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A delay-decomposition approach for stability of neural network with time-varying delay*

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This paper studies delay-dependent asymptotical stability problems for the neural system with time-varying delay. By dividing the whole interval into multiple segments such that each segment has a different Lyapunov matrix, some improved delay-dependent stability conditions are derived by employing an integral equality technique. A numerical example is given to demonstrate the effectiveness and less conservativeness of the proposed methods.

Keywords: neural system, global asymptotical stability, time-varying delay

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1. Introduction

In recent decades, much attention has been devoted to the studies of neural networks due to the fact that neural networks can be applied to signal processing, image processing, pattern recognition, control and optimization problems. It is well known that the existence of time delay may cause oscillations and instability in neural networks. Therefore, the stability analysis of neural networks with time delay is an important issue.[1−23] The criteria for such systems can be classified into two categories, namely, delay-independent criteria[1−4] and delay-dependent criteria.[5−23] Generally speaking, the former cases tend to be conservative especially for small delays.

Recently, various approaches have been proposed to obtain stability criteria for time-delay neural networks. In Refs.[15]–[17], some sufficient conditions for global stability of neural networks have been provided, yet only constant delays are allowed in their results. But in practice, time-delay is usually time-varying, which can even largely change the dynamics of system in some cases. Therefore, the stability of neural networks with time varying delays has become more interesting than that of networks with constant time delays. Although stability criteria for neural networks with time-varying delay were derived in Refs.[5], [18]–[20], the constraint $\dot{h}(t) < 1$ on time-varying delay was imposed. Such a restriction is very conservative and has physical limitations. Recently, He *et al*[10,24] and Wu *et al*[25] proposed a new method for dealing with time-delay systems, which employs free weighting matrices to express the relationships between the terms in the Leibniz–Newton formula, and all the negative terms in the derivative of the Lyapunov functional are retained. This approach avoids the restriction on the derivative of a time-varying delay. Moveover, in Ref.[15], by dividing the delay interval into multiple segments and choosing proper functions with different weighted matrices corresponding to different segments, a novel delay-dependent stability criterion is derived. However, the author in Ref.[15] employed an integral inequality technique to handle the term $-\int_{t-i\tau}^{t-(i-1)\tau} \dot{x}^{\mathrm{T}}(s) Z_i \dot{x}(s) \mathrm{d}s$, which may lead to considerable conservativeness.

In this paper, the asymptotic stability of the neural network with time-varying delay is studied. By employing a special Lyapunov functional form based on delay-decomposition approach, some new delay-dependent stability conditions are derived in terms of integral equality technique. The proposed criterion is both delay-dependent and rate-dependent. Therefore the methods in this paper are less conservative than those by previous approaches. Finally, a numerical example is given to show the effectiveness of the proposed methods.

Note: Throughout this paper, the T stands for matrix transposition. R^n is the n-dimensional Euclidean space. $R^{n\times m}$ is the set of all $(n \times m)$-dimensional matrices. I denotes the identity matrix of ap-

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propriate dimensions. $P > 0$ means that P is positive-definite. The $*$ represents the elements below the main diagonal of a symmetric matrix.

2. Problem statement

The neural network with time-varying delay in this paper can be described by the following normalized equations

$$\begin{aligned}\dot{u}(t) = &-Cu(t) + Af(u(t)) \\ &+A^{\tau} f(u(t-h(t))) + J, \end{aligned} \tag{1}$$

where $u(\cdot) = (u_1(\cdot), u_2(\cdot), \ldots, u_n(\cdot))^{\mathrm{T}} \in R^n$ is the state vector with the n neurons; $f(u(\cdot)) = (f_1(u(\cdot)), f_2(u(\cdot)), \ldots, f_n(u(\cdot)))^{\mathrm{T}} \in R^n$ is called an activation function indicating how the j-th neuron responses to its input; $C = \mathrm{diag}(c_1, \ldots, c_n)$ is a diagonal matrix with each $c_i > 0$ controlling the rate with which the i-th unit will reset its potential to the resting state in isolation when disconnected from the network and external inputs; $A = (a_{ij})_{n\times n}$ and $A^{\tau} = (a_{ij}^{\tau})_{n\times n}$ are the feedback and the delayed feedback matrix, respectively; $J = (J_1, J_2, \ldots, J_n)^{\mathrm{T}} \in R^n$ is a constant input vector, and $h(t)$ is time-varying delay, which satisfies

$$0 \le h(t) \le h < \infty, \quad |\dot{h}(t)| \le \mu, \tag{2}$$

where h and μ are constants. Throughout this paper, it is assumed that each of the activation functions f_j $(j = 1, 2, \ldots, n)$ possess the following condition

$$\begin{aligned}&0 \le \frac{f_i(u) - f_i(v)}{u - v} \le k_i, \\ &u \ne v \in R, \quad i = 1, 2, \ldots, n, \end{aligned} \tag{3}$$

where k_i $(i = 1, 2, \ldots, n)$ are positive constants.

Next, the equilibrium point $u^* = (u_1^*, u_2^*, \ldots, u_n^*)$ of system (1) is shifted to the origin through the transformation $x(t) = u(t) - u^*$, then system (1) can be equivalently written as the following system

$$\dot{x}(t) = -Cx(t) + Ag(x(t)) + A^{\tau} g(x(t-h(t))), \tag{4}$$

where $x(\cdot) = (x_1(\cdot), x_2(\cdot), \ldots, x_n(\cdot))^{\mathrm{T}}$, $g(x(\cdot)) = (g_1(x_1(\cdot)), g_2(x_2(\cdot)), \ldots, g_n(x_n(\cdot)))^{\mathrm{T}}$, and $g_i(x_i(\cdot)) = f_i(x_i(\cdot) + u_i^*) - f_i(u_i^*), i = 1, 2, \ldots, n$. It is obvious that the function $g_j(\cdot)$ $(j = 1, 2, \ldots, n)$ satisfies the following condition,

$$0 \le \frac{g_i(x_i)}{x_i} \le k_i, \quad g_i(0) = 0,$$

$$\forall x_i \ne 0, \quad i = 1, 2, ..., n, \tag{5}$$

which is equivalent to

$$\begin{aligned}&g_i(x_i)(g_i(x_i) - k_i x_i) \le 0, \quad g_i(0) = 0, \\ &\forall x_i \ne 0, \quad i = 1, 2, ..., n. \end{aligned} \tag{6}$$

In order to derive the maximum allowable delay for stability, we divide $h(t)$ into multiple segments by an integer N with the length of each segment denoted by $\tau(t)$, i.e.,

$$\tau(t) = \frac{h(t)}{N}.$$

Therefore, $\tau(t)$ represents a partition of the time-varying delay $h(t)$, which satisfies

$$0 \le \tau(t) \le \tau < \infty, \quad |\dot{\tau}(t)| \le \nu, \tag{7}$$

where $\tau = h/N, \nu = \mu/N$.

Before processing further, the following lemma is introduced which will be used in the proof of the main results.

Lemma 1[26] Given any constant appropriately dimensioned matrices R, N, X, a scalar $h > 0$ and vector value function $f(x)$ and η such that the following integration is well defined, then

$$\begin{aligned}&-\int_{t-h}^{t} f^{\mathrm{T}}(s) R f(s) \\ &= 2\eta^{\mathrm{T}} N \int_{t-h}^{t} f(s)\mathrm{d}s + h\eta^{\mathrm{T}} X \eta \\ &\quad -\int_{t-h}^{t} \begin{pmatrix} \eta \\ f(s) \end{pmatrix}^{\mathrm{T}} \begin{pmatrix} X & N \\ N^{\mathrm{T}} & R \end{pmatrix} \begin{pmatrix} \eta \\ f(s) \end{pmatrix} \mathrm{d}s. \end{aligned} \tag{8}$$

3. Stability analysis

In this Section, we first present a delay-dependent asymptotical stability criterion for the system (1) with time-varying delay (2).

Theorem 1 For given scalars τ, ν and μ, the origin of system (1) with time-varying delay (2) is globally asymptotically stable if there exist positive definite matrices $P > 0$, $Q > 0$, $Q_i > 0$, $Z_i > 0$, diagonal matrices $S \ge 0$, $\Lambda_j \ge 0$ and appropriately dimensioned matrices $M_i = [M_{i1}^{\mathrm{T}}, M_{i2}^{\mathrm{T}}, \ldots, M_{i,N+3}^{\mathrm{T}}]^{\mathrm{T}}, i = 1, 2, \ldots, N; j = 1, 2$, such that the following symmetric linear matrix inequality (LMI) holds:

$$\begin{pmatrix} \Theta & \Gamma^{\mathrm{T}}\Lambda & \tau M_1 & \tau M_2 & \dots & \tau M_N \\ \Lambda^{\mathrm{T}}\Gamma & -\Lambda & 0 & 0 & \dots & 0 \\ \tau M_1^{\mathrm{T}} & 0 & -\tau Z_1 & 0 & \dots & 0 \\ \tau M_2^{\mathrm{T}} & 0 & 0 & -\tau Z_2 & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots \\ \tau M_N^{\mathrm{T}} & 0 & 0 & 0 & \dots & -\tau Z_N \end{pmatrix} < 0, \tag{9}$$

where

$$\Gamma = [-C, 0, A, A^{\tau}, 0, \dots, 0], \quad \Lambda = \tau \sum_{i=1}^{N} Z_i,$$

$$\Theta = \Theta_1 + \Theta_2 + \Theta_2^{\mathrm{T}},$$

$$\Theta_1 = \begin{pmatrix} \Theta_{11} & 0 & \Theta_{13} & PA^{\tau} & 0 & 0 & \dots & 0 \\ * & \Theta_{22} & 0 & K\Lambda_2 & 0 & 0 & \dots & 0 \\ * & * & \Theta_{33} & SA^{\tau} & 0 & 0 & \dots & 0 \\ * & * & * & \Theta_{44} & 0 & 0 & \dots & 0 \\ * & * & * & * & \Theta_{55} & 0 & \dots & 0 \\ * & * & * & * & * & \Theta_{66} & \dots & 0 \\ \dots & \dots & \dots & \dots & \dots & \dots & \dots & \dots \\ * & * & * & * & * & * & \dots & \Theta_{i+4,i+4} \end{pmatrix},$$

$$K = \mathrm{diag}\{k_1, k_2, \dots, k_n\},$$

$$\Theta_{11} = -PC - C^{\mathrm{T}}P + Q_1,$$

$$\Theta_{13} = PA - CS + K\Lambda_1,$$

$$\Theta_{22} = -(1 - N\nu)Q_N,$$

$$\Theta_{33} = SA + A^{\mathrm{T}}S + Q - 2\Lambda_1,$$

$$\Theta_{44} = -(1 - \mu)Q - 2\Lambda_2,$$

$$\Theta_{i+4,i+4} = -(1 - i\nu)Q_i + (1 + i\nu)Q_{i+1}, \quad i = 1, 2, \dots, N-1,$$

$$\Theta_2 = (M_1, -M_N, 0, 0, -M_1 + M_2, \dots, -M_{N-1} + M_N).$$

Proof Choose a Lyapunov functional candidate $V(t)$ for the system (1)

$$V(t) = V_1(t) + V_2(t) + V_3(t) + V_4(t), \tag{10}$$

where

$$V_1(t) = x^{\mathrm{T}}(t)Px(t), \tag{11}$$

$$V_2(t) = 2\sum_{i=1}^{n} s_i \int_0^{x_i(t)} g_i(s)\mathrm{d}s + \int_{t-h(t)}^{t} g^{\mathrm{T}}(x(s))Qg(x(s))\mathrm{d}s, \tag{12}$$

$$V_3(t) = \sum_{i=1}^{N} \int_{t-i\tau(t)}^{t-(i-1)\tau(t)} x^{\mathrm{T}}(s)Q_i x(s)\mathrm{d}s, \tag{13}$$

$$V_4(t) = \sum_{i=1}^{N} \int_{-i\tau(t)}^{-(i-1)\tau(t)} \int_{t+\Theta}^{t} \dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\mathrm{d}s\mathrm{d}\Theta. \tag{14}$$

Here $P = P^{\mathrm{T}} > 0$, $Q = Q^{\mathrm{T}} > 0$, $Q_i = Q_i^{\mathrm{T}} > 0$, $Z_i = Z_i^{\mathrm{T}} > 0$ $(i = 1, 2, \dots, N)$ and diagonal matrice $S = \mathrm{diag}\{s_1, s_2, \dots, s_n\} \geq 0$ are to be determined.

In addition, for given appropriately dimensioned matrices M_i, X_i and vector value function $\xi(t)$, by Lemma 1 we can derive

$$-\sum_{i=1}^{N} \int_{t-i\tau(t)}^{t-(i-1)\tau(t)} \dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\mathrm{d}s = 2\sum_{i=1}^{N} \xi^{\mathrm{T}}(t)M_i \int_{t-i\tau(t)}^{t-(i-1)\tau(t)} \dot{x}(s)\mathrm{d}s + \tau(t)\sum_{i=1}^{N} \xi^{\mathrm{T}}(t)X_i\xi(t)$$

$$-\sum_{i=1}^{N}\int_{t-i\tau(t)}^{t-(i-1)\tau(t)}\begin{pmatrix}\xi(t)\\ \dot{x}(s)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}X_i & M_i\\ M_i^{\mathrm{T}} & Z_i\end{pmatrix}\begin{pmatrix}\xi(t)\\ \dot{x}(s)\end{pmatrix}\mathrm{d}s.$$

Now, calculating the derivative of $V(t)$ with respect to t along the solution of system (1) yields

$$\begin{aligned}
\dot{V}_1(t) &= 2x^{\mathrm{T}}(t)P\dot{x}(t)\\
&= 2x^{\mathrm{T}}(t)P\left(-Cx(t)+Ag(x(t))+A^{\tau}g\left(x(t-h(t))\right)\right)\\
&= x^{\mathrm{T}}(t)(-PC-C^{\mathrm{T}}P)x(t)+2x^{\mathrm{T}}(t)PAg(x(t))+2x^{\mathrm{T}}(t)PA^{\tau}g\left(x(t-h(t))\right), \qquad (15)\\
\dot{V}_2(t) &\le 2g^{\mathrm{T}}(x(t))S\dot{x}(t)+g^{\mathrm{T}}(x(t))Qg(x(t))-(1-\mu)g^{\mathrm{T}}(x(t-h(t)))Qg(x(t-h(t)))\\
&= 2g^{\mathrm{T}}(x(t))S\left(-Cx(t)+Ag(x(t))+A^{\tau}g\left(x(t-h(t))\right)\right)\\
&\quad +g^{\mathrm{T}}(x(t))Qg(x(t))-(1-\mu)g^{\mathrm{T}}(x(t-h(t)))Qg(x(t-h(t))), \qquad (16)\\
\dot{V}_3(t) &= \sum_{i=1}^{N}\left[(1-(i-1)\dot{\tau}(t))\,x^{\mathrm{T}}\left(t-(i-1)\tau(t)\right)Q_ix\left(t-(i-1)\tau(t)\right)-(1-i\dot{\tau}(t))\,x^{\mathrm{T}}\left(t-i\tau(t)\right)Q_ix\left(t-i\tau(t)\right)\right]\\
&\le \sum_{i=1}^{N}\left[(1+(i-1)\nu)\,x^{\mathrm{T}}\left(t-(i-1)\tau(t)\right)Q_ix\left(t-(i-1)\tau(t)\right)\right. \qquad (17)\\
&\quad \left.-\,(1-i\nu)\,x^{\mathrm{T}}\left(t-i\tau(t)\right)Q_ix\left(t-i\tau(t)\right)\right]\\
&= x^{\mathrm{T}}(t)Q_1x(t)+x^{\mathrm{T}}(t-\tau(t))\left(-(1-\nu)Q_1+(1+\nu)Q_2\right)x(t-\tau(t))+\dots\\
&\quad +x^{\mathrm{T}}(t-\tau(t))\left(-(1-(N-1)\nu)Q_{N-1}+(1+(N-1)\nu)Q_N\right)x(t-\tau(t))\\
&\quad -(1-N\nu)x^{\mathrm{T}}(t-h(t))Q_Nx(t-h(t)), \qquad (18)\\
\dot{V}_4(t) &\le \tau\dot{x}^{\mathrm{T}}(t)\sum_{i=1}^{N}Z_i\dot{x}(t)-\sum_{i=1}^{N}\int_{t-i\tau(t)}^{t-(i-1)\tau(t)}\dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\mathrm{d}s\\
&= \tau\dot{x}^{\mathrm{T}}(t)\sum_{i=1}^{N}Z_i\dot{x}(t)+\tau(t)\sum_{i=1}^{N}\xi^{\mathrm{T}}(t)X_i\xi(t)+2\sum_{i=1}^{N}\xi^{\mathrm{T}}(t)M_i\int_{t-i\tau(t)}^{t-(i-1)\tau(t)}\dot{x}(s)\mathrm{d}s\\
&\quad -\sum_{i=1}^{N}\int_{t-i\tau(t)}^{t-(i-1)\tau(t)}\begin{pmatrix}\xi(t)\\ \dot{x}(s)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}X_i & M_i\\ M_i^{\mathrm{T}} & Z_i\end{pmatrix}\begin{pmatrix}\xi(t)\\ \dot{x}(s)\end{pmatrix}\mathrm{d}s\\
&\le \tau\dot{x}^{\mathrm{T}}(t)\sum_{i=1}^{N}Z_i\dot{x}(t)+2\sum_{i=1}^{N}\xi^{\mathrm{T}}(t)M_i\left[x\left(t-(i-1)\tau(t)\right)-x\left(t-i\tau(t)\right)\right]\\
&\quad +\tau\sum_{i=1}^{N}\xi^{\mathrm{T}}(t)X_i\xi(t)-\sum_{i=1}^{N}\int_{t-i\tau(t)}^{t-(i-1)\tau(t)}\begin{pmatrix}\xi(t)\\ \dot{x}(s)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}X_i & M_i\\ M_i^{\mathrm{T}} & Z_i\end{pmatrix}\begin{pmatrix}\xi(t)\\ \dot{x}(s)\end{pmatrix}\mathrm{d}s, \qquad (19)
\end{aligned}$$

where

$$\xi(t)=\left[x^{\mathrm{T}}(t),x^{\mathrm{T}}(t-h(t)),g(x(t)),g(x(t-h(t))),\dots x^{\mathrm{T}}(t-\tau(t)),\dots,x^{\mathrm{T}}\left(t-(N-1)\tau(t)\right)\right]^{\mathrm{T}}. \qquad (20)$$

From Eq.(6), for appropriately dimensioned diagonal matrices $\Lambda_j\ge 0$ $(j=1,2)$, we have

$$-2g^{\mathrm{T}}(x(t))\Lambda_1\left[g(x(t))-Kx(t)\right]\ge 0, \qquad (21)$$

$$-2g^{\mathrm{T}}(x(t-h(t)))\Lambda_2\left[g(x(t-h(t)))-Kx(t-h(t))\right]\ge 0. \qquad (22)$$

In addition, choosing $X_i=M_iZ_i^{-1}M_i^{\mathrm{T}}$ can guarantee

$$\begin{pmatrix}X_i & M_i\\ M_i^{\mathrm{T}} & Z_i\end{pmatrix}\ge 0.$$

Then combining Eqs.(15)–(22) with $X_i=M_iZ_i^{-1}M_i^{\mathrm{T}}$, we can derive

$$\dot{V}(t)\le\dot{V}_1(t)+\dot{V}_2(t)+\dot{V}_3(t)+\dot{V}_4(t)-2g^{\mathrm{T}}(x(t))\Lambda_1\left[g(x(t))-Kx(t)\right]$$

$$
\begin{aligned}
&- 2g^{\mathrm{T}}(x(t-h(t)))\Lambda_2\left[g(x(t-h(t)))-Kx(t-h(t))\right]\\
\le\ & x^{\mathrm{T}}(t)(-PC-C^{\mathrm{T}}P+Q_1)x(t)+2x^{\mathrm{T}}(t)(PA-C^{\mathrm{T}}S+K^{\mathrm{T}}\Lambda_1)g(x(t))\\
&+2x^{\mathrm{T}}(t)PA^{\tau}g(x(t-h(t)))+2g^{\mathrm{T}}(x(t-h(t)))(\Lambda_2K)x(t-h(t))\\
&+2g^{\mathrm{T}}(x(t))SA^{\tau}g(x(t-h(t)))+g^{\mathrm{T}}(x(t-h(t)))(-(1-\mu)Q-2\Lambda_2)g(x(t-h(t)))\\
&+g^{\mathrm{T}}(x(t))(SA+A^{\mathrm{T}}S+Q-2\Lambda_1)g(x(t))+x^{\mathrm{T}}(t-h(t))(-(1-N\nu)Q_N)x(t-h(t))\\
&+x^{\mathrm{T}}(t-\tau(t))\left(-(1-\nu)Q_1+(1+\nu)Q_2\right)x(t-\tau(t))+\dots\\
&+x^{\mathrm{T}}(t-(N-1)\tau(t))\left(-(1-(N-1)\nu)Q_{N-1}+(1+(N-1)\nu)Q_N\right)x(t-(N-1)\tau(t))\\
&+\tau\dot{x}^{\mathrm{T}}(t)\sum_{i=1}^{N}Z_i\dot{x}(t)+\tau\sum_{i=1}^{N}\xi^{\mathrm{T}}(t)M_iZ_i^{-1}M_i^{\mathrm{T}}\xi(t)+2\sum_{i=1}^{N}\xi^{\mathrm{T}}(t)M_i\left[x\left(t-(i-1)\tau(t)\right)-x\left(t-i\tau(t)\right)\right]\\
&-\sum_{i=1}^{N}\int_{t-i\tau(t)}^{t-(i-1)\tau(t)}\begin{pmatrix}\xi(t)\\ \dot{x}(s)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}X_i & M_i\\ M_i^{\mathrm{T}} & Z_i\end{pmatrix}\begin{pmatrix}\xi(t)\\ \dot{x}(s)\end{pmatrix}\mathrm{d}s\\
\le\ & \xi^{\mathrm{T}}(t)\left[\Theta_1+\Theta_2+\Theta_2^{\mathrm{T}}+\Gamma^{\mathrm{T}}\Lambda\Gamma+\tau\sum_{i=1}^{N}M_iZ_i^{-1}M_i^{\mathrm{T}}\right]\xi(t). \qquad (23)
\end{aligned}
$$

Denote

$$\Xi=\Theta_1+\Theta_2+\Theta_2^{\mathrm{T}}+\Gamma^{\mathrm{T}}\Lambda\Gamma+\tau\sum_{i=1}^{N}M_iZ_i^{-1}M_i^{\mathrm{T}}.$$

If matrix inequality (9) is feasible, then by Schur complement we can obtain $\Xi<0$. Next, let $\lambda=\lambda_{\min}(-\Xi)$, from Eq.(23) it follows that $\dot{V}(t)\le-\lambda\parallel x(t)\parallel^2$. Therefore, according to Ref.[27], if there exist symmetric positive-definite matrices $P>0$, $Q>0$, $Q_i>0$, $Z_i>0$, diagonal matrices $S\ge0$, $\Lambda_j\ge0$ and appropriately dimensioned matrices $M_i=[M_{i1}^{\mathrm{T}},M_{i2}^{\mathrm{T}},\dots,M_{iN}^{\mathrm{T}}]^{\mathrm{T}}$, $(i=1,2,\dots,N;j=1,2)$ such that the LMI (9) is satisfied, then system (1) with time-varying delay (2) is asymptotically stable. This completes the proof.

Remark 1 Theorem 1 gives a delay-dependent and rate-dependent asymptotical stability criterion for system (1) by employing a delay-decomposition approach as in Ref.[15]. However, in Ref.[15], $-\int_{t-i\tau}^{t-(i-1)\tau}\dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\mathrm{d}s$ is often enlarged as

$$-\tau\left(\int_{t-i\tau}^{t-(i-1)\tau}\dot{x}(s)\mathrm{d}s\right)^{\mathrm{T}}Z_i\left(\int_{t-i\tau}^{t-(i-1)\tau}\dot{x}(s)\mathrm{d}s\right),$$

which may lead to considerable conservativeness. In this work, by employing an integral equality technique, $-\int_{t-i\tau(t)}^{t-(i-1)\tau(t)}\dot{x}^{\mathrm{T}}(s)Z_i\dot{x}(s)\mathrm{d}s$ can be written as

$$
\begin{aligned}
&2\sum_{i=1}^{N}\xi^{\mathrm{T}}(t)M_i\int_{t-i\tau(t)}^{t-(i-1)\tau(t)}\dot{x}(s)\mathrm{d}s\\
&+\tau(t)\sum_{i=1}^{N}\xi^{\mathrm{T}}(t)X_i\xi(t)\\
&-\sum_{i=1}^{N}\int_{t-i\tau(t)}^{t-(i-1)\tau(t)}\begin{pmatrix}\xi(t)\\ \dot{x}(s)\end{pmatrix}^{\mathrm{T}}\begin{pmatrix}X_i & M_i\\ M_i^{\mathrm{T}} & Z_i\end{pmatrix}\begin{pmatrix}\xi(t)\\ \dot{x}(s)\end{pmatrix}\mathrm{d}s,
\end{aligned}
$$

which have potential to yield less conservative results. Compared with those in Refs.[10], [13]–[15], the method in this paper is more useful.

For $N=1$, Theorem 1 will reduce to the following Corollary.

Corollary 1 For given scalars h and μ, the origin of system (1) with time-varying delay (2) is globally asymptotically stable if there exist positive-definite matrices $P>0$, $Q>0$, $Q_1>0$, $Z_1>0$, diagonal matrices $S\ge0$, $\Lambda_j\ge0$ $(j=1,2)$ and appropriately dimensioned matrices $M_1=[M_{11}^{\mathrm{T}},M_{12}^{\mathrm{T}},M_{13}^{\mathrm{T}},M_{14}^{\mathrm{T}}]^{\mathrm{T}}$, such that the following symmetric linear matrix inequality holds

$$
\begin{pmatrix}
\Phi_{11} & \Phi_{12} & \Phi_{13} & \Phi_{14} & hM_{11} & -hC^{\mathrm{T}}Z_1\\
* & \Phi_{22} & -M_{13}^{\mathrm{T}} & \Phi_{24} & hM_{12} & 0\\
* & * & \Phi_{33} & SA^{\tau} & hM_{13} & hA^{\mathrm{T}}Z_1\\
* & * & * & \Phi_{44} & hM_{14} & hA^{\tau\mathrm{T}}Z_1\\
* & * & * & * & -hZ_1 & 0\\
* & * & * & * & * & -hZ_1
\end{pmatrix}<0, \qquad (24)
$$

where

$$K = \mathrm{diag}\{k_1, k_2, \ldots, k_n\},$$
$$\Phi_{11} = -PC - C^{\mathrm{T}}P + Q_1 + M_{11} + M_{11}^{\mathrm{T}},$$
$$\Phi_{12} = -M_{11} + M_{12}^{\mathrm{T}},$$
$$\Phi_{13} = PA - CS + K\Lambda_1 + M_{13},$$
$$\Phi_{14} = PA^{\tau} + M_{14},$$
$$\Phi_{24} = K\Lambda_2 - M_{14}^{\mathrm{T}},$$
$$\Phi_{22} = -(1-\mu)Q_1 - M_{12} - M_{12}^{\mathrm{T}},$$
$$\Phi_{33} = SA + A^{\mathrm{T}}S + Q - 2\Lambda_1,$$
$$\Theta_{44} = -(1-\mu)Q - 2\Lambda_2.$$

Remark 2 Obviously, Corollary 1 in this work is equivalent to Corollary 1 in Ref.[10], then Corollary 1 in Ref.[10] is a special case of Theorem 1 in this paper. In addition, as pointed out in Ref.[10], Corollary 1 in Ref.[10] is equivalent to Theorem 1 in Ref.[14], so Theorem 1 in this work also includes Theorem 1 in Ref.[14] as a special case.

When $h(t) \equiv h$ (h is a constant), we can easily get a delay-dependent asymptotical stability criterion for neural system Σ with constant delay. We will show the obtained stability criterion for this case in Corollary 2.

Corollary 2 For given scalars $\tau(t) \equiv \tau$, the origin of system (1) is globally asymptotically stable if there exist positive-definite matrices $P > 0$, $Q > 0$, $Q_i > 0$, $Z_i > 0$, diagonal matrices $S \geq 0$, $\Lambda_j \geq 0$ and appropriately dimensioned matrices $M_i = [M_{i1}^{\mathrm{T}}, M_{i2}^{\mathrm{T}}, \ldots, M_{i,N+3}^{\mathrm{T}}]^{\mathrm{T}}$, $i = 1, 2, \ldots, N; j = 1, 2$, such that the following symmetric linear matrix inequality holds

$$\begin{pmatrix} \widehat{\Theta} & \Gamma^{\mathrm{T}}\Lambda & \tau M_1 & \tau M_2 & \ldots & \tau M_N \\ \Lambda^{\mathrm{T}}\Gamma & -\Lambda & 0 & 0 & \ldots & 0 \\ \tau M_1^{\mathrm{T}} & 0 & -\tau Z_1 & 0 & \ldots & 0 \\ \tau M_2^{\mathrm{T}} & 0 & 0 & -\tau Z_2 & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ \tau M_N^{\mathrm{T}} & 0 & 0 & 0 & \ldots & -\tau Z_N \end{pmatrix} < 0, \quad (25)$$

where

$$\widehat{\Theta} = \widehat{\Theta}_1 + \Theta_2 + \Theta_2^{\mathrm{T}},$$
$$\widehat{\Theta}_1 = \begin{pmatrix} \Theta_{11} & 0 & \Theta_{13} & PA^{\tau} & 0 & 0 & \ldots & 0 \\ * & -Q_N & 0 & K\Lambda_2 & 0 & 0 & \ldots & 0 \\ * & * & \Theta_{33} & SA^{\tau} & 0 & 0 & \ldots & 0 \\ * & * & * & -Q - 2\Lambda_2 & 0 & 0 & \ldots & 0 \\ * & * & * & * & \Theta_{55} & 0 & \ldots & 0 \\ * & * & * & * & * & \Theta_{66} & \ldots & 0 \\ \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots & \ldots \\ * & * & * & * & * & * & \ldots & \widehat{\Theta}_{i+4,i+4} \end{pmatrix},$$
$$\widehat{\Theta}_{i+4,i+4} = -Q_i + Q_{i+1}, \quad i = 1, 2, \ldots, N-1,$$

and other parameters are defined in Theorem 1.

Remark 3 Note that Theorem 1 in Ref.[15] is a special case of Corollary 2 in this paper. We only consider the case when $N = 2$. In fact, for $N = 2$, the LMI (25) can be reduced to

$$\begin{pmatrix} \widetilde{\Theta}_{11} & -M_{21} + M_{12}^{\mathrm{T}} & \widetilde{\Theta}_{13} & PA^{\tau} + M_{14}^{\mathrm{T}} & -M_{11} + M_{15}^{\mathrm{T}} + M_{21} & -C^{\mathrm{T}}\Lambda & \tau M_{11} & \tau M_{21} \\ * & \widetilde{\Theta}_{22} & -M_{23}^{\mathrm{T}} & K\Lambda_2 - M_{24}^{\mathrm{T}} & -M_{12} - M_{25}^{\mathrm{T}} + M_{22} & 0 & \tau M_{12} & \tau M_{22} \\ * & * & \Theta_{33} & SA^{\tau} & -M_{13} + M_{23} & A^{\mathrm{T}}\Lambda & \tau M_{13} & \tau M_{23} \\ * & * & * & -Q - 2\Lambda_2 & -M_{14} + M_{24} & A^{\tau\mathrm{T}}\Lambda & \tau M_{14} & \tau M_{24} \\ * & * & * & * & \widetilde{\Theta}_{55} & 0 & \tau M_{15} & \tau M_{25} \\ * & * & * & * & * & -\Lambda & 0 & 0 \\ * & * & * & * & * & * & -\tau Z_1 & 0 \\ * & * & * & * & * & * & * & -\tau Z_2 \end{pmatrix} < 0, \quad (26)$$

where

$$\begin{aligned}
\widetilde{\Theta}_{11} &= -PC - C^{\mathrm{T}}P + Q_1 + M_{11} + M_{11}^{\mathrm{T}},\\
\widetilde{\Theta}_{13} &= PA - CS + K\Lambda_1 + M_{13}^{\mathrm{T}},\\
\widetilde{\Theta}_{22} &= -Q_2 - M_{22} - M_{22}^{\mathrm{T}},\\
\widetilde{\Theta}_{55} &= -Q_1 + Q_2 - M_{15} - M_{15}^{\mathrm{T}} + M_{25} + M_{25}^{\mathrm{T}},
\end{aligned}$$

and Θ_{33} are defined in Theorem 1. If we denote $M_{13} = M_{14} = M_{23} = M_{24} = 0, M_{21} = M_{12}^{\mathrm{T}}, M_{11} = -M_{15} - M_{12}, M_{22} = -M_{21} - M_{25}$ and $Z_1 = -M_{11} - M_{11}^{\mathrm{T}}, Z_2 = M_{22} + M_{22}^{\mathrm{T}}$, then the LMI (26) can be reduced to

$$\overline{\Xi} = \begin{pmatrix}
\overline{\Theta}_{11} & 0 & \Theta_{13} & PA^{\tau} & Z_1 & -C^{\mathrm{T}}\Lambda & \tau M_{11} & \tau M_{21}\\
* & -Q_2 - Z_2 & 0 & K\Lambda_2 & Z_2 & 0 & \tau M_{12} & \tau M_{22}\\
* & * & \Theta_{33} & SA^{\tau} & 0 & A^{\mathrm{T}}\Lambda & 0 & 0\\
* & * & * & -Q - 2\Lambda_2 & 0 & A^{\tau\mathrm{T}}\Lambda & 0 & 0\\
* & * & * & * & \overline{\Theta}_{55} & 0 & \tau M_{15} & \tau M_{25}\\
* & * & * & * & * & -\Lambda & 0 & 0\\
* & * & * & * & * & * & -\tau Z_1 & 0\\
* & * & * & * & * & * & * & -\tau Z_2
\end{pmatrix} < 0,$$

where

$$\begin{aligned}
\overline{\Theta}_{11} &= -PC - C^{\mathrm{T}}P + Q_1 - Z_1,\\
\overline{\Theta}_{55} &= -Q_1 + Q_2 - Z_1 - Z_2.
\end{aligned}$$

Obviously, $\overline{\Xi} < 0$ can result in

$$\begin{pmatrix}
\overline{\Theta}_{11} & 0 & \Theta_{13} & PA^{\tau} & Z_1 & -C^{\mathrm{T}}\Lambda\\
* & -Q_2 - Z_2 & 0 & K\Lambda_2 & Z_2 & 0\\
* & * & \Theta_{33} & SA^{\tau} & 0 & A^{\mathrm{T}}\Lambda\\
* & * & * & -Q - 2\Lambda_2 & 0 & A^{\tau\mathrm{T}}\Lambda\\
* & * & * & * & \overline{\Theta}_{55} & 0\\
* & * & * & * & * & -\Lambda
\end{pmatrix} < 0,$$

which is equivalent to Theorem 1 in Ref.[15]. For $N > 2$, we can arrive at the same conclusion by following a similar line as the above.

Remark 4 A delay-decomposition approach combined with an integral equality technique in this paper can be extended to solve many problems, such as robust stability, absolute stability, exponential stability and so on, for neural network with constant delay or time-varying delay.

4. Illustrative example

In this Section, a numerical example is given to demonstrate the utility of the proposed stability theories.

Example Consider the following delayed neural network as in Ref.[10] with

$$C = \mathrm{diag}\{1.2769, 0.6231, 0.9230, 0.4480\},$$

$$A = \begin{pmatrix}
-0.0373 & 0.4852 & -0.3351 & 0.2336\\
-1.6033 & 0.5988 & -0.3224 & 1.2352\\
0.3394 & -0.0860 & -0.3824 & -0.5785\\
-0.1311 & 0.3253 & -0.9534 & -0.5015
\end{pmatrix},$$

$$A^{\tau} = \begin{pmatrix}
0.8674 & -1.2405 & -0.5325 & 0.0220\\
0.0474 & -0.9164 & 0.0360 & 0.9816\\
1.8495 & 2.6117 & -0.3788 & 0.8428\\
-2.0413 & 0.5179 & 1.1734 & -0.2775
\end{pmatrix},$$

$$K = \mathrm{diag}\{0.1137, 0.1279, 0.7994, 0.2386\}.$$

Case 1 We will consider the case that the delay is time-varying. For different values of μ, we use Theorem 1 to calculate the maximal allowable value h that guarantees the asymptotical stability of the system. For example, when $\mu = 0.1$, by Theorem 1 we obtain the maximum bound $h = 5.7016$ with the following solutions

$$P = \begin{pmatrix}
3.4388 & -0.0728 & -0.3299 & 0.1205\\
-0.0728 & 3.5646 & -1.7025 & 0.2733\\
-0.3299 & -1.7025 & 6.9937 & -0.4737\\
0.1205 & 0.2733 & -0.4737 & 2.1166
\end{pmatrix},$$

$$Q=\begin{pmatrix} 412.9123 & 3.9929 & -16.2635 & -2.6661 \\ 3.9929 & 55.4610 & -7.5810 & -12.6088 \\ -16.2635 & -7.5810 & 5.1847 & 4.3443 \\ -2.6661 & -12.6088 & 4.3443 & 9.3775 \end{pmatrix},$$

$$Q_1=\begin{pmatrix} 0.3070 & -0.1615 & 0.2337 & 0.0946 \\ -0.1615 & 1.2154 & -0.5412 & -0.4193 \\ 0.2337 & -0.5412 & 3.6070 & 1.1366 \\ 0.0946 & -0.4193 & 1.1366 & 0.4130 \end{pmatrix},$$

$$Q_2=\begin{pmatrix} 0.2736 & -0.1347 & 0.1988 & 0.0941 \\ -0.1347 & 1.3170 & -0.5975 & -0.2876 \\ 0.1988 & -0.5975 & 3.5624 & 0.7953 \\ 0.0941 & -0.2876 & 0.7953 & 0.5569 \end{pmatrix},$$

$$S=\mathrm{diag}\{7.1863, 0.0008, 0.0003, 0.0001\},$$

$$\Lambda_1=\mathrm{diag}\{502.4479, 74.6305, 5.8397, 12.0829\},$$

$$\Lambda_2=\mathrm{diag}\{0.1661, 100.09, 5.0746, 8.4451\}.$$

$$Z_1=\begin{pmatrix} 0.4935 & -0.0895 & 0.0405 & 0.0981 \\ -0.0895 & 1.1006 & -0.7355 & 0.5728 \\ 0.0405 & -0.7355 & 1.5647 & -1.1982 \\ 0.0981 & 0.5728 & -1.1982 & 0.9596 \end{pmatrix},$$

$$Z_2=\begin{pmatrix} 0.2100 & -0.0006 & -0.0011 & 0.0569 \\ -0.0006 & 0.8744 & -0.4779 & 0.3929 \\ -0.0011 & -0.4779 & 1.0085 & -0.7836 \\ 0.0569 & 0.3929 & -0.7836 & 0.6249 \end{pmatrix},$$

For other μ, we can also obtain the maximum bound h with the solutions which we omit here. Table 1 demonstrates the numerical results for different μ, from which it can be seen that the maximum allowable delay h decreases as μ increases. Moveover, it is obvious that the bound obtained with our approach is better than those in Ref.[10] and [13].

Table 1. Maximum upper bound of h for different values of μ.

method	Hua[13]	He[10]	Theorem 1 ($N=2$)
h ($\mu=0.1$)	3.2775	3.2793	5.7016
h ($\mu=0.5$)	2.1502	2.2245	3.8560
h ($\mu=0.8$)	1.3164	1.7463	1.9568

Case 2 Next we will consider this case that the delay is a constant. Applying the methods in Refs.[14] and [13], the stability of this system is guaranteed for $h=1.4224$ and $h=3.5841$, respectively. In Ref.[10], by employing free weighting matrices technique, the stability is guaranteed for $h=3.5841$. In Ref.[15], by dividing the delay interval into multiple segments, for $N=2$ and $N=3$, the maximal allowable value of h ensuring asymptotic stability of this system is 4.0204 and 4.1032, respectively. In this work, based on delay-decomposition approach combined with an integral equality technique, the stability is guaranteed for $h=6.2278$ even with $N=2$.

Then, as shown in Table 2, our proposed stability criterion gives a much less conservative result than those in Refs.[10], [13]–[15]. And when $h_{\max}=6.2278$, the solutions given by the LMI control tool-box are

$$P=\begin{pmatrix} 0.0258 & -0.0024 & 0.0013 & 0.0021 \\ -0.0024 & 0.0294 & -0.0150 & 0.0019 \\ 0.0013 & -0.0150 & 0.0570 & -0.0026 \\ 0.0021 & 0.0019 & -0.0026 & 0.0174 \end{pmatrix},$$

$$Q=\begin{pmatrix} 3.3039 & -0.0117 & -0.1322 & -0.0447 \\ -0.0117 & 0.3709 & -0.0608 & -0.1002 \\ -0.1322 & -0.0608 & 0.0357 & 0.0295 \\ -0.0447 & -0.1002 & 0.0295 & 0.0594 \end{pmatrix},$$

$$Q_1=\begin{pmatrix} 0.0149 & -0.0032 & 0.0034 & 0.0019 \\ -0.0032 & 0.0116 & -0.0056 & -0.0036 \\ 0.0034 & -0.0056 & 0.0314 & 0.0092 \\ 0.0019 & -0.0036 & 0.0092 & 0.0036 \end{pmatrix},$$

$$Q_2=\begin{pmatrix} 0.0145 & -0.0030 & 0.0032 & 0.0020 \\ -0.0030 & 0.0135 & -0.0066 & -0.0028 \\ 0.0032 & -0.0066 & 0.0336 & 0.0073 \\ 0.0020 & -0.0028 & 0.0073 & 0.0052 \end{pmatrix},$$

$$Z_1=\begin{pmatrix} 0.0032 & -0.0005 & 0.0004 & 0.0007 \\ -0.0005 & 0.0082 & -0.0056 & 0.0043 \\ 0.0004 & -0.0056 & 0.0107 & -0.0086 \\ 0.0007 & 0.0043 & -0.0086 & 0.0073 \end{pmatrix},$$

$$Z_2=\begin{pmatrix} 0.0014 & -0.0001 & 0.0000 & 0.0004 \\ -0.0001 & 0.0065 & -0.0034 & 0.0026 \\ 0.0000 & -0.0034 & 0.0065 & -0.0054 \\ 0.0004 & 0.0026 & -0.0054 & 0.0045 \end{pmatrix},$$

$$S=\mathrm{diag}\{0.1629, 0, 0, 0\},$$

$$\Lambda_1=\mathrm{diag}\{3.6048, 0.5341, 0.0447, 0.0891\},$$

$$\Lambda_2=\mathrm{diag}\{0.5924, 1.0469, 0.0475, 0.0722\}.$$

Table 2. Maximum upper bound of h.

method	Xu[14]	Hua[13]	He[10]	Shao[15]($N=2$)	Shao[15]($N=3$)	Corollary 2 ($N=2$)
h	1.4224	3.5841	3.5841	4.0204	4.1032	6.2278

5. Conclusion

This paper has discussed stability criteria for the neural system with a time-varying delay. With the delay divided into multiple segments, a novel Lyapunov functional is introduced and the less conservative delay-dependent stability conditions are derived. It is shown that the proposed criteria are both delay-dependent and rate-dependent, and the stability criteria are less conservative than some existing ones. A numerical example has shown the effectiveness of the proposed methods.

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