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Exact solution for the thermo Jaynes–Cummings model*

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Based on the construction of supersymmetric generators, we use the Lewis–Riesenfeld invariant method to deduce the exact and explicit eigen-energy spectrum with the time-dependent thermo Jaynes–Cummings model. One of the advantages of this approach is that it can transform the hidden form, related to the chronological product, of the time evolution operator into an explicit expression. Moreover, the dynamical and statistics properties of physical quantities are obtained for the given initial states in the thermo Jaynes–Cummings system.

Keywords: eigen-energy spectrum, Lewis–Riesenfeld invariant method, thermo Jaynes–Cummings model

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1. Introduction

It is well known that the Jaynes–Cummings (JC) model[1] plays an important role in the study of a single two-level atom interacting with a single quantized electromagnetic field, which is shown to exhibit interesting nonclassical effects, such as the collapse and revival of the Rabi oscillations of the atomic inversion, anti-bunched light, and squeezing. Over the past many years, various exactly solvable modifications and generalizations of the original JC model have been studied widely such as the Kerr effect, the Stark effect, the Doppler effect, and so on.[2−6] In fact, all these JC models are discussed at a zero-temperature environment. Since most systems are immersed in a thermal reservoir and thermal excitation and de-excitation always happen, Fan and Lu[7] have proposed a thermo JC model at a finite temperature by analysing Takahashi–Umezawa thermo field dynamics (TFD)[8−10] and constructing new thermal excitation states.[11] When a model is investigated, the basic task is to obtain the energy spectra, eigen-states and various quantum evolution problems.[12−16] Recently, the dynamical invariant method, introduced firstly by Lewis–Riesefeld,[17] has been extensively employed to investigate various quantum evolution problems and the geometric phase factor.[18−23] One of the advantages of this approach is that it can transform the hidden form, related to the chronological product, of the time evolution operator $U(t)$ into an explicit expression.

In the present work, based on the above ideas, we shall apply the Lewis–Riesenfeld invariant method (LRIM) to exactly solve the thermo JC model as follows:[7]

$$H = \omega h + \frac{1}{2}\Omega\sigma_z + \lambda\left[\sigma_+\sqrt{\frac{\alpha-\widetilde{\alpha}^\dagger}{\alpha^\dagger-\widetilde{\alpha}}}\sqrt{h} + \sqrt{h}\sqrt{\frac{\alpha^\dagger-\widetilde{\alpha}}{\alpha-\widetilde{\alpha}^\dagger}}\sigma_-\right], \quad (1)$$

where ω and Ω are the thermal field frequency and the atom transition frequency, respectively, Pauli matrices σ_z, $\sigma_\pm$ represent the two-level atomic system. $h = \alpha^\dagger\alpha - \widetilde{\alpha}^\dagger\widetilde{\alpha}$, the bosonic creation operator $\widetilde{\alpha}^\dagger$ in the fictitious mode accompanies the operator $\alpha^\dagger$ denoted by the excitations in the finite temperature system, which satisfies $\left[\widetilde{\alpha},\widetilde{\alpha}^\dagger\right] = \left[\alpha,\alpha^\dagger\right] = 1$. $\sqrt{(\alpha-\widetilde{\alpha}^\dagger)/(\alpha^\dagger-\widetilde{\alpha})}$ is the de-excitation and $\sqrt{h}$ means that the interaction is field-intensity dependent. $\sigma_+\sqrt{(\alpha-\widetilde{\alpha}^\dagger)/(\alpha^\dagger-\widetilde{\alpha})}\sqrt{h}$ describes the interaction between the thermo photon field and a two-level atom excited from the lower level to the upper level. While the term $\sqrt{h}\sqrt{(\alpha^\dagger-\widetilde{\alpha})/(\alpha-\widetilde{\alpha}^\dagger)}\sigma_-$ denotes the opposite process. Our main task is to derive the exact and explicit eigen-energy spectrum with time-dependent thermo JC model and get dynamical and statistics properties of physical quantities for the given initial states. In the following section, for the convenience of completeness, we shall review the thermo excitation and de-excitation operator and representation in the TFD. We devote Section 3 to deal with the thermo JC model by using the LRIM and Lie algebra theory. We obtain its exact eigen-energy spectrum with the time-dependent thermo JC model

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and the explicit expressions for the evolution operator of the wavefunctions. Finally, in Section 4, a brief summary is given.

2. Thermo excitation representation

In this section, in order to understand clearly the physics meaning of the thermo JC model, we briefly review and explain the thermo excitation and de-excitation operator and representation in the TFD.

Begin with the thermo excitation Hamiltonian of the total system including the effects of the reservoir, expressed as[10]

$$H = \hbar \int \mathrm{d}^3k\omega(\boldsymbol{k},\beta) \times \left[\alpha^\dagger(\boldsymbol{k},\beta)\alpha(\boldsymbol{k},\beta) - \tilde{\alpha}^\dagger(\boldsymbol{k},\beta)\tilde{\alpha}(\boldsymbol{k},\beta)\right], \quad (2)$$

where the creation operator $\alpha^\dagger(\boldsymbol{k},\beta)$ describes the excitation of an additional quantum with positive energy $\hbar\omega$ and momentum $\hbar\boldsymbol{k}$ and $\beta = 1/\kappa T$, κ is the Boltzmann constant, T is the temperature. h in Eq.(1) is equal to

$$\begin{aligned} h &= \alpha^\dagger(\boldsymbol{k},\beta)\alpha(\boldsymbol{k},\beta) - \tilde{\alpha}^\dagger(\boldsymbol{k},\beta)\tilde{\alpha}(\boldsymbol{k},\beta) \\ &\equiv \alpha^\dagger\alpha - \tilde{\alpha}^\dagger\tilde{\alpha}, \end{aligned} \quad (3)$$

which is invariant under the thermo operator transformation,[24] here the notations $\boldsymbol{k}$ and β are omitted for simplicity.

One can construct the following excitation state[11]

$$||e,n\rangle = \left(\sqrt{\frac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}}\right)^e |n,\tilde{n}\rangle, \quad (4)$$

where e is the excitation number of the total system's energy, and the number state $|n\rangle = \alpha^{\dagger n}/\sqrt{n!}\,|0\rangle$ in the original Fock space is accompanied by the tilde state $|\tilde{n}\rangle = \tilde{\alpha}^{\dagger n}/\sqrt{n!}\,|\tilde{0}\rangle$ and $|n,\tilde{n}\rangle$ spans the whole system (original+tilde). The state $||e,n\rangle$ spans a complete and orthogonal set, i.e.,

$$\langle e',n'\,||e,n\rangle = \delta_{e'e}\delta_{n'n} \quad (5)$$

and

$$\sum_{e=-\infty}^{\infty}\sum_{n=0}^{\infty} ||e,n\rangle\ \langle e,n|| = 1, \quad (6)$$

which make it a useful representation, the detailed deduction could be found in Ref.[25].

It is easily proved that

$$\sqrt{\frac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}}\,||e,n\rangle = ||e+1,n\rangle \quad (7)$$

and

$$\sqrt{\frac{\alpha - \widetilde{\alpha}^\dagger}{\alpha^\dagger - \widetilde{\alpha}}}\,||e,n\rangle = ||e-1,n\rangle\,, \quad (8)$$

which indicate that $\sqrt{\frac{\alpha - \widetilde{\alpha}^\dagger}{\alpha^\dagger - \widetilde{\alpha}}}$ and $\sqrt{\frac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}}$ play the roles of lowering (de-excitation) and ascending (excitation), respectively, when they are operating on the state $||e,n\rangle$. So the physics process of the term $\sigma_+\sqrt{\frac{\alpha - \widetilde{\alpha}^\dagger}{\alpha^\dagger - \widetilde{\alpha}}}\sqrt{h}$ in Eq.(1) is that the thermal photon of thermal field is de-excitation and the atom is excited from the lower level to the upper level. $\sqrt{h}\sqrt{\frac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}}\sigma_-$ describes the opposite process, namely, the thermal photon of thermal field is excitation and the atom is excited from the upper level to the lower level.

On the other hand, due to $\left[h,(\alpha - \widetilde{\alpha}^\dagger)(\alpha^\dagger - \widetilde{\alpha})\right] = 0$, $\left[h,\alpha - \widetilde{\alpha}^\dagger\right] = -\left(\alpha - \widetilde{\alpha}^\dagger\right)$ and $\left[h,\alpha^\dagger - \widetilde{\alpha}\right] = \alpha^\dagger - \widetilde{\alpha}$, one can obtain that

$$\left[h,\sqrt{\frac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}}\right] = \sqrt{\frac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}} \quad (9)$$

and

$$\left[h,\sqrt{\frac{\alpha - \widetilde{\alpha}^\dagger}{\alpha^\dagger - \widetilde{\alpha}}}\right] = -\sqrt{\frac{\alpha - \widetilde{\alpha}^\dagger}{\alpha^\dagger - \widetilde{\alpha}}}. \quad (10)$$

By using Eqs.(9) and (10) and $h\,|n,\tilde{n}\rangle = 0$, it is proved that $||e,n\rangle$ is the eigen-state of the operator h with the eigenvalue e as follows:

$$\begin{aligned} h\,||e,n\rangle &= \left[h,\left(\sqrt{\frac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}}\right)^e\right]|n,\tilde{n}\rangle \\ &= e\,||e,n\rangle\,. \end{aligned} \quad (11)$$

From Eqs.(7), (8) and (11), $||e,n\rangle$ is called the thermo excitation.

3. Solving the thermo JC model via the LRIM

In the following section, we shall use another approach to solve the above thermo JC model by virtue of the LRIM theory (see the Appendix in detail).

We first redescribe the thermo JC Hamiltonian via the supersymmetric operators, denoted as

$$Q = \left(\sqrt{h}\sqrt{\frac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}}\right)\sigma_- = \begin{pmatrix} 0 & 0 \\ \sqrt{h}\sqrt{\dfrac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}} & 0 \end{pmatrix}, \tag{12}$$

$$Q^\dagger = \sigma_+ \left(\sqrt{\frac{\alpha - \widetilde{\alpha}^\dagger}{\alpha^\dagger - \widetilde{\alpha}}}\sqrt{h}\right) = \begin{pmatrix} 0 & \sqrt{\dfrac{\alpha - \widetilde{\alpha}^\dagger}{\alpha^\dagger - \widetilde{\alpha}}}\sqrt{h} \\ 0 & 0 \end{pmatrix}, \tag{13}$$

and

$$N = h + \frac{1}{2}\sigma_z + \frac{1}{2} = \begin{pmatrix} \alpha\alpha^\dagger - \widetilde{\alpha}^\dagger\widetilde{\alpha} & 0 \\ 0 & \alpha^\dagger\alpha - \widetilde{\alpha}^\dagger\widetilde{\alpha} \end{pmatrix}. \tag{14}$$

It is easily seen that N, Q, $Q^\dagger$ form the supersymmetric generators and have Lie superalgebra properties,[26]

$$Q^2 = Q^{\dagger 2} = 0, \quad [Q^\dagger, Q] = N\sigma_z, \tag{15}$$

$$(Q^\dagger - Q)^2 = -N, \quad [Q^\dagger, \sigma_z] = -2Q^\dagger, \tag{16}$$

$$[N, Q] = [N, Q^\dagger] = 0, \quad [Q, \sigma_z] = 2Q, \tag{17}$$

$$\{Q, \sigma_z\} = \{Q^\dagger, \sigma_z\} = 0, \quad \{Q, Q^\dagger\} = N. \tag{18}$$

In terms of the above generators, we can rewrite the Hamiltonian Eq.(1) as

$$H = \omega N + \frac{1}{2}\delta\sigma_z + \lambda\left(Q^\dagger + Q\right) - \frac{1}{2}\omega, \tag{19}$$

where $\delta = \Omega - \omega$ is called the detuning quantity.

Since the LRIM can only be applied to the study of such a system, in which there exists a closed quasi-algebra, it is easy to see from Eqs.(15)–(18) that there does not exist such an algebra for the system. However, considering $\sigma_z|\pm\rangle = \pm|\pm\rangle$ and Eqs.(11) and (14), we get that the following eigenvalue equation of N

$$N\begin{pmatrix} \|e, n\rangle \\ \|e+1, n\rangle \end{pmatrix} = (e+1)\begin{pmatrix} \|e, n\rangle \\ \|e+1, n\rangle \end{pmatrix}, \tag{20}$$

where

$$\begin{pmatrix} \|e, n\rangle \\ 0 \end{pmatrix} \equiv \|e, n\rangle \otimes |+\rangle = \|e, n\rangle \otimes \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{21}$$

and

$$\begin{pmatrix} 0 \\ \|e+1, n\rangle \end{pmatrix} \equiv \|e+1, n\rangle \otimes |-\rangle = \|e+1, n\rangle \otimes \begin{pmatrix} 0 \\ 1 \end{pmatrix}. \tag{22}$$

In these eigen-spaces of the invariant operator N, one finds that operators Q, $Q^\dagger$ and σ_z form a closed quasi-algebra,

$$[Q^\dagger, Q] = (e+1)\sigma_z, \quad (Q^\dagger - Q)^2 = -(e+1), \tag{23}$$

and

$$\{Q, \sigma_z\} = \{Q^\dagger, \sigma_z\} = 0, \quad \{Q, Q^\dagger\} = e+1, \tag{24}$$

where N may be considered as constant, i.e. its eigenvalue is $e+1$.

Due to the closure property of Eqs.(21) and (22), the Hermitian invariant I can be constructed as

$$I = f\left(Q^\dagger + Q\right) + g\sigma_z, \tag{25}$$

where f and g are time-independent real quantities to be determined. According to the LRIM theory, substituting Eqs.(19) and (25) into Eq.(A1) in the Appendix and identifying the coefficients of Q, $Q^\dagger$ and σ_z on both sides, we derive the following identities

$$\frac{\mathrm{d}f}{\mathrm{d}t} = \mathrm{i}\left(-f\delta + 2g\lambda\right), \quad \frac{\mathrm{d}g}{\mathrm{d}t} = 0, \tag{26}$$

which yields

$$f = \frac{2g\lambda}{\delta}. \tag{27}$$

Using the matrix representation of I, whose eigen-equation may be written in the form

$$\begin{pmatrix} g & f\sqrt{\dfrac{\alpha - \widetilde{\alpha}^\dagger}{\alpha^\dagger - \widetilde{\alpha}}}\sqrt{h} \\ f\sqrt{h}\sqrt{\dfrac{\alpha^\dagger - \widetilde{\alpha}}{\alpha - \widetilde{\alpha}^\dagger}} & -g \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix} = \beta\begin{pmatrix} x \\ y \end{pmatrix}, \tag{28}$$

one can easily obtain that equation (28) should have the following four types of solutions,

$$\begin{aligned} &x = \cos\frac{\theta}{2}\|e, n\rangle, \quad y = \sin\frac{\theta}{2}\|e+1, n\rangle; \\ &x = -\sin\frac{\theta}{2}\|e, n\rangle, \quad y = \cos\frac{\theta}{2}\|e+1, n\rangle; \\ &x = 0, \quad y = \|0, n\rangle; \\ &x = \|-1, n\rangle, \quad y = 0. \end{aligned} \tag{29}$$

Note that in Eq.(29), we have used the property of the eigenvalues β being real quantities. By substitution of Eq.(29) into Eq.(28), then the eigenvalues β_i

and normalized eigenfunctions $|\phi_i\rangle$ of the invariant I can be obtained as

$$\beta_1 = \frac{g}{\delta}\sqrt{\delta^2 + 4\lambda^2 (e+1)},$$
$$|\phi_1\rangle = \begin{pmatrix} \cos\dfrac{\theta}{2}\,\|e,n\rangle \\ \sin\dfrac{\theta}{2}\,\|e+1,n\rangle \end{pmatrix}, \tag{30}$$
$$\beta_1 = -\frac{g}{\delta}\sqrt{\delta^2 + 4\lambda^2 (e+1)},$$
$$|\phi_2\rangle = \begin{pmatrix} -\sin\dfrac{\theta}{2}\,\|e,n\rangle \\ \cos\dfrac{\theta}{2}\,\|e+1,n\rangle \end{pmatrix}, \tag{31}$$
$$\beta_3 = -g, \quad |\phi_3\rangle = \begin{pmatrix} 0 \\ \|0,n\rangle \end{pmatrix}, \tag{32}$$

and

$$\beta_4 = g, \quad |\phi_4\rangle = \begin{pmatrix} \|-1,n\rangle \\ 0 \end{pmatrix}, \tag{33}$$

where $\theta = \tan^{-1}\left(\dfrac{2\lambda\sqrt{e+1}}{\delta}\right)$, β_i $(i = 1,2,3,4)$ are independent of time.

Next we proceed to calculate the time evolution of the state vector of the system and its evolution operator. It is straightforward to obtain the corresponding eigen-state $|\Phi_j(t)\rangle$ with time evolution

$$\begin{aligned} |\Phi_1(t)\rangle &= \exp\left[\mathrm{i}\varphi_1(t)\right]|\phi_1\rangle, \\ |\Phi_2(t)\rangle &= \exp\left[\mathrm{i}\varphi_2(t)\right]|\phi_2\rangle, \\ |\Phi_3(t)\rangle &= \exp\left[\mathrm{i}\varphi_3(t)\right]|\phi_3\rangle, \\ |\Phi_4(t)\rangle &= \exp\left[\mathrm{i}\varphi_4(t)\right]|\phi_4\rangle \end{aligned} \tag{34}$$

with $\varphi_1(t)$, $\varphi_2(t)$, $\varphi_3(t)$ and $\varphi_4(t)$ being given by

$$\begin{aligned} \varphi_1(t) &= -\langle\Phi_1(t)|\,H\,|\Phi_1(t)\rangle\, t \\ &= -E_1 t = -\left(\omega e + \frac{\delta}{2\cos\theta} + \frac{\omega}{2}\right)t, \end{aligned} \tag{35}$$
$$\begin{aligned} \varphi_2(t) &= -\langle\Phi_2(t)|\,H\,|\Phi_2(t)\rangle\, t \\ &= -E_2 t = -\left(\omega e - \frac{\delta}{2\cos\theta} + \frac{\omega}{2}\right)t, \end{aligned} \tag{36}$$
$$\begin{aligned} \varphi_3(t) &= -\langle\Phi_3(t)|\,H\,|\Phi_3(t)\rangle\, t \\ &= -E_3 t = -\left(\omega - \frac{\Omega}{2}\right)t, \end{aligned} \tag{37}$$

and

$$\begin{aligned} \varphi_4(t) &= -\langle\Phi_4(t)|\,H\,|\Phi_4(t)\rangle\, t \\ &= -E_4 t = -\left(-\omega + \frac{\Omega}{2}\right)t, \end{aligned} \tag{38}$$

where E_j $(j = 1,2,3,4)$ are the energy eigenvalues of this system. Since $|\Phi_3(t)\rangle$ and $|\Phi_4(t)\rangle$ represent the states without the interaction between the thermo field and the atom, the total wavefunction $|\Psi(t)\rangle$ for the system is expressed as

$$|\Psi(t)\rangle = C_1\,|\Phi_1(t)\rangle + C_2\,|\Phi_2(t)\rangle, \tag{39}$$

where the coefficients $C_j = \langle\phi_j|\,\Psi(0)\rangle$, $(j = 1,2)$ and $|\Psi(0)\rangle$ is a given initial state of the system. Comparing with $|\Psi(t)\rangle = U(t,0)\,|\Psi(0)\rangle$ and considering Eqs.(34) and (39), we can deduce the evolution operator

$$U(t,0) = \mathrm{e}^{\mathrm{i}\varphi_1(t)}\,|\phi_1\rangle\langle\phi_1| + \mathrm{e}^{\mathrm{i}\varphi_2(t)}\,|\phi_2\rangle\langle\phi_2|. \tag{40}$$

Based on the general solution $|\Psi(t)\rangle$ and the time-development operator $U(t,0)$, further we are able to discuss the dynamical and statistics properties of physical quantities in the thermo JC system for different initial states $|\Psi(0)\rangle$ of the system.

For example, the time evolution of both the thermo excitation number and the atomic inversion operator can be calculated as, respectively

$$\begin{aligned} &\langle h(t)\rangle \\ &= \sum_e \left[|C_1|^2 \left(e + \sin^2\frac{\theta}{2}\right) + |C_2|^2 \left(e + \cos^2\frac{\theta}{2}\right) \right. \\ &\quad \left. + \mathrm{Re}(C_1^* C_2 \sin\theta\, \mathrm{e}^{\mathrm{i}\delta t/\cos\theta}) \right], \end{aligned} \tag{41}$$

and

$$\begin{aligned} \langle\sigma_z(t)\rangle = \sum_e \Big[&\left(|C_1|^2 - |C_2|^2\right)\cos\theta \\ &- 2\mathrm{Re}(C_1^* C_2 \sin\theta\, \mathrm{e}^{\mathrm{i}\delta t/\cos\theta}) \Big], \end{aligned} \tag{42}$$

where

$$\begin{aligned} \sin\theta &= \frac{2\lambda\sqrt{e+1}}{\sqrt{\delta^2 + 4\lambda^2(e+1)}}, \\ \cos\theta &= \frac{\delta}{\sqrt{\delta^2 + 4\lambda^2(e+1)}}. \end{aligned} \tag{43}$$

Here the summation considers the relative weight for each value of e, for the Rabi oscillations in summation with different e have different frequencies.

When the initial state $|\Psi(0)\rangle = \|e,n\rangle \otimes |+\rangle \equiv \|e,n,+\rangle$, equations (41) and (42) are also simplified as

$$\langle h(t)\rangle = \sum_e \left[e + \frac{\sin^2\dfrac{\theta}{2}}{2}\left(1 - \cos\left(\frac{\delta}{\cos\theta}t\right)\right)\right] \tag{44}$$

and

$$\langle\sigma_z(t)\rangle = \sum_e \left[\cos^2\theta + \sin^2\theta\cos\left(\frac{\delta}{\cos\theta}t\right)\right]. \tag{45}$$

Obviously, it is easy to calculate the probability amplitude $\|e,n,+\rangle$ and $\|e+1,n,-\rangle$ for $|\Psi(0)\rangle = \|e,n,+\rangle$ as follows:

$$\begin{aligned} & |C_+(t)|^2 \\ \equiv & \left|\langle e,n,+\|\,\Psi(t)\rangle\right|^2 \\ = & \sum_e \left[1-\frac{\sin^2\theta}{2}\left(1-\cos\left(\frac{\delta}{\cos\theta}t\right)\right)\right], \quad (46) \\ & |C_-(t)|^2 \\ \equiv & \left|\langle e+1,n,-\|\,\Psi(t)\rangle\right|^2 \\ = & \sum_e \left[\frac{\sin^2\theta}{2}\left(1-\cos\left(\frac{\delta}{\cos\theta}t\right)\right)\right]. \quad (47) \end{aligned}$$

In fact, the inversion $\langle\sigma_z(t)\rangle$ is related to the probability amplitude $\|e,n,+\rangle$ and $\|e+1,n,-\rangle$, i.e.

$$\begin{aligned} & \langle\sigma_z(t)\rangle \\ = & \left[|C_+(t)|^2-|C_-(t)|^2\right] \\ = & \sum_e \left[\cos^2\theta+\sin^2\theta\cos\left(\frac{\delta}{\cos\theta}t\right)\right], \quad (48) \end{aligned}$$

which is in agreement with Eq.(45).

4. Conclusions

In summary, based on the construction of supersymmetric generators, we have used the LRIM method to derive the exact and explicit eigen-energy spectrum with the time-dependent thermo JC model. The obtained results are the same as those given in Ref.[7]. Moreover, we also obtained dynamical and statistics properties of physical quantities for the given initial states in the thermo JC system. In view of the above calculations, one can see clearly that this approach has some useful applications, for instance, it can be used to solve the time-dependent systems and obtain the explicit expressions, instead of the hidden form, for the evolution operator of the wavefunctions. Therefore, we hope our work can stimulate some further research or itself can enrich the photon–atom interaction theory at a finite temperature for the JC system.

Appendix

In this Appendix, we begin with some related results for the Lewis–Riesenfeld invariant method (LRIM).[17−20] For a given time-dependent Hamiltonian $H(t)$, it can be supposed that there is a Hermitian invariant operator $I(t)$ which satisfies

$$\frac{\partial I}{\partial t}-\mathrm{i}\,[I,H]=0. \quad (A1)$$

If there exists a complete orthogonal set of the eigenstates $|\Phi_m(t)\rangle$ of $I(t)$ with eigenvalues β_m, i.e.,

$$I\,|\Phi_m(t)\rangle = \beta_m\,|\Phi_m(t)\rangle\,, \quad (A2)$$

then we can express the exact solution $|\Psi(t)\rangle$ and the time evolution operator $U(t)$ for the system in terms of instantaneous eigen-states of $I(t)$ as

$$|\Psi(t)\rangle = \sum_m C_m \exp\left[\mathrm{i}\varphi_m(t)\right]|\Phi_m(t)\rangle\,, \quad (A3)$$

$$U(t,0) = \sum_m \exp\left[\mathrm{i}\varphi_m(t)\right]|\Phi_m(t)\rangle\langle\Phi_m(0)| \quad (A4)$$

with

$$\begin{aligned} & \varphi_m(t) \\ = & \int_0^t \langle\Phi_m(t')|\left[\mathrm{i}\frac{\partial}{\partial t'}-H(t')\right]|\Phi_m(t')\rangle\,\mathrm{d}t', \quad (A5) \end{aligned}$$

$$C_m = \langle\Phi_m(0)|\,\Psi(0)\rangle. \quad (A6)$$

Thus the problems of solving the time-dependent Schrödinger equation are turned into finding the instantaneous eigen-spectra of the invariant chosen adequately. Obviously, the invariant method is also applicable for the cases with time-independent Hamiltonian.

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