Vol 18 No 12, December 2009
1674-1056/2009/18(12)/5139-05

Chinese Physics B

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The N-mode squeezed state with enhanced squeezing*

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(Received 22 May 2009; revised manuscript received 8 June 2009)

It is known that $\exp\left[\mathrm{i}\lambda\left(Q_1P_1-\mathrm{i}/2\right)\right]$ is a unitary single-mode squeezing operator, where Q_1, P_1 are the coordinate and momentum operators, respectively. In this paper we employ Dirac's coordinate representation to prove that the exponential operator $S_n \equiv \exp\left[\mathrm{i}\lambda\sum_{i=1}^{n}\left(Q_iP_{i+1}+Q_{i+1}P_i\right)\right)\Big]$, $(Q_{n+1}=Q_1,\ P_{n+1}=P_1)$, is an n-mode squeezing operator which enhances the standard squeezing. By virtue of the technique of integration within an ordered product of operators we derive S_n's normally ordered expansion and obtain new n-mode squeezed vacuum states, its Wigner function is calculated by using the Weyl ordering invariance under similar transformations.

Keywords: Dirac's representation, integration within an ordered product technique, squeezing enhanced operator, squeezed sate

PACC: 0365, 4250

1. Introduction

Squeezed state has been a hot topic in quantum optics since Stoler[1] put forward the concept of the optical squeezing in 1970's. $S_1 = \exp\left[\mathrm{i}\lambda\left(Q_1P_1-\mathrm{i}/2\right)\right]$ is a unitary single-mode squeezing operator, where Q_1, P_1 are the coordinate and momentum operators, respectively, λ is a squeezing parameter. Their variances in the squeezed state $S_1\left|0\right\rangle = \mathrm{sech}^{1/2}\lambda\exp\left[-\frac{1}{2}a_1^{\dagger 2}\tanh\lambda\right]\left|0\right\rangle$ are $\Delta Q_1 = \frac{1}{2}\mathrm{e}^{2\lambda}, \Delta P_1 = \frac{1}{2}\mathrm{e}^{-2\lambda}, (\Delta Q_1)(\Delta P_1) = \frac{1}{4}$. Some generalized squeezed states have been proposed since then. Among them the two-mode squeezed state not only exhibits squeezing, but also quantum entanglement between the idle-mode and the signal-mode in the frequency domain, therefore it is a typical entangled state of continuous variable. In recent years, various entangled states have attracted considerable attention and interests of physists because of their potential uses in quantum communication.[2] Theoretically, the two-mode squeezed state is constructed by acting the two-mode squeezing operator $S_2 = \exp[\lambda(a_1a_2 - a_1^{\dagger}a_2^{\dagger})]$ on the two-mode vacuum state $\left|00\right\rangle$,[3−5]

$$S_2\left|00\right\rangle = \mathrm{sech}\lambda\exp\left[-a_1^{\dagger}a_2^{\dagger}\tanh\lambda\right]\left|00\right\rangle. \quad (1)$$

We also have $S_2 = \exp\left[\mathrm{i}\lambda\left(Q_1P_2+Q_2P_1\right)\right]$, where Q_i and P_i are the coordinate and momentum operators related to Bose operators $(a_i, a_i^{\dagger})$ by $Q_i = (a_i+a_i^{\dagger})/\sqrt{2},\ P_i = (a_i-a_i^{\dagger})/(\sqrt{2}\mathrm{i})$.

In the state $S_2\left|00\right\rangle$, the variances of the two-mode quadrature operators of light field,

$$\mathfrak{X} = (Q_1+Q_2)/2,\ \mathfrak{P} = (P_1+P_2)/2,\ [\mathfrak{X},\mathfrak{P}] = \frac{\mathrm{i}}{2}, \quad (2)$$

take the standard form, i.e.,

$$\begin{aligned}\left\langle 00\right|S_2^{\dagger}\mathfrak{X}^2S_2\left|00\right\rangle &= \frac{1}{4}\mathrm{e}^{-2\lambda},\\ \left\langle 00\right|S_2^{\dagger}\mathfrak{P}^2S_2\left|00\right\rangle &= \frac{1}{4}\mathrm{e}^{2\lambda},\\ (\Delta\mathfrak{X})(\Delta\mathfrak{P}) &= \frac{1}{4}. \end{aligned} \quad (3)$$

On the other hand, the two-mode squeezing operator has a neat and natural representation in the entangled state $\left|\eta\right\rangle$ representation,[6−8]

$$S_2 = \int\frac{\mathrm{d}^2\eta}{\pi\mu}\left|\frac{\eta}{\mu}\right\rangle\left\langle\eta\right|,$$

where

$$\left|\eta\right\rangle = \exp\left(-\frac{1}{2}\left|\eta\right|^2+\eta a_1^{\dagger}-\eta^*a_2^{\dagger}+a_1^{\dagger}a_2^{\dagger}\right)\left|00\right\rangle$$

makes up a complete set

$$\int\frac{\mathrm{d}^2\eta}{\pi}\left|\eta\right\rangle\left\langle\eta\right| = 1.$$

The $\left|\eta\right\rangle$ was constructed according to the idea of quantum entanglement initiated by Einstein, Podolsky and Rosen in their argument that quantum mechanics is incomplete.[9]

*Project supported by the National Natural Science Foundation of China (Grant Nos 10775097 and 10874174), and the Research Foundation of the Education Department of Jiangxi Province of China.

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An interesting question naturally arises: is the n-mode exponential operator

$$S_n \equiv \exp\left[\mathrm{i}\lambda\sum_{i=1}^{n}(Q_iP_{i+1}+Q_{i+1}P_i)\right], \quad (Q_{n+1}=Q_1,\ P_{n+1}=P_1),\ n\geqslant 2, \tag{4}$$

a squeezing operator? If yes, what kind of squeezing for n-mode quadratures of field can engenders? To answer these questions we must know what is the normally ordered expansion of S_n and what is the state $S_n|\mathbf{0}\rangle$ ($|\mathbf{0}\rangle$ is the n-mode vacuum state)? In this work we shall analyse S_n in detail. But how to disentangle the exponential of S_n? Since the terms in the set Q_iP_{i+1} and $Q_{i+1}P_i$ $(i=1,2,\ldots,n)$ do not make up a closed Lie algebra, the problem of what is S_n's normally ordered form seems difficult. Thus we appeal to Dirac's coordinate representation and the technique of integration within an ordered product (IWOP) of operators[10,11] to solve this problem. Our work is arranged as follows: firstly we use the IWOP technique to derive the normally ordered expansion of S_n and obtain the explicit form of $S_n|\mathbf{0}\rangle$; then we examine the variances of the n-mode quadrature operators in the state $S_n|\mathbf{0}\rangle$. We find that S_n causes squeezing which is stronger than the standard squeezing. Thus S_n is an n-mode squeezing-enhanced operator. The Wigner function of $S_n|\mathbf{0}\rangle$ is calculated by using the Weyl ordering invariance under similar transformations. Some examples are discussed in the last section.

2. Normal product form of S_n derived by Dirac's coordinate representation

In order to disentangle operator S_n, let A be

$$A=\begin{pmatrix} 0 & 1 & 0 & \cdots & 1\\ 1 & 0 & 1 & \cdots & 0\\ 0 & 1 & 0 & \ddots & 0\\ \vdots & \vdots & \ddots & \ddots & \vdots\\ 1 & 0 & \cdots & 1 & 0 \end{pmatrix}, \tag{5}$$

then S_n in expression (4) is compactly expressed as

$$S_n=\exp[\mathrm{i}\lambda Q_iA_{ij}P_j], \tag{6}$$

here and henceforth the repeated indices represent Einstein's summation notation. Using the Baker–Hausdorff formula,

$$\mathrm{e}^{A}B\mathrm{e}^{-A}=B+[A,B]+\frac{1}{2!}[A,[A,B]]+\frac{1}{3!}[A,[A,[A,B]]]+\cdots,$$

we have

$$\begin{aligned} S_n^{-1}Q_kS_n &= Q_k-\lambda Q_iA_{ik}+\frac{1}{2!}\mathrm{i}\lambda^2[Q_iA_{ij}P_j,Q_lA_{lk}]+\cdots\\ &= Q_i(\mathrm{e}^{-\lambda A})_{ik}=(\mathrm{e}^{-\lambda\tilde{A}})_{ki}Q_i, \end{aligned} \tag{7}$$

$$\begin{aligned} S_n^{-1}P_kS_n &= P_k+\lambda A_{ki}P_i+\frac{1}{2!}\mathrm{i}\lambda^2[A_{ki}P_j,Q_lA_{lm}P_m]+\cdots\\ &= (\mathrm{e}^{\lambda A})_{ki}P_i. \end{aligned} \tag{8}$$

From Eq.(7) we see that when S_n acts on the n-mode coordinate eigenstate $|\boldsymbol{q}\rangle$, where $\widetilde{\boldsymbol{q}}=(q_1,q_2,\ldots,q_n)$, it squeezes $|\boldsymbol{q}\rangle$ in this way:

$$S_n|\boldsymbol{q}\rangle=|\Lambda|^{1/2}|\Lambda\boldsymbol{q}\rangle,\quad \Lambda=\mathrm{e}^{-\lambda\tilde{A}},\quad |\Lambda|\equiv\det\Lambda. \tag{9}$$

Thus S_n has the representation on the Dirac's coordinate basis $\langle\boldsymbol{q}|$,[12]

$$S_n=\int\mathrm{d}^nqS_n|\boldsymbol{q}\rangle\langle\boldsymbol{q}|=|\Lambda|^{1/2}\int\mathrm{d}^nq|\Lambda\boldsymbol{q}\rangle\langle\boldsymbol{q}|,\quad S_n^{\dagger}=S_n^{-1}, \tag{10}$$

since $\int\mathrm{d}^nq|\boldsymbol{q}\rangle\langle\boldsymbol{q}|=1$. Using the expression of $|\boldsymbol{q}\rangle$ in Fock space

$$|\boldsymbol{q}\rangle=\pi^{-n/4}\exp\left[-\frac{1}{2}\widetilde{\boldsymbol{q}}\boldsymbol{q}+\sqrt{2}\widetilde{\boldsymbol{q}}a^{\dagger}-\frac{1}{2}\tilde{a}^{\dagger}a^{\dagger}\right]|\mathbf{0}\rangle,\quad \tilde{a}^{\dagger}=(a_1^{\dagger},a_2^{\dagger},\ldots,a_n^{\dagger}), \tag{11}$$

and the normally ordered form of n-mode vacuum projector $|\mathbf{0}\rangle\langle\mathbf{0}|=:\exp[-\tilde{a}^{\dagger}a]:$, we can put S_n into the normal ordering form,

$$S_n=\pi^{-n/2}|\Lambda|^{1/2}\int\mathrm{d}^nq{:}\exp\left[-\frac{1}{2}\widetilde{\boldsymbol{q}}(1+\widetilde{\Lambda}\Lambda)\boldsymbol{q}+\sqrt{2}\widetilde{\boldsymbol{q}}(\widetilde{\Lambda}a^{\dagger}+a)-\frac{1}{2}(\widetilde{a}a+\tilde{a}^{\dagger}a^{\dagger})-\tilde{a}^{\dagger}a\right]{:}. \tag{12}$$

To perform the integration in Eq.(12) by virtue of the IWOP technique, using the mathematical formula

$$\int\mathrm{d}^nx\,\mathrm{e}^{-\widetilde{x}Fx+\widetilde{x}v}=\pi^{n/2}(\det F)^{-1/2}\mathrm{e}^{\frac{1}{4}\widetilde{v}F^{-1}v}, \tag{13}$$

then we derive

$$S_n=\left(\frac{\det\Lambda}{\det N}\right)^{1/2}\mathrm{e}^{\frac{1}{2}\tilde{a}^{\dagger}(\Lambda N^{-1}\widetilde{\Lambda}-I)a^{\dagger}}\times{:}\mathrm{e}^{\tilde{a}^{\dagger}(\Lambda N^{-1}-I)a}{:}\,\mathrm{e}^{\frac{1}{2}\tilde{a}(N^{-1}-I)a}, \tag{14}$$

where $N=(1+\widetilde{\Lambda}\Lambda)/2$. Equation (14) is just the normal product form of S_n.

3. Squeezing property of $S_n\left|\mathbf{0}\right\rangle$

Operating S_n on the n-mode vacuum state $\left|\mathbf{0}\right\rangle$, we obtain the squeezed vacuum state

$$S_n\left|\mathbf{0}\right\rangle=\left(\frac{\det\Lambda}{\det N}\right)^{1/2}\mathrm{e}^{\frac{1}{2}\tilde{a}^{\dagger}\left(\Lambda N^{-1}\widetilde{\Lambda}-I\right)a^{\dagger}}\left|\mathbf{0}\right\rangle. \quad (15)$$

Now we evaluate the variances of the n-mode quadratures. The quadratures in the n-mode case are defined as

$$X_1=\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}Q_i, X_2=\frac{1}{\sqrt{2n}}\sum_{i=1}^{n}P_i, \quad (16)$$

obeying $[X_1,X_2]=\frac{\mathrm{i}}{2}$. Their variances are $(\Delta X_i)^2=\left\langle X_i^2\right\rangle-\left\langle X_i\right\rangle^2$, $i=1,2$. Noting the expectation values of X_1 and X_2 in the state $S_n\left|\mathbf{0}\right\rangle$, $\left\langle X_1\right\rangle=\left\langle X_2\right\rangle=0$, then using Eqs.(7) and (8) we see that the variances are

$$\begin{aligned}(\Delta X_1)^2&=\left\langle\mathbf{0}\right|S_n^{-1}X_1^2S_n\left|\mathbf{0}\right\rangle\\&=\frac{1}{2n}\left\langle\mathbf{0}\right|S_n^{-1}\sum_{i=1}^{n}Q_i\sum_{j=1}^{n}Q_jS_n\left|\mathbf{0}\right\rangle\\&=\frac{1}{2n}\left\langle\mathbf{0}\right|\sum_{i=1}^{n}Q_k(\mathrm{e}^{-\lambda A})_{ki}\sum_{j=1}^{n}(\mathrm{e}^{-\lambda\tilde{A}})_{jl}Q_l\left|\mathbf{0}\right\rangle\\&=\frac{1}{2n}\sum_{i,j}^{n}(\mathrm{e}^{-\lambda A})_{ki}(\mathrm{e}^{-\lambda\tilde{A}})_{jl}\left\langle\mathbf{0}\right|Q_kQ_l\left|\mathbf{0}\right\rangle\\&=\frac{1}{4n}\sum_{i,j}^{n}(\mathrm{e}^{-\lambda A})_{ki}(\mathrm{e}^{-\lambda\tilde{A}})_{jl}\left\langle\mathbf{0}\right|a_ka_l^{\dagger}\left|\mathbf{0}\right\rangle\\&=\frac{1}{4n}\sum_{i,j}^{n}(\mathrm{e}^{-\lambda A})_{ki}(\mathrm{e}^{-\lambda\tilde{A}})_{jl}\delta_{kl}\\&=\frac{1}{4n}\sum_{i,j}^{n}(\widetilde{\Lambda}\Lambda)_{ij}, \quad (17)\end{aligned}$$

similarly we have

$$(\Delta X_2)^2=\left\langle\mathbf{0}\right|S_n^{-1}X_2^2S_n\left|\mathbf{0}\right\rangle=\frac{1}{4n}\sum_{i,j}^{n}\left[(\widetilde{\Lambda}\Lambda)^{-1}\right]_{ij}. \quad (18)$$

Equations (17) and (18) are the quadrature variance formulas in the transformed vacuum state acted by the operator $\exp[\mathrm{i}\lambda Q_iA_{ij}P_j]$. By observing that A in Eq.(6) is a symmetric matrix, we see

$$\sum_{i,j}^{n}\left[(A+\tilde{A})^l\right]_{ij}=2^{2l}n, \quad (19)$$

then using $A\tilde{A}=\tilde{A}A$, so $\widetilde{\Lambda}\Lambda=\mathrm{e}^{-\lambda(A+\tilde{A})}$, a symmetric matrix, we have

$$\begin{aligned}\sum_{i,j=1}^{n}(\widetilde{\Lambda}\Lambda)_{ij}&=\sum_{l=0}^{\infty}\frac{(-\lambda)^l}{l!}\sum_{i,j}^{n}\left[(A+\tilde{A})^l\right]_{ij}\\&=n\sum_{l=0}^{\infty}\frac{(-\lambda)^l}{l!}2^{2l}=n\mathrm{e}^{-4\lambda}, \quad (20)\end{aligned}$$

and

$$\sum_{i,j=1}^{n}(\widetilde{\Lambda}\Lambda)_{ij}^{-1}=n\mathrm{e}^{4\lambda}. \quad (21)$$

It then follows that

$$(\Delta X_1)^2=\frac{1}{4n}\sum_{i,j}^{n}(\widetilde{\Lambda}\Lambda)_{ij}=\frac{\mathrm{e}^{-4\lambda}}{4}, \quad (22)$$

$$(\Delta X_2)^2=\frac{1}{4n}\sum_{i,j}^{n}\left[(\widetilde{\Lambda}\Lambda)^{-1}\right]_{ij}=\frac{\mathrm{e}^{4\lambda}}{4}. \quad (23)$$

This leads to $(\Delta X_1)(\Delta X_2)=\frac{1}{4}$, which shows that S_n is a correct n-mode squeezing operator for the n-mode quadratures in Eq.(16). Furthermore, equations (22) and (23) clearly indicate that the squeezed vacuum state $S_n\left|\mathbf{0}\right\rangle$ may exhibit stronger squeezing ($\mathrm{e}^{-4\lambda}$) in one quadrature than that ($\mathrm{e}^{-2\lambda}$) of the usual two-mode squeezed vacuum state. This is a way of enhancing squeezing.

4. The Wigner function of $S_n\left|\mathbf{0}\right\rangle$

Wigner distribution functions[13–16] of quantum states are widely studied in quantum statistics and quantum optics. Now we derive the expression of the Wigner function of $S_n\left|\mathbf{0}\right\rangle$. Here we take a new method to do it. Recalling that in Ref.[17] we have introduced the Weyl ordering form of single-mode Wigner operator $\Delta_1(q_1,p_1)$,

$$\Delta_1(q_1,p_1)=\text{:}\,\delta(q_1-Q_1)\,\delta(p_1-P_1)\,\text{:}, \quad (24)$$

its normal ordering form is

$$\Delta_1(q_1,p_1)=\frac{1}{\pi}\text{: }\mathrm{e}^{-(q_1-Q_1)^2-(p_1-P_1)^2}\text{:} \quad (25)$$

where the symbols : : and $\text{:}\ \text{:}$ denote the normal ordering and the Weyl ordering, respectively. Note that the order of Bose operators a_1 and $a_1^{\dagger}$ within a normally ordered product and a Weyl ordered product can be permuted. That is to say, even though $[a_1,a_1^{\dagger}]=1$, we can have $:a_1a_1^{\dagger}:=:a_1^{\dagger}a_1:$ and $\text{:}a_1a_1^{\dagger}\text{:}=\text{:}a_1^{\dagger}a_1\text{:}$. The Weyl ordering has a remarkable property, i.e., the order-invariance of Weyl ordered operators under similar transformations, which means

$$U\,\text{:}\,(\circ\circ\circ)\,\text{:}\,U^{-1}=\text{:}\,U(\circ\circ\circ)U^{-1}\,\text{:}, \quad (26)$$

as if the "fence" $\text{:}\ \text{:}$ did not exist.

For n-mode case, the Weyl ordering form of the Wigner operator is

$$\Delta_n(\boldsymbol{q},\boldsymbol{p}) = {:}\,\delta(\boldsymbol{q}-\boldsymbol{Q})\,\delta(\boldsymbol{p}-\boldsymbol{P})\,{:}\,, \tag{27}$$

where $\widetilde{\boldsymbol{Q}} = (Q_1, Q_2, \ldots, Q_n)$ and $\widetilde{\boldsymbol{P}} = (P_1, P_2, \ldots, P_n)$. Then according to the Weyl ordering invariance under similar transformations and Eqs.(7) and (8) we have

$$\begin{aligned}
&S_n^{-1}\Delta_n(\boldsymbol{q},\boldsymbol{p})\,S_n\\
&= S_n^{-1}\,{:}\,\delta(\boldsymbol{q}-\boldsymbol{Q})\,\delta(\boldsymbol{p}-\boldsymbol{P})\,{:}\,S_n\\
&= {:}\,\delta\left(q_k - (\mathrm{e}^{-\lambda\tilde{A}})_{ki}Q_i\right)\delta\left(p_k - (\mathrm{e}^{\lambda A})_{ki}P_i\right)\,{:}\\
&= {:}\,\delta\left(\mathrm{e}^{\lambda\tilde{A}}\boldsymbol{q}-\boldsymbol{Q}\right)\delta\left(\mathrm{e}^{-\lambda A}\boldsymbol{p}-\boldsymbol{P}\right)\,{:}\\
&= \Delta\left(\mathrm{e}^{\lambda\tilde{A}}\boldsymbol{q}, \mathrm{e}^{-\lambda A}\boldsymbol{p}\right).
\end{aligned} \tag{28}$$

Thus using Eqs.(24) and (28) the Wigner function of $S_n|\boldsymbol{0}\rangle$ is

$$\begin{aligned}
&\langle\boldsymbol{0}|\,S_n^{-1}\Delta_n(\boldsymbol{q},\boldsymbol{p})\,S_n|\boldsymbol{0}\rangle\\
&= \frac{1}{\pi^n}\langle\boldsymbol{0}|\,{:}\exp[-(\mathrm{e}^{\lambda\tilde{A}}\boldsymbol{q}-\boldsymbol{Q})^2 - (\mathrm{e}^{-\lambda A}\boldsymbol{p}-\boldsymbol{P})^2]{:}\,|\boldsymbol{0}\rangle\\
&= \frac{1}{\pi^n}\exp[-(\mathrm{e}^{\lambda\tilde{A}}\boldsymbol{q})^2 - \left(\mathrm{e}^{-\lambda A}\boldsymbol{p}\right)^2]\\
&= \frac{1}{\pi^n}\exp\left[-\widetilde{\boldsymbol{q}}\,\mathrm{e}^{\lambda A}\mathrm{e}^{\lambda\tilde{A}}\boldsymbol{q} - \widetilde{\boldsymbol{p}}\,\mathrm{e}^{-\lambda\tilde{A}}\mathrm{e}^{-\lambda A}\boldsymbol{p}\right]\\
&= \frac{1}{\pi^n}\exp\left[-\widetilde{\boldsymbol{q}}\left(\Lambda\tilde{\Lambda}\right)^{-1}\boldsymbol{q} - \widetilde{\boldsymbol{p}}\Lambda\tilde{\Lambda}\boldsymbol{p}\right].
\end{aligned} \tag{29}$$

From Eq.(29) we see that once the explicit expression of $\Lambda\tilde{\Lambda} = \exp[-\lambda(A+\tilde{A})]$ is deduced, the Wigner function of $S_n|\boldsymbol{0}\rangle$ can be calculated.

5. Some examples of calculating the Wigner function

For $n = 2$, form Eq.(4) we have $S_2' = \exp[\mathrm{i}2\lambda(Q_1P_2 + Q_2P_1)]$ which exhibits clearly the stronger squeezing than the usual two-mode squeezing operator S_2'. For $n = 3$, the three-mode operator S_3, from Eq.(6) we see that the matrix A is $\begin{pmatrix} 0 & 1 & 1\\ 1 & 0 & 1\\ 1 & 1 & 0 \end{pmatrix}$, thus we have

$$\Lambda\tilde{\Lambda} = \begin{pmatrix} u & v & v\\ v & u & v\\ v & v & u \end{pmatrix}, \tag{30}$$

where $u = \frac{2}{3}\mathrm{e}^{2\lambda} + \frac{1}{3\mathrm{e}^{4\lambda}}$, $v = \frac{1}{3\mathrm{e}^{4\lambda}} - \frac{1}{3}\mathrm{e}^{2\lambda}$, and $(\Lambda\tilde{\Lambda})^{-1}$ is obtained by replacing λ with $-\lambda$ in $\Lambda\tilde{\Lambda}$. Thus the squeezing state $S_3|000\rangle$ is[18]

$$S_3|000\rangle = A_3\mathrm{e}^{\frac{1}{6}A_1\sum_{i=1}^3 a_i^{\dagger 2} - \frac{2}{3}A_2\sum_{i<j}^3 a_i^\dagger a_j^\dagger}|000\rangle, \tag{31}$$

where $A_1 = (1-\mathrm{sech}2\lambda)\tanh\lambda$, $A_2 = \frac{\sinh 3\lambda}{2\cosh\lambda\cosh 2\lambda}$, $A_3 = \mathrm{sech}\,\lambda\cosh^{-1/2}2\lambda$.

In particular, for the case of the infinite squeezing $\lambda\to\infty$, equation (31) reduces to

$$S_3|000\rangle \sim \mathrm{e}^{\frac{1}{6}\left[\sum_{i=1}^3 a_i^{\dagger 2} - 4\sum_{i<j}^3 a_i^\dagger a_j^\dagger\right]}|000\rangle \equiv |\,\rangle_{s_3}, \tag{32}$$

which is just the common eigenvector of the three compatible Jacobian operators in the three-body case with zero eigenvalues,[19] i.e.,

$$\begin{aligned}
&(P_1+P_2+P_3)\,|\,\rangle_{s_3} = 0,\\
&(Q_3-Q_2)\,|\,\rangle_{s_3} = 0,\\
&\left(\frac{\mu_3Q_3+\mu_2Q_2}{\mu_3+\mu_2} - Q_1\right)|\,\rangle_{s_3} = 0,\\
&\left(\mu_i = \frac{m_i}{m_1+m_2+m_3}\right),
\end{aligned} \tag{33}$$

as common eigenvector $[P_1+P_2+P_3, Q_3-Q_2] = 0$,

$$\left[\frac{\mu_3Q_3+\mu_2Q_2}{\mu_3+\mu_2} - Q_1, P_1+P_2+P_3\right] = 0.$$

Since the common eigenvector of three compatible Jacobian operators is an entangled state, the state $|\,\rangle_{s_3}$ is also an entangled state. By using Eq.(29) the Wigner function is

$$\begin{aligned}
&\langle\boldsymbol{0}|\,S_3^{-1}\Delta_3(\boldsymbol{q},\boldsymbol{p})\,S_3|\boldsymbol{0}\rangle\\
&= \frac{1}{\pi^3}\exp\left[-\frac{2}{3}(\cosh 4\lambda + 2\cosh 2\lambda)\sum_{i=1}^3|\alpha_i|^2\right]\\
&\quad\times\exp\{-\frac{1}{3}(\sinh 4\lambda - 2\sinh 2\lambda)\sum_{i=1}^3\alpha_i^2\\
&\quad-\frac{2}{3}\sum_{j>i=1}^3\left[(\cosh 4\lambda - \cosh 2\lambda)\,\alpha_i\alpha_j^*\right.\\
&\quad\left.+(\sinh 2\lambda + \sinh 4\lambda)\,\alpha_i\alpha_j\right] + \mathrm{c.c.}\}.
\end{aligned} \tag{34}$$

For the case of $n = 4$, the four-mode operator S_4 is

$$S_4 = \mathrm{e}^{\mathrm{i}\lambda[(Q_1+Q_3)(P_4+P_2)+(Q_2+Q_4)(P_1+P_3)]}, \tag{35}$$

the matrix $A = \begin{pmatrix} 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \end{pmatrix}$, thus we have $\Lambda\tilde{A} = \begin{pmatrix} r & t & s & t \\ t & r & t & s \\ s & t & r & t \\ t & s & t & r \end{pmatrix}$, (36)

where $r = \cosh^2 2\lambda, s = \sinh^2 2\lambda, t = -\sinh 2\lambda \cosh 2\lambda$. Then substituting Eq.(36) into Eq.(29) we obtain

$$\langle \mathbf{0}| S_4^{-1} \Delta_4 (\boldsymbol{q}, \boldsymbol{p}) S_4 |\mathbf{0}\rangle = \frac{1}{\pi^4} \mathrm{e}^{-2\cosh^2 2\lambda \left[\sum_{i=1}^4 |\alpha_i|^2 + (M+M^*)\tanh^2 2\lambda + (R^*+R)\tanh 2\lambda\right]}, \tag{37}$$

where $M = \alpha_1\alpha_3^* + \alpha_2\alpha_4^*$, $R = \alpha_1\alpha_2 + \alpha_1\alpha_4 + \alpha_2\alpha_3 + \alpha_3\alpha_4$. This form differs evidently from the Wigner function of the direct-product of usual two two-mode squeezed states' Wigner functions. In addition, using Eq.(36) we can check Eqs.(22) and (23). Further, we have

$$N^{-1} = \frac{1}{2} \begin{pmatrix} 2 & \tanh 2\lambda & 0 & \tanh 2\lambda \\ \tanh 2\lambda & 2 & \tanh 2\lambda & 0 \\ 0 & \tanh 2\lambda & 2 & \tanh 2\lambda \\ \tanh 2\lambda & 0 & \tanh 2\lambda & 2 \end{pmatrix}, \tag{38}$$

$$\det N = \cosh^2 2\lambda.$$

Then substituting Eq.(38) into Eq.(14) yields the four-mode squeezed state,

$$S_4 |0000\rangle = \mathrm{sech} 2\lambda \, \mathrm{e}^{-\frac{1}{2}\left(a_1^\dagger + a_3^\dagger\right)\left(a_2^\dagger + a_4^\dagger\right)\tanh 2\lambda} |0000\rangle, \tag{39}$$

from which one can see that the four-mode squeezed state is not the same as the direct product of two two-mode squeezed states in Eq.(1). For more discussions about this state, we refer to Ref.[20].

In summary, by virtue of Dirac's coordinate representation and the IWOP technique we have shown that an n-mode squeezing operator $S_n \equiv \exp[\mathrm{i}\,\lambda \sum_{i=1}^n (Q_i P_{i+1} + Q_{i+1} P_i)]$, $(Q_{n+1} = Q_1,\ P_{n+1} = P_1)$, is an n-mode squeezing operator which enhances the stronger squeezing for the n-mode quadratures.[21] S_n's normally ordered expansion and new n-mode squeezed vacuum states are obtained.

References

[1] Stoler D 1970 *Phys. Rev.* D **1** 3217

[2] Bouwmeester D, Ekert A and Zeilinger A 2000 *The Physics of Quantum Information* (Berlin: Springer)

Nielsen M A and Chuang I L 2000 *Quantum Computation and Quantum Information* (Cambridge: Cambridge University Press)

[3] Buzek V 1990 *J. Mod. Opt.* **37** 303

[4] Loudon R and Knight P L 1987 *J. Mod. Opt.* **34** 709

[5] Dodonov V V 2002 *J. Opt.* B: *Quantum Semiclass. Opt.* **4** R1

[6] Fan H Y and Klauder J R 1994 *Phys. Rev.* A **49** 704

[7] Fan H Y and Fan Y 1996 *Phys. Rev.* A **54** 958

[8] Wang F Q, Zhang Z M and Liang R S 2009 *Chin. Phys.* B **18** 597

Xu X F and Zhu S Q 2009 *Chin. Phys.* B **18** 1512

[9] Einstein A, Poldolsky B and Rosen N 1935 *Phys. Rev.* **47** 777

[10] Fan H Y 2008 *Ann. Phys.* **323** 500

[11] Fan H Y 2003 *J. Opt.* B: *Quantum Semiclass. Opt.* **5** R147

[12] Dirac P A M 1958 *The Principle of Quantum Mechanics* Fourth ed. (Oxford: Oxford University Press)

[13] Wigner E P 1932 *Phys. Rev.* **40** 749

[14] O'Connell R F and Wigner E P 1981 *Phys. Lett.* A **83** 145

[15] Schleich W 2001 *Quantum Optics* (New York: Wiley)

[16] Hu L Y and Fan H Y 2009 *Chin. Phys.* B **18** 902

[17] Fan H Y 1992 *J. Phys.* A **25** 3443

[18] Hu L Y and Fan H Y 2008 *Mod. Phys. Lett.* B **22** 2055

[19] Fan H Y and Zhang Y 1998 *Phys. Rev.* A **57** 3225

[20] Hu L Y and Fan H Y 2008 *J. Mod. Opt.* **55** 1065

[21] Hu L Y and Fan H Y 2009 *Europhys. Lett.* **85** 60001