuitive result of this paper is that the disturbance rejection level of the dynamical network with a diffusive coupling will never be better than that of an isolated node. To improve this, local feedback injections are applied to a small fraction of the nodes in the network. Some criteria for possible performance improvement are derived in terms of linear matrix inequalities. It is further demonstrated via a simulation example that one can indeed improve the disturbance rejection level of the network by pinning the nodes with higher degrees than pinning those with lower degrees.

The rest of this paper is organized as follows: A linear dynamical network perturbed by additional disturbances is proposed in Section 2. The H_∞ performance (i.e., the disturbance rejection level) of the network is analysed in Section 3. The pinning control problem of the dynamical network is investigated in Section 4. In Section 5, a simulation example is presented to illustrate the effectiveness of the conditions derived. Section 6 concludes the paper.

2. Dynamical network model

Consider a complex network consisting of N identical linearly and diffusively coupled nodes, with each node being a linear dynamical system perturbed by additional disturbances:

$$\begin{aligned}\dot{x}_i &= Ax_i + \sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}L_1(y_j - y_i) + Bw_i,\\ z_i &= Cx_i + \sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}L_2(y_j - y_i) + Dw_i,\\ i &= 1, 2, \ldots, N,\end{aligned} \tag{1}$$

where $x_i = [x_{i1}, x_{i2}, \ldots, x_{in}]^T \in \boldsymbol{R}^n$ is the state variable, $y_i \in \boldsymbol{R}$ is the output variable given by

$$y_i = Hx_i, \tag{2}$$

where $H = [h_1, h_2, \ldots, h_n] \in \boldsymbol{R}^{1\times n}$ is the output matrix, $w_i \in \boldsymbol{R}^{m_1}$ denotes the external disturbance, $z_i \in \boldsymbol{R}^{m_2}$ is the performance variable of node i, denoting the parameter or a combination of parameters one is interested in; $L_1 = [l_{11}, l_{12}, \ldots, l_{1n}]^{\mathrm{T}}$ and $L_2 = [l_{21}, l_{22}, \ldots, l_{2n}]^{\mathrm{T}} \in \boldsymbol{R}^{n\times 1}$ are the gain matrices to be designed to meet some performance index, and $G = (G_{ij})_{N\times N}$ is the coupling matrix. The entries of G are defined as follows: If there is a connection between node i and node j $(i \neq j)$, then $G_{ij} = G_{ji} = 1$; otherwise, $G_{ij} = G_{ji} = 0$ $(i \neq j)$. The diagonal elements of the coupling matrix (i.e., the degrees of the corresponding nodes) are $G_{ii} = -\sum_{\substack{j=1\\ j\neq i}}^{N} G_{ij}$, for $i = 1, 2, \ldots, N$, which define the diffusive connection of the network. Clearly, all the rows of the coupling matrix G have a zero sum, which implies that zero is an eigenvalue of G, and the corresponding eigenvector is an all-one vector. Assume, hereafter, that the network has no isolated clusters, i.e., the network is connected. Then, the coupling matrix G is symmetric and irreducible. In this case, it can be shown that zero is a simple eigenvalue of G and all the other eigenvalues of G are strictly negative, namely, $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$.[32]

Substituting Eq.(2) into Eq.(1) gives

$$\begin{aligned}\dot{x}_i &= Ax_i + \sum_{j=1}^{N} G_{ij}L_1Hx_j + Bw_i,\\ z_i &= Cx_i + \sum_{j=1}^{N} G_{ij}L_2Hx_j + Dw_i,\\ i &= 1, 2, \ldots, N.\end{aligned} \tag{3}$$

Let $x = [x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}}$, $z = [z_1^{\mathrm{T}}, \ldots, z_N^{\mathrm{T}}]^{\mathrm{T}}$, $w = [w_1^{\mathrm{T}}, \ldots, w_N^{\mathrm{T}}]^{\mathrm{T}}$. Then, one can write the network model in a compact form, as

$$\begin{aligned}\dot{x} &= (I_N \otimes A + G \otimes L_1H)x + (I_N \otimes B)w,\\ z &= (I_N \otimes C + G \otimes L_2H)x + (I_N \otimes D)w,\end{aligned} \tag{4}$$

where $\otimes$ denotes the Kronecker product and I_N means the identity matrix of dimension N.

3. H_∞ performance analysis

The intention of this paper is to investigate the disturbance rejection ability of the complex dynami-

cal network (3) or (4). In order to quantify the disturbance rejection level of the network, the H_∞ norm will be used as the measure. A system is said to possess a better level of disturbance rejection, if the H_∞ norm of its transfer function matrix from the disturbance to the performance variable is smaller. Denote by T_{wz} the transfer function matrix from w to z of system (4). Then, the robustness to disturbance problem of the complex dynamical system (4) can be stated as follows: For a given (small) $\gamma > 0$, find appropriate gain matrices L_1 and L_2, such that (i) the dynamical network (4) is asymptotically stable; (ii) $\|T_{wz}\|_\infty < \gamma$.

Let $0 = \lambda_1 > \lambda_2 \geq \cdots \geq \lambda_N$ be the eigenvalues of the coupling matrix G. Since G is symmetric, there exists a unitary matrix U such that $U^{-1}GU = \Lambda = \mathrm{diag}(\lambda_1, \ldots, \lambda_{\mathrm{N}})$. Let $x = (U \otimes I_n)\xi$, where $\xi = [\xi_1^{\mathrm{T}}, \ldots, \xi_N^{\mathrm{T}}]^{\mathrm{T}}$, and ξ_i has the same dimension as x_i. Then, network (4) can be rewritten in terms of ξ as

$$\begin{aligned} \dot{\xi} &= (I_N \otimes A + \Lambda \otimes L_1 H)\xi + (U^{-1} \otimes B)w, \\ z &= (U \otimes C + GU \otimes L_2 H)\xi + (I_N \otimes D)w. \end{aligned} \quad (5)$$

Further, reformulate the disturbance variable w and the performance variable z via

$$w = (U \otimes I_{m_1})\tilde{w}, \quad z = (U \otimes I_{m_2})\tilde{z}, \quad (6)$$

where $\tilde{w} = [\tilde{w}_1^{\mathrm{T}}, \ldots, \tilde{w}_N^{\mathrm{T}}]^{\mathrm{T}}, \tilde{z} = [\tilde{z}_1^{\mathrm{T}}, \ldots, \tilde{z}_N^{T}]^{T}$. Then, substituting Eq.(6) into Eq.(5) gives

$$\begin{aligned} \dot{\xi} &= (I_N \otimes A + \Lambda \otimes L_1 H)\xi + (I_N \otimes B)\tilde{w}, \\ \tilde{z} &= (I_N \otimes C + \Lambda \otimes L_2 H)\xi + (I_N \otimes D)\tilde{w}. \end{aligned} \quad (7)$$

It is worth while noting that equation (7) is composed of the following N isolated systems

$$\begin{aligned} \dot{\xi}_i &= (A + \lambda_i L_1 H)\xi_i + B\tilde{w}_i, \\ \tilde{z}_i &= (C + \lambda_i L_2 H)\xi_i + D\tilde{w}_i \quad i = 1, 2, \ldots, N. \end{aligned} \quad (8)$$

Denote by $T_{\tilde{w}\tilde{z}}$ and $T_{\tilde{w}_i\tilde{z}_i}$ the transfer function matrices of systems (7) and (8), respectively. Then, it follows from Eqs.(4), (6)–(8) that

$$\begin{aligned} T_{\tilde{w}\tilde{z}} &= \mathrm{diag}(T_{\tilde{w}_1\tilde{z}_1}, \ldots, T_{\tilde{w}_N\tilde{z}_N}), \\ &= (U^{-1} \otimes I_{m_2})T_{wz}(U \otimes I_{m_1}). \end{aligned} \quad (9)$$

Recall that the H_∞ norm of a transfer function matrix $F(s)$ is defined as $\|F(s)\|_\infty = \sup_{w \in \boldsymbol{R}} \bar{\sigma}(F(jw))$,[33] where $\bar{\sigma}(F)$ represents the maximal singular value of F. Then, by observing that both $U \otimes I_{m_1}$ and $U \otimes I_{m_2}$ are unitary matrices, one obtains the relationships between the H_∞ norms of T_{wz}, $T_{\tilde{w}\tilde{z}}$, $T_{\tilde{w}_i\tilde{z}_i}$ from Eq.(9) as follows:

$$\|T_{\tilde{w}\tilde{z}}\|_\infty = \|T_{wz}\|_\infty = \max_{i=1,\ldots,N} \|T_{\tilde{w}_i\tilde{z}_i}\|_\infty. \quad (10)$$

Equation (10) shows that the H_∞ norm of the transformed system (7), which is represented by the maximum of the H_∞ norms of the set of N systems (8), is equal to the H_∞ norm of the original network (4). Hence, the disturbance rejection problem of network (4) is cast into designing some common gain matrices L_1 and L_2 such that all the systems in expression (8) are simultaneously asymptotically stable, and that all the H_∞ norms of their transfer function matrices are less than a desired value. The above discussions are summarized in the following theorem.

Theorem 1 For a given scalar $\gamma > 0$, the dynamical network (4) is asymptotically stable and $\|T_{wz}\|_\infty < \gamma$, if and only if there exist gain matrices L_1 and L_2 such that all the systems in expression (8) are simultaneously asymptotically stable, with $\|T_{\tilde{w}_i\tilde{z}_i}\|_\infty < \gamma$, for $i = 1, 2, \ldots, N$.

Remark 1 When the number N of nodes is large, the stability and disturbance rejection level of the dynamical network is generally very hard to determine. Theorem 1 relaxes this difficulty significantly by converting the disturbance rejection problem of the network into the disturbance rejection problem of a set of independent systems whose dimensions are the same as that of an isolated node. The key tools leading to this result rely on the input and the output transformations, apart from the state transformation. The number of systems to be verified is equal to the number of distinct eigenvalues of the coupling matrix G, sometimes much less than N. For example, for the globally-coupled network studied in Ref.[14], only two systems need to be tested, since in this case G has only two distinct eigenvalues, $-N$ and 0.

Remark 2 Since $\lambda_1 = 0$ in expression (8), the corresponding system is the same as an isolated node, i.e.,

$$\begin{aligned} \dot{\xi}_1 &= A\xi_1 + B\tilde{w}_1, \\ \tilde{z}_1 &= C\xi_1 + D\tilde{w}_1. \end{aligned} \quad (11)$$

Therefore, one can see that the H_∞ norm of network (4) is impossible to be less than γ for any gain matrices L_1 and L_2, if the H_∞ norm of the isolated node (11) is not less than γ. Thus, Theorem 1 yields a counterintuitive result that the H_∞ norm of the whole network (4) is bigger than or equal to that of an isolated

node, which means that the disturbance rejection level of the network as a whole does not become any better, as compared to that of an isolated node. This implies that coupling will not benefit the robustness in the sense of disturbance rejection for the network model under consideration.

Remark 3 A necessary condition for the existence of the desired gain matrix L_1 in Theorem 1 is that (A, H) is observable, or slightly weaker, detectable, which follows immediately from expression (8). Therefore, it will be assumed hereafter that (A, H) is observable.

4. Pinning control for improving robustness

In the last section, it was shown that the disturbance rejection level of the dynamical network (4) is not any better than that of an isolated node. In this section, this is improved by applying local control injections to a small fraction of nodes in the dynamical network, which is known as the pinning control strategy.[1,22−26]

Assume that one applies the pinning control strategy to a small fraction δ $(0 < \delta \ll 1)$ of the nodes in network (4). Without loss of generality, one can rearrange, if necessary, the order of the nodes in the network such that nodes $1, 2, \ldots, q$ in the rearranged network are selected to pin, where $q = [\delta N]$ stands for the smaller but nearest integer to the real number δN. Then, the controlled network can be described by

$$\begin{aligned} \dot{x}_i &= Ax_i + \sum_{j=1}^{N} G_{ij} L_1 H x_j + Bw_i - d_i L_1 H x_i, \\ z_i &= Cx_i + \sum_{j=1}^{N} G_{ij} L_2 H x_j + Dw_i - d_i L_2 H x_i, \end{aligned} \tag{12}$$

where $d_i > 0$ are the feedback gains to be determined, satisfying

$$\begin{aligned} &d_i > 0,\ i \in \{1, 2, \ldots, q\}; \\ &d_i = 0,\ i \in \{q+1, \ldots, N\}. \end{aligned} \tag{13}$$

Let $\hat{D} = \mathrm{diag}(d_1, d_2, \ldots, d_N)$, $x = [x_1^{\mathrm{T}}, \ldots, x_N^{\mathrm{T}}]^{\mathrm{T}}$, $z = [z_1^{\mathrm{T}}, \ldots, z_N^{\mathrm{T}}]^{\mathrm{T}}$, and $w = [w_1^{\mathrm{T}}, \ldots, w_N^{\mathrm{T}}]^{\mathrm{T}}$. Then, equation (12) can be rewritten in a compact form by using the Kronecker product as did in the last section:

$$\begin{aligned} \dot{x} &= (I_N \otimes A + \hat{G} \otimes L_1 H)x + (I_N \otimes B)w, \\ z &= (I_N \otimes C + \hat{G} \otimes L_2 H)x + (I_N \otimes D)w, \end{aligned} \tag{14}$$

where $\hat{G} = G - \hat{D}$.

Lemma 1[22,23] Let $M = (M_{ij})_{N\times N}$ be a symmetric matrix satisfying $M_{ij} \geq 0$, $i \neq j$, and $\sum_{j=1}^{N} M_{ij} = 0$, for $i = 1, 2, \ldots, N$, and $R = \mathrm{diag}(r_1, r_2, \ldots, r_N)$ be a nonzero matrix with $r_i \geq 0$, $i = 1, 2, \ldots, N$. Then, all the eigenvalues of matrix $M - R$ are negative.

From Lemma 1 and the definitions of matrices G and $\hat{D}$, it is obvious that $\hat{G} = G - \hat{D}$ is negative-definite. Let $0 > \hat{\lambda}_1 \geq \hat{\lambda}_2 \geq \cdots \geq \hat{\lambda}_N$ be the eigenvalues of $\hat{G}$. Denote by $\hat{T}_{wz}$ the transfer function matrix from w to z of the controlled network (14). By following the same steps in deriving Theorem 1, one can obtain the following result.

Theorem 2 For a given scalar $\gamma > 0$, the controlled network (14) is asymptotically stable and $\|\hat{T}_{wz}\|_\infty < \gamma$, if and only if there exist gain matrices L_1 and L_2 such that the following N systems are simultaneously asymptotically stable and the H_∞ norms of their transfer function matrices are all less than γ:

$$\begin{aligned} \dot{x}_i &= (A + \hat{\lambda}_i L_1 H)x_i + Bw_i, \\ z_i &= (C + \hat{\lambda}_i L_2 H)x_i + Dw_i, \\ &i = 1, 2, \ldots, N. \end{aligned} \tag{15}$$

Remark 4 In this case, $\hat{\lambda}_i$, $i = 1, \ldots, N$, are all nonzero, which makes it possible that, for some gain matrices L_1 and L_2, the H_∞ norm of the controlled network (14) is less than that of an isolated node, i.e., the disturbance rejection level of the controlled network does become better than that of an isolated node. It is easy to see that one can even achieve this by pinning only one single node of the network.

In what follows, some sufficient conditions are derived for the existence of the required gain matrices L_1 and L_2 in Theorem 2. The next lemma is another representation of the bounded real Lemma.[32]

Lemma 2[34] Consider system (11), and denote by $T_{\tilde{w}_1\tilde{z}_1}$ the transfer function matrix from $\tilde{w}_1$ to $\tilde{z}_1$. Then, $\|T_{\tilde{w}_1\tilde{z}_1}\|_\infty < \gamma$, where $\gamma > 0$ is a prescribed value, if and only if there exist matrices $P = P^{\mathrm{T}} > 0$, F, and K such that

$$\begin{bmatrix} -K^{\mathrm{T}}-K & P-F+A^{\mathrm{T}}K^{\mathrm{T}} & KB & 0 \\ P-F^{\mathrm{T}}+KA & A^{\mathrm{T}}F^{\mathrm{T}}+FA & FB & C^{\mathrm{T}} \\ B^{\mathrm{T}}K^{\mathrm{T}} & B^{\mathrm{T}}F^{\mathrm{T}} & -\gamma I & D^{\mathrm{T}} \\ 0 & C & D & -\gamma I \end{bmatrix} < 0. \tag{16}$$

By introducing two slack variables, F and K, the Lyapunov matrix P in LMI (16) is rendered independent of the dynamical matrices of systems (11). Matrices F and G need not be symmetric nor positive-definite, thus providing extra degrees of freedom for the design. With Lemma 2, an LMI method can be applied to the design of the gain matrices L_1 and L_2.

Theorem 3 The controlled network (14) is asymptotically stable, with $\|T_{wz}\|_\infty < \gamma$, if for some scalar μ, there exist matrices $P_i > 0 \in \boldsymbol{R}^{n\times n}$, $K \in \boldsymbol{R}^{n\times n}$, $S \in \boldsymbol{R}^{n\times 1}$ and $Q \in \boldsymbol{R}^{n\times 1}$ that together satisfy the following inequalities for $i = 1, N$:

$$\begin{bmatrix} -K-K^{\mathrm{T}} & \Phi_{1,i} & KB & 0 \\ \Phi_{1,i}^{\mathrm{T}} & \Phi_{2,i} & \mu KB & C^{\mathrm{T}}+\hat{\lambda}_i H^{\mathrm{T}}Q^{\mathrm{T}} \\ B^{\mathrm{T}} & \mu B^{\mathrm{T}} & -\gamma I & D^{\mathrm{T}} \\ 0 & C+\hat{\lambda}_i QH & D & -\gamma I \end{bmatrix} < 0, \tag{17}$$

where

$$\Phi_{1,i} = P_i - \mu K + AK^{\mathrm{T}} + \hat{\lambda}_i H^{\mathrm{T}}S^{\mathrm{T}},$$
$$\Phi_{2,i} = \mu(KA + \hat{\lambda}_i SH + K^{\mathrm{T}}A^{\mathrm{T}} + \hat{\lambda}_i H^{\mathrm{T}}S^{\mathrm{T}}).$$

If inequality (17) holds for $i = 1, N$, then the gain matrices are given by $L_1 = K^{-1}S$ and $L_2 = Q$.

Proof From Lemma 2, the set of systems in Eq.(15) are asymptotically stable and the H_∞ norms of their transfer function matrices are less than γ, if and only if

$$\begin{bmatrix} -K_i-K_i^{\mathrm{T}} & \Omega_{1,i} & K_iB & 0 \\ \Omega_{1,i}^{\mathrm{T}} & \Omega_{2,i} & F_iB & (C+\hat{\lambda}_i L_2H)^{\mathrm{T}} \\ B^{\mathrm{T}}K_i^{\mathrm{T}} & B^{\mathrm{T}}F_i^{\mathrm{T}} & -\gamma I & D^{\mathrm{T}} \\ 0 & C+\hat{\lambda}_i L_2H & D & -\gamma I \end{bmatrix} < 0, \quad i = 1, 2, \ldots, N, \tag{18}$$

where $\Omega_{1,i} = P_i - F_i + (A+\hat{\lambda}_i L_1H)^{\mathrm{T}}K_i^{\mathrm{T}}$ and $\Omega_{2,i} = (A+\hat{\lambda}_i L_1H)^{\mathrm{T}}F_i^{\mathrm{T}} + F_i(A+\hat{\lambda}_i L_1H)$. Note that in Ref.[18], the gain matrix L_1 is coupled with matrices F_i and K_i, $i = 1, 2, \ldots, N$, so it is necessary to impose the following additional constraint in order that the problem is tractable:

$$K_1 = K_2 = \cdots = K_N = K,$$
$$F_1 = F_2 = \cdots = F_N = \mu K, \tag{19}$$

where μ is a scalar to be determined. Let $S = KL_1$, so one obtains the set of inequalities in (17) for $i = 1, 2, \ldots, N$. For any fixed μ, the inequalities (17) are LMIs about the variables $P_i > 0$ ($i = 1, 2, \ldots, N$), K, S and $Q = L_2$, and thus have the convex property. Therefore, only the two LMIs in inequalities (17) corresponding to the largest eigenvalue $\hat{\lambda}_1$ and the smallest eigenvalue $\hat{\lambda}_N$, need to be verified. If these two LMIs hold, then the other $N-2$ LMIs in inequalities (17), for $i = 2, \ldots, N-1$, hold as well.

If inequalities (17) are feasible for $i = 1, N$, then $K + K^{\mathrm{T}} > 0$, which implies that K is nonsingular. Therefore, one obtains the gain matrices as $L_1 = K^{-1}S$ and $L_2 = Q$.

Remark 5 A tuning parameter μ is contained in inequalities (17), so it is actually a quasi-convex optimization problem. The optimum value in Theorem 3 can be ascertained by following the approach proposed in Remark 5 of Ref.[35].

5. A simulation example

In this section, a simple simulation example is shown, which demonstrates the effectiveness of the conditions derived in this paper. Consider a dynamical network composed of 8 nodes, described by system (1) with

$$A = \begin{bmatrix} 0 & 5 \\ -2 & -6 \end{bmatrix}, \quad B = \begin{bmatrix} 4 \\ 2 \end{bmatrix}, \\ C = [\,2 \quad 0\,], \quad D = 0. \tag{20}$$

Choose the output matrix $H = [2 \;\; 3]$. The H_∞ norm of an isolated node is 6.8, obtained by the LMI Toolbox in MATLAB.[36]

Suppose the coupling matrix is

$$G = \begin{bmatrix} -6 & 1 & 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & -3 & 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & -3 & 0 & 1 & 0 & 0 \\ 1 & 0 & 1 & 0 & -4 & 0 & 1 & 1 \\ 1 & 0 & 0 & 1 & 0 & -2 & 0 & 0 \\ 1 & 0 & 1 & 0 & 1 & 0 & -4 & 1 \\ 1 & 0 & 0 & 0 & 1 & 0 & 1 & -3 \end{bmatrix}. \tag{21}$$

The objective is to stabilize this dynamical network and meanwhile render the disturbance level γ to be less than 2 by first applying local linear feedback injections to only one node, and then designing the gain matrices L_1 and L_2.

Consider here two different pinning control schemes: to pin the node with the biggest degree, i.e. node 1, and to pin the node with the smallest degree, i.e. node 2.

To pin node 1, choose $d_1 = 1$. The eigenvalues of $\hat{G}$ are $\{-0.11, -0.9593, -1.4259, -2.5938, -3.8916, -5, -5.0248, -7.9946\}$. Let $\gamma = 2$, $\mu = 1$ in Theorem 3. By employing the LMI Toolbox in MATLAB, one can find feasible gain matrices $L_1 = [250.8463, 136.8075]^{\mathrm{T}}$ and $L_2 = 0.7438$, along with the following two common positive-definite matrices:

$$P_1 = \begin{bmatrix} 294.9981 & 396.1785 \\ 396.1785 & 589.5869 \end{bmatrix}, \\ P_N = \begin{bmatrix} 11.1679 & -2.2124 \\ -2.2124 & 14.1850 \end{bmatrix}, \tag{22}$$

such that conditions (17) are satisfied, for $i = 1, 8$. Therefore, by Theorem 3, the controlled network (14) is asymptotically stable, and the disturbance rejection level of the controlled dynamical network is 0.28973, which is less than $\gamma = 2$.

To pin node 2, choose $d_2 = 1$. The eigenvalues of $\hat{G}$ in this case are $\{-0.0642, -1.4221, -1.8460, -2.5947, -3.8862, -4.9818, -5, -7.2051\}$. Let the gain matrices be as before, i.e., $L_1 = [250.8463, 136.8075]^{\mathrm{T}}$ and $L_2 = [0.7438]$. The stability of the controlled network is guaranteed and the disturbance rejection level of the dynamical network is 0.42914 in this case, which is also less than $\gamma = 2$ but larger than the one obtained above.

Comparing the above two pinning schemes, one can see that the dynamical network yields a better level of disturbance rejection by applying a pinning control strategy to the nodes with higher degrees than to those with lower degrees. This is in accordance with the intuition that the "big" nodes with higher degrees play a more important role in stabilizing the dynamical network and in attenuating external disturbances.

6. Conclusions

In this paper, the disturbance rejection problem of a dynamical network perturbed by external disturbances has been addressed. It is shown that the disturbance rejection problem of a high-dimensional dynamical network can be solved by converting it into the H_∞ control problem of a set of independent systems whose dimensions are equal to that of a single node. A surprising result is presented that the disturbance rejection level of the whole network with a diffusive coupling will never be better than that of an isolated node. To improve this, pinning control strategy is applied to a small fraction of nodes in the network. It is then shown, via a simulation example, that the dynamical network will yield a better level of disturbance rejection by pinning the nodes with higher degrees than pinning those with lower degrees. It implies that the "big" nodes with higher degrees play a more important role in stabilizing the dynamical network and in attenuating external disturbances. This result may be further extended to networks with inhomogeneous connectivity distributions, such as scale-free dynamical networks. The disturbance rejection problem of complex dynamical networks with coupling delays is yet another interesting topic for future research.

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