Vol 18 No 12, December 2009
1674-1056/2009/18(12)/5123-10

Chinese Physics B

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# Solving mKdV–sinh-Gordon equation by a modified variable separated ordinary differential equation method*

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(Received 11 April 2009; revised manuscript received 21 June 2009)

By introducing a more general auxiliary ordinary differential equation (ODE), a modified variable separated ODE method is developed for solving the mKdV–sinh-Gordon equation. As a result, many explicit and exact solutions including some new formal solutions are successfully picked up for the mKdV–sinh-Gordon equation by this approach.

**Keywords:** modified variable separated ODE method, mKdV–sinh-Gordon equation, explicit and exact solution

**PACC:** 0340K, 0290

## 1. Introduction

The investigation of the explicit and exact solutions of nonlinear evolution equations (NEEs) plays a very important and significant role in the study of nonlinear problems partly because these solutions may well describe or explain various complex phenomena in nature, such as vibrations, solitons and propagation with a finite speed and partly because they may give more insight into the physics aspects of problems. In recent years, much effort has been spent on the construction of the explicit and exact solutions of NEEs and a vast variety of powerful and efficient methods have been proposed and developed. They are the hyperbolic tangent function expansion method,[1−5] the homogeneous balance method,[6,7] the trial function method,[8−11] the Jacobi elliptic function expansion method,[12−16] the trial equation method,[17,18] the homotopy method,[19−23] the variable coefficient algebraic method,[24] the combination method,[25−27] the $(\omega/g)$-expansion method,[28] the additional generating condition method,[29] the transplantation method of solution,[30] and so on. Recently, Sirendaoreji and Sun[31] developed a direct algebraic method named as the variable separated ordinary differential equation (ODE) method of solving sine–Gordon type equations, which has scientific value and reliability and works effectively if the equation involves sine, cosine, hyperbolic sine and hyperbolic cosine functions. More recently, Wazwaz[32] extended the study conducted by Sirendaoreji and Sun to solve the mKdV–sinh-Gordon equation. In the present paper, we would like to further modify Wazwaz's method and to present a modified variable separated ODE method to pick up more explicit and exact solutions (especially some new solutions) of the mKdV–sinh-Gordon equation by introducing a more general auxiliary ODE and with the aid of its manifold types of solutions.

## 2. Solutions of the mKdV–sinh-Gordon equation

The well-known mKdV–sinh-Gordon equation reads

$$u_{xt} - \frac{3}{2}u_x^2 u_{xx} + u_{xxxx} = \sinh u, \tag{1}$$

which was solved by Wazwaz[32] by the variable separated ODE method given by Sirendaoreji and Sun.[31] As stated above, the motivation of this paper is to further modify Wazwaz's study to construct more explicit and exact solutions (especially some new solutions) of the mKdV–sinh-Gordon equation. To this end, we take the following travelling wave transformation

$$u = u(\xi), \quad \xi = x - vt. \tag{2}$$

Putting expression (2) into Eq.(1) engenders the following ODE for $\xi$

$$-vu'' + \frac{3}{2}(u')^2 u'' + u^{(4)} = \sinh u, \tag{3}$$

where primes denote the derivatives for $\xi$.

*Project supported by the National Natural Science Foundation of China (Grant No 10672053).

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In order to facilitate the solution of Eq.(3), here we suppose that $u(\xi)$ in Eq.(3) satisfies the following additional ODE

$$u' = a_0 + \sum_{j=1}^{n} \left( a_j \cosh^j \frac{1}{2}u + b_j \sinh^j \frac{1}{2}u \right), \tag{4}$$

where $a_j$ $(j = 0, 1, 2, \ldots)$ and $b_j$ $(j = 1, 2, \ldots)$ are constants to be determined, and the positive integer $n$ can be determined by the leading-order analysis method.

Using the following formulae

$$\sinh u = \frac{\mathrm{e}^u - \mathrm{e}^{-u}}{2}, \quad \cosh u = \frac{\mathrm{e}^u + \mathrm{e}^{-u}}{2}, \tag{5}$$

and inserting Eqs.(5) and (4) into (3) and employing the leading-order analysis method, we get $n = 1$, and therefore

$$u' = a_0 + a_1 \cosh \frac{1}{2}u + b_1 \sinh \frac{1}{2}u. \tag{6}$$

For the sake of generality, in the following we first seek the solutions to Eq.(6). By the method of variable separation and with the aid of Mathematica, we can find the solutions of Eq.(6), which are listed as follows.

**Case 1** When $a_1 = b_1 = 0,\ a_0 \neq 0$

$$u(\xi) = a_0\xi + \xi_0. \tag{7}$$

**Case 2** When $a_0 = b_1 = 0,\ a_1 \neq 0$

$$u(\xi) = 2\ln\left[\tan\left(\frac{1}{4}a_1\xi + \xi_0\right)\right], \tag{8}$$

$$u(\xi) = 2\ln\left[-\cot\left(\frac{1}{4}a_1\xi + \xi_0\right)\right], \tag{9}$$

$$u(\xi) = 2\ln\left[\tan\left(\frac{1}{2}a_1\xi + \xi_0\right) \pm \sec\left(\frac{1}{2}a_1\xi + \xi_0\right)\right], \tag{10}$$

$$u(\xi) = 2\ln\left[-\cot\left(\frac{1}{2}a_1\xi + \xi_0\right) \mp \csc\left(\frac{1}{2}a_1\xi + \xi_0\right)\right], \tag{11}$$

$$u(\xi) = 2\ln\left[\frac{\tan\left(\frac{1}{2}a_1\xi + \xi_0\right)}{1 \pm \sec\left(\frac{1}{2}a_1\xi + \xi_0\right)}\right], \tag{12}$$

$$u(\xi) = 2\ln\left[\frac{-\cot\left(\frac{1}{2}a_1\xi + \xi_0\right)}{1 \pm \csc\left(\frac{1}{2}a_1\xi + \xi_0\right)}\right], \tag{13}$$

$$u(\xi) = 2\ln\left[\frac{-A\cos\left(\frac{1}{2}a_1\xi + \xi_0\right) \mp \sqrt{A^2 - B^2}}{A\sin\left(\frac{1}{2}a_1\xi + \xi_0\right) \pm B}\right], \tag{14}$$

$$u(\xi) = 2\ln\left[\frac{A\sin\left(\frac{1}{2}a_1\xi + \xi_0\right) \mp \sqrt{A^2 - B^2}}{A\cos\left(\frac{1}{2}a_1\xi + \xi_0\right) \pm B}\right], \tag{15}$$

$$u(\xi) = 2\,\mathrm{arcsinh}\left[\tan\left(\frac{1}{2}a_1\xi + \xi_0\right)\right], \tag{16}$$

$$u(\xi) = 2\,\mathrm{arcsinh}\left[-\cot\left(\frac{1}{2}a_1\xi + \xi_0\right)\right], \tag{17}$$

$$u(\xi) = 2\,\mathrm{arcsinh}[\tan(a_1\xi + \xi_0) \pm \sec(a_1\xi + \xi_0)], \tag{18}$$

$$u(\xi) = 2\,\mathrm{arcsinh}[-\cot(a_1\xi + \xi_0) \mp \csc(a_1\xi + \xi_0)], \tag{19}$$

$$u(\xi) = 2\,\mathrm{arcsinh}\left[\frac{\tan(a_1\xi + \xi_0)}{1 \pm \sec(a_1\xi + \xi_0)}\right], \tag{20}$$

$$u(\xi) = 2\,\mathrm{arcsinh}\left[\frac{-\cot(a_1\xi + \xi_0)}{1 \pm \csc(a_1\xi + \xi_0)}\right], \tag{21}$$

$$u(\xi) = 2\,\mathrm{arcsinh}\left[\frac{-A\cos(a_1\xi + \xi_0) \mp \sqrt{A^2 - B^2}}{A\sin(a_1\xi + \xi_0) \pm B}\right], \tag{22}$$

$$u(\xi) = 2\,\mathrm{arcsinh}\left[\frac{A\sin(a_1\xi + \xi_0) \mp \sqrt{A^2 - B^2}}{A\cos(a_1\xi + \xi_0) \pm B}\right], \tag{23}$$

$$u(\xi) = 2\,\mathrm{arccosh}\left[\pm\sec\left(\frac{1}{2}a_1\xi + \xi_0\right)\right], \tag{24}$$

$$u(\xi) = 2\,\mathrm{arccosh}\left[\pm\csc\left(\frac{1}{2}a_1\xi + \xi_0\right)\right], \tag{25}$$

$$u(\xi) = 4\,\mathrm{arctanh}\left[\tan\left(\frac{1}{4}a_1\xi + \xi_0\right)\right], \tag{26}$$

$$u(\xi) = 4\,\mathrm{arctanh}\left[-\cot\left(\frac{1}{4}a_1\xi + \xi_0\right)\right], \tag{27}$$

$$u(\xi) = 4\,\mathrm{arctanh}\left[\tan\left(\frac{1}{2}a_1\xi + \xi_0\right) \pm \sec\left(\frac{1}{2}a_1\xi + \xi_0\right)\right], \tag{28}$$

$$u(\xi) = 4\,\mathrm{arctanh}\left[-\cot\left(\frac{1}{2}a_1\xi + \xi_0\right) \mp \csc\left(\frac{1}{2}a_1\xi + \xi_0\right)\right], \tag{29}$$

$$u(\xi) = 4\,\mathrm{arctanh}\left[\frac{\tan\left(\frac{1}{2}a_1\xi + \xi_0\right)}{1 \pm \sec\left(\frac{1}{2}a_1\xi + \xi_0\right)}\right], \tag{30}$$

$$u(\xi) = 4\,\mathrm{arctanh}\left[\frac{-\cot\left(\frac{1}{2}a_1\xi + \xi_0\right)}{1 \pm \csc\left(\frac{1}{2}a_1\xi + \xi_0\right)}\right], \tag{31}$$

$$u(\xi) = 4\,\mathrm{arctanh}\left[\frac{-A\cos\left(\frac{1}{2}a_1\xi + \xi_0\right) \mp \sqrt{A^2 - B^2}}{A\sin\left(\frac{1}{2}a_1\xi + \xi_0\right) \pm B}\right], \tag{32}$$

$$u(\xi) = 4\,\mathrm{arctanh}\left[\frac{A\sin\left(\frac{1}{2}a_1\xi + \xi_0\right) \mp \sqrt{A^2 - B^2}}{A\cos\left(\frac{1}{2}a_1\xi + \xi_0\right) \pm B}\right], \tag{33}$$

$$u(\xi) = 4\,\mathrm{arccoth}\left[-\tan\left(\frac{1}{4}a_1\xi + \xi_0\right)\right], \tag{34}$$

$$u(\xi) = 4\,\mathrm{arccoth}\left[\cot\left(\frac{1}{4}a_1\xi + \xi_0\right)\right], \tag{35}$$

$$u(\xi)=4\,\mathrm{arccoth}\left[-\tan\left(\frac{1}{2}a_1\xi+\xi_0\right)\mp\sec\left(\frac{1}{2}a_1\xi+\xi_0\right)\right], \tag{36}$$

$$u(\xi)=4\,\mathrm{arccoth}\left[\cot\left(\frac{1}{2}a_1\xi+\xi_0\right)\pm\csc\left(\frac{1}{2}a_1\xi+\xi_0\right)\right], \tag{37}$$

$$u(\xi)=4\,\mathrm{arccoth}\left[\frac{-\tan\left(\frac{1}{2}a_1\xi+\xi_0\right)}{1\pm\sec\left(\frac{1}{2}a_1\xi+\xi_0\right)}\right], \tag{38}$$

$$u(\xi)=4\,\mathrm{arccoth}\left[\frac{\cot\left(\frac{1}{2}a_1\xi+\xi_0\right)}{1\pm\csc\left(\frac{1}{2}a_1\xi+\xi_0\right)}\right], \tag{39}$$

$$u(\xi)=4\,\mathrm{arccoth}\left[\frac{A\cos\left(\frac{1}{2}a_1\xi+\xi_0\right)\mp\sqrt{A^2-B^2}}{A\sin\left(\frac{1}{2}a_1\xi+\xi_0\right)\pm B}\right], \tag{40}$$

$$u(\xi)=4\,\mathrm{arccoth}\left[\frac{-A\sin\left(\frac{1}{2}a_1\xi+\xi_0\right)\mp\sqrt{A^2-B^2}}{A\cos\left(\frac{1}{2}a_1\xi+\xi_0\right)\pm B}\right]. \tag{41}$$

In the above equations, $A$ and $B$ are two non-zero real constants and satisfy the constraint condition $A^2>B^2$.

**Case 3** When $a_0=a_1=0,\ b_1\neq 0$

$$u(\xi)=4\,\mathrm{arctanh}\left[\pm\exp\left(\frac{1}{2}b_1\xi+\xi_0\right)\right], \tag{42}$$

$$u(\xi)=4\,\mathrm{arccoth}\left[-\exp\left(\frac{1}{2}b_1\xi+\xi_0\right)\right], \tag{43}$$

$$u(\xi)=2\ln\left[-\tanh\left(\frac{1}{4}b_1\xi+\xi_0\right)\right], \tag{44}$$

$$u(\xi)=2\ln\left[-\coth\left(\frac{1}{4}b_1\xi+\xi_0\right)\right], \tag{45}$$

$$u(\xi)=2\ln\left[-\tanh\left(\frac{1}{2}b_1\xi+\xi_0\right)\mp\mathrm{i}\,\mathrm{sech}\left(\frac{1}{2}b_1\xi+\xi_0\right)\right], \tag{46}$$

$$u(\xi)=2\ln\left[-\coth\left(\frac{1}{2}b_1\xi+\xi_0\right)\mp\mathrm{csch}\left(\frac{1}{2}b_1\xi+\xi_0\right)\right], \tag{47}$$

$$u(\xi)=2\ln\left[\frac{-\tanh\left(\frac{1}{2}b_1\xi+\xi_0\right)}{1\pm\mathrm{sech}\left(\frac{1}{2}b_1\xi+\xi_0\right)}\right], \tag{48}$$

$$u(\xi)=2\ln\left[\frac{-\coth\left(\frac{1}{2}b_1\xi+\xi_0\right)}{1\pm\mathrm{i}\,\mathrm{csch}\left(\frac{1}{2}b_1\xi+\xi_0\right)}\right], \tag{49}$$

$$u(\xi)=2\ln\left[\frac{-A\cosh\left(\frac{1}{2}b_1\xi+\xi_0\right)\mp\sqrt{A^2+B^2}}{A\sinh\left(\frac{1}{2}b_1\xi+\xi_0\right)\pm B}\right], \tag{50}$$

$$u(\xi)=2\ln\left[\frac{-A\sinh\left(\frac{1}{2}b_1\xi+\xi_0\right)\mp\sqrt{B^2-A^2}}{A\cosh\left(\frac{1}{2}b_1\xi+\xi_0\right)\pm B}\right], \tag{51}$$

$$u(\xi)=2\,\mathrm{arcsinh}\left[\pm\mathrm{csch}\left(\frac{1}{2}b_1\xi+\xi_0\right)\right], \tag{52}$$

$$u(\xi)=2\,\mathrm{arcsinh}\left[\pm\mathrm{i}\,\mathrm{sech}\left(\frac{1}{2}b_1\xi+\xi_0\right)\right], \tag{53}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\pm\tanh\left(\frac{1}{2}b_1\xi+\xi_0\right)\right], \tag{54}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\pm\coth\left(\frac{1}{2}b_1\xi+\xi_0\right)\right], \tag{55}$$

$$u(\xi)=2\,\mathrm{arccosh}[\pm\tanh(b_1\xi+\xi_0)\mp\mathrm{i}\,\mathrm{sech}(b_1\xi+\xi_0)], \tag{56}$$

$$u(\xi)=2\,\mathrm{arccosh}[\pm\coth(b_1\xi+\xi_0)\mp\mathrm{csch}(b_1\xi+\xi_0)], \tag{57}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{\pm\tanh(b_1\xi+\xi_0)}{1\pm\mathrm{sech}(b_1\xi+\xi_0)}\right], \tag{58}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{\pm\coth(b_1\xi+\xi_0)}{1\pm\mathrm{i}\,\mathrm{csch}(b_1\xi+\xi_0)}\right], \tag{59}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{\pm A\cosh(b_1\xi+\xi_0)\mp\sqrt{A^2+B^2}}{A\sinh(b_1\xi+\xi_0)\pm B}\right], \tag{60}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{\pm A\sinh(b_1\xi+\xi_0)\mp\sqrt{B^2-A^2}}{A\cosh(b_1\xi+\xi_0)\pm B}\right]. \tag{61}$$

In the above equations, $A$ and $B$ are two non-zero real constants and satisfy the constraint condition $B^2>A^2$.

**Case 4** When $a_1=0,\ a_0\neq 0,\ b_1\neq 0$

$$u(\xi)=2\ln\left[-\frac{1}{b_1}\left(a_0+\sqrt{a_0^2+b_1^2}\tanh\left(\frac{\sqrt{a_0^2+b_1^2}}{4}\xi+\xi_0\right)\right)\right], \tag{62}$$

$$u(\xi)=2\ln\left[-\frac{1}{b_1}\left(a_0+\sqrt{a_0^2+b_1^2}\coth\left(\frac{\sqrt{a_0^2+b_1^2}}{4}\xi+\xi_0\right)\right)\right], \tag{63}$$

$$u(\xi)=2\,\mathrm{arcsinh}\left[\frac{b_1^2-\left(a_0+\sqrt{a_0^2+b_1^2}\tanh\left((\sqrt{a_0^2+b_1^2}\xi)/4+\xi_0\right)\right)^2}{2b_1\left(a_0+\sqrt{a_0^2+b_1^2}\tanh\left((\sqrt{a_0^2+b_1^2}\xi)/4+\xi_0\right)\right)}\right], \tag{64}$$

$$u(\xi)=2\,\mathrm{arcsinh}\left[\frac{b_1^2-\left(a_0+\sqrt{a_0^2+b_1^2}\coth\left((\sqrt{a_0^2+b_1^2}\xi)/4+\xi_0\right)\right)^2}{2b_1\left(a_0+\sqrt{a_0^2+b_1^2}\coth\left((\sqrt{a_0^2+b_1^2}\xi)/4+\xi_0\right)\right)}\right], \tag{65}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{-b_1^2-\left(a_0+\sqrt{a_0^2+b_1^2}\tanh\left((\sqrt{a_0^2+b_1^2}\xi)/4+\xi_0\right)\right)^2}{2b_1\left(a_0+\sqrt{a_0^2+b_1^2}\tanh\left((\sqrt{a_0^2+b_1^2}\xi)/4+\xi_0\right)\right)}\right], \tag{66}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{-b_1^2-\left(a_0+\sqrt{a_0^2+b_1^2}\coth\left((\sqrt{a_0^2+b_1^2}\xi)/4+\xi_0\right)\right)^2}{2b_1\left(a_0+\sqrt{a_0^2+b_1^2}\coth\left((\sqrt{a_0^2+b_1^2}\xi)/4+\xi_0\right)\right)}\right]. \tag{67}$$

**Case 5** When $b_1=0,\ a_0\neq 0,\ a_1\neq 0$

**Case 5.1** For $a_0^2>a_1^2$

$$u(\xi)=2\ln\left[\frac{1}{a_1}\left(-a_0-\sqrt{a_0^2-a_1^2}\tanh\left(\frac{\sqrt{a_0^2-a_1^2}}{4}\xi+\xi_0\right)\right)\right], \tag{68}$$

$$u(\xi)=2\ln\left[\frac{1}{a_1}\left(-a_0-\sqrt{a_0^2-a_1^2}\coth\left(\frac{\sqrt{a_0^2-a_1^2}}{4}\xi+\xi_0\right)\right)\right], \tag{69}$$

$$u(\xi)=2\,\mathrm{arcsinh}\left[\frac{a_1^2-\left(a_0+\sqrt{a_0^2-a_1^2}\tanh\left((\sqrt{a_0^2-a_1^2}\xi)/4+\xi_0\right)\right)^2}{2a_1\left(a_0+\sqrt{a_0^2-a_1^2}\tanh\left((\sqrt{a_0^2-a_1^2}\xi)/4+\xi_0\right)\right)}\right], \tag{70}$$

$$u(\xi)=2\,\mathrm{arcsinh}\left[\frac{a_1^2-\left(a_0+\sqrt{a_0^2-a_1^2}\coth\left((\sqrt{a_0^2-a_1^2}\xi)/4+\xi_0\right)\right)^2}{2a_1\left(a_0+\sqrt{a_0^2-a_1^2}\coth\left((\sqrt{a_0^2-a_1^2}\xi)/4+\xi_0\right)\right)}\right], \tag{71}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{-a_1^2-\left(a_0+\sqrt{a_0^2-a_1^2}\tanh\left((\sqrt{a_0^2-a_1^2}\xi)/4+\xi_0\right)\right)^2}{2a_1\left(a_0+\sqrt{a_0^2-a_1^2}\tanh\left((\sqrt{a_0^2-a_1^2}\xi)/4+\xi_0\right)\right)}\right], \tag{72}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{-a_1^2-\left(a_0+\sqrt{a_0^2-a_1^2}\coth\left((\sqrt{a_0^2-a_1^2}\xi)/4+\xi_0\right)\right)^2}{2a_1\left(a_0+\sqrt{a_0^2-a_1^2}\coth\left((\sqrt{a_0^2-a_1^2}\xi)/4+\xi_0\right)\right)}\right]. \tag{73}$$

**Case 5.2** For $a_0^2<a_1^2$

$$u(\xi)=2\ln\left[\frac{1}{a_1}\left(-a_0+\sqrt{a_1^2-a_0^2}\tan\left(\frac{\sqrt{a_1^2-a_0^2}}{4}\xi+\xi_0\right)\right)\right], \tag{74}$$

$$u(\xi)=2\ln\left[\frac{1}{a_1}\left(-a_0-\sqrt{a_1^2-a_0^2}\cot\left(\frac{\sqrt{a_1^2-a_0^2}}{4}\xi+\xi_0\right)\right)\right], \tag{75}$$

$$u(\xi)=2\,\mathrm{arcsinh}\left[\frac{-a_1^2+\left(-a_1+\sqrt{a_1^2-a_0^2}\tan\left((\sqrt{a_1^2-a_0^2}\xi)/4+\xi_0\right)\right)^2}{2a_1\left(-a_0+\sqrt{a_1^2-a_0^2}\tan\left((\sqrt{a_1^2-a_0^2}\xi)/4+\xi_0\right)\right)}\right], \tag{76}$$

$$u(\xi)=2\,\mathrm{arcsinh}\left[\frac{a_1^2-\left(a_0+\sqrt{a_1^2-a_0^2}\cot\left((\sqrt{a_1^2-a_0^2}\xi)/4+\xi_0\right)\right)^2}{2a_1\left(a_0+\sqrt{a_1^2-a_0^2}\cot\left((\sqrt{a_1^2-a_0^2}\xi)/4+\xi_0\right)\right)}\right], \tag{77}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{a_1^2+\left(-a_0+\sqrt{a_1^2-a_0^2}\tan\left((\sqrt{a_1^2-a_0^2}\xi)/4+\xi_0\right)\right)^2}{2a_1\left(-a_0+\sqrt{a_1^2-a_0^2}\tan\left((\sqrt{a_1^2-a_0^2}\xi)/4+\xi_0\right)\right)}\right], \tag{78}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{-a_1^2-\left(a_0+\sqrt{a_1^2-a_0^2}\cot\left((\sqrt{a_1^2-a_0^2}\xi)/4+\xi_0\right)\right)^2}{2a_1\left(a_0+\sqrt{a_1^2-a_0^2}\cot\left((\sqrt{a_1^2-a_0^2}\xi)/4+\xi_0\right)\right)}\right]. \tag{79}$$

**Case 5.3** For $a_1=a_0$

$$u(\xi)=2\ln\Big(-1-\frac{4}{a_0\xi+\xi_0}\Big), \tag{80}$$

$$u(\xi)=2\,\mathrm{arcsinh}\left[\frac{-4(2+a_0\xi+\xi_0)}{(a_0\xi+\xi_0)(4+a_0\xi+\xi_0)}\right], \tag{81}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{-(a_0\xi+\xi_0)^2-4(a_0\xi+\xi_0)-8}{(a_0\xi+\xi_0)(4+a_0\xi+\xi_0)}\right]. \tag{82}$$

**Case 5.4** For $a_1=-a_0$

$$u(\xi)=2\ln\Big(1+\frac{4}{a_0\xi+\xi_0}\Big), \tag{83}$$

$$u(\xi)=2\,\mathrm{arcsinh}\left[\frac{4(2+a_0\xi+\xi_0)}{(a_0\xi+\xi_0)(4+a_0\xi+\xi_0)}\right], \tag{84}$$

$$u(\xi)=2\,\mathrm{arccosh}\left[\frac{(a_0\xi+\xi_0)^2+4(a_0\xi+\xi_0)+8}{(a_0\xi+\xi_0)(4+a_0\xi+\xi_0)}\right]. \tag{85}$$

Substituting Eq.(6) into Eq.(3) and equating the coefficients of those hyperbolic sine and hyperbolic cosine functions result in the following set of algebraic equations with respect to the undetermined constants $a_0$, $a_1$, $b_1$ and $v$

$$-\frac{1}{2}a_0a_1v-\frac{5}{8}a_0^3a_1-\frac{5}{16}a_0a_1^3+\frac{5}{16}a_0a_1b_1^2=0, \tag{86}$$

$$-\frac{1}{2}a_0b_1v-\frac{5}{8}a_0^3b_1+\frac{5}{16}a_0b_1^3-\frac{5}{16}a_0a_1^2b_1=0, \tag{87}$$

$$-2-\frac{1}{2}a_1^2v-\frac{11}{8}a_0^2a_1^2-\frac{1}{8}a_1^4-\frac{1}{2}b_1^2v-\frac{11}{8}a_0^2b_1^2+\frac{1}{8}b_1^4=0, \tag{88}$$

$$-\frac{1}{2}a_0b_1v-\frac{1}{8}a_1^3b_1+\frac{1}{8}a_1b_1^3-\frac{11}{8}a_0^2a_1b_1=0, \tag{89}$$

$$-\frac{9}{16}a_0a_1^3-\frac{27}{16}a_0a_1b_1^2=0, \tag{90}$$

$$-\frac{9}{16}a_0b_1^3-\frac{27}{16}a_0a_1^2b_1=0, \tag{91}$$

$$-\frac{3}{16}a_0a_1^3-\frac{9}{16}a_0a_1b_1^2=0, \tag{92}$$

$$-\frac{3}{16}a_0b_1^3-\frac{9}{16}a_0a_1^2b_1=0. \tag{93}$$

Solving the above set of algebraic equations with the aid of Mathematica yields the following results:

$$a_0=0,\ b_1=0,\quad a_1\neq 0,\text{ and }\ v=-\frac{a_1^4+16}{4a_1^2}, \tag{94}$$

$$a_0=0,\ a_1=0,\quad b_1\neq 0,\ \text{ and }\ v=\frac{b_1^4-16}{4b_1^2}. \tag{95}$$

Using Eqs.(94) and (95) and considering case 2 and case 3, we have the following solutions to the mKdV–sinh-Gordon equation (1)

$$u(x,t)=2\ln\left[\tan\left(\frac{1}{4}a_1(x+\frac{a_1^4+16}{4a_1^2}t)+\xi_0\right)\right], \tag{96}$$

$$u(x,t)=2\ln\left[-\cot\left(\frac{1}{4}a_1(x+\frac{a_1^4+16}{4a_1^2}t)+\xi_0\right)\right], \tag{97}$$

$$u(x,t)=2\ln\left[\tan\left(\frac{1}{2}a_1(x+\frac{a_1^4+16}{4a_1^2}t)+\xi_0\right)\pm\sec\left(\frac{1}{2}a_1(x+\frac{a_1^4+16}{4a_1^2}t)+\xi_0\right)\right], \tag{98}$$

$$u(x,t)=2\ln\left[-\cot\left(\frac{1}{2}a_1(x+\frac{a_1^4+16}{4a_1^2}t)+\xi_0\right)\mp\csc\left(\frac{1}{2}a_1(x+\frac{a_1^4+16}{4a_1^2}t)+\xi_0\right)\right], \tag{99}$$

$$u(x,t)=2\ln\left[\frac{\tan\left(\frac{1}{2}a_1(x+\frac{a_1^4+16}{4a_1^2}t)+\xi_0\right)}{1\pm\sec\left(\frac{1}{2}a_1(x+\frac{a_1^4+16}{4a_1^2}t)+\xi_0\right)}\right], \tag{100}$$

$$u(x,t)=2\ln\left[\frac{-\cot\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}{1\pm\csc\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}\right], \tag{101}$$

$$u(x,t)=2\ln\left[\frac{-A\cos\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\sqrt{A^2-B^2}}{A\sin\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{102}$$

$$u(x,t)=2\ln\left[\frac{A\sin\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\sqrt{A^2-B^2}}{A\cos\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{103}$$

$$u(x,t)=2\,\mathrm{arcsinh}\left[\tan\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{104}$$

$$u(x,t)=2\,\mathrm{arcsinh}\left[-\cot\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{105}$$

$$u(x,t)=2\,\mathrm{arcsinh}\left[\tan\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm\sec\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{106}$$

$$u(x,t)=2\,\mathrm{arcsinh}\left[-\cot\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\csc\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{107}$$

$$u(x,t)=2\,\mathrm{arcsinh}\left[\frac{\tan\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}{1\pm\sec\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}\right], \tag{108}$$

$$u(x,t)=2\,\mathrm{arcsinh}\left[\frac{-\cot\left(a_1(x+\frac{a_1^4+16}{4a_1^2}t)+\xi_0\right)}{1\pm\csc\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}\right], \tag{109}$$

$$u(x,t)=2\,\mathrm{arcsinh}\left[\frac{-A\cos\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\sqrt{A^2-B^2}}{A\sin\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{110}$$

$$u(x,t)=2\,\mathrm{arcsinh}\left[\frac{A\sin\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\sqrt{A^2-B^2}}{A\cos\left(a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{111}$$

$$u(x,t)=2\,\mathrm{arccosh}\left[\pm\sec\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{112}$$

$$u(x,t)=2\,\mathrm{arccosh}\left[\pm\csc\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{113}$$

$$u(x,t)=4\,\mathrm{arctanh}\left[\tan\left(\frac{1}{4}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{114}$$

$$u(x,t)=4\,\mathrm{arctanh}\left[-\cot\left(\frac{1}{4}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{115}$$

$$u(x,t)=4\,\mathrm{arctanh}\left[\tan\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm\sec\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{116}$$

$$u(x,t)=4\,\mathrm{arctanh}\left[-\cot\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\csc\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{117}$$

$$u(x,t)=4\,\mathrm{arctanh}\left[\frac{\tan\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}{1\pm\sec\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}\right], \tag{118}$$

$$u(x,t)=4\,\mathrm{arctanh}\left[\frac{-\cot\left(\frac{1}{2}a_1(x+\frac{a_1^4+16}{4a_1^2}t)+\xi_0\right)}{1\pm\csc\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}\right], \tag{119}$$

$$u(x,t)=4\,\mathrm{arctanh}\left[\frac{-A\cos\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\sqrt{A^2-B^2}}{A\sin\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{120}$$

$$u(x,t)=4\,\mathrm{arctanh}\left[\frac{A\sin\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\sqrt{A^2-B^2}}{A\cos\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{121}$$

$$u(x,t)=4\,\mathrm{arccoth}\left[-\tan\left(\frac{1}{4}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{122}$$

$$u(x,t)=4\,\mathrm{arccoth}\left[\cot\left(\frac{1}{4}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{123}$$

$$u(x,t)=4\,\mathrm{arccoth}\left[-\tan\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\sec\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{124}$$

$$u(x,t)=4\,\mathrm{arccoth}\left[\cot\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm\csc\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\right], \tag{125}$$

$$u(x,t)=4\,\mathrm{arccoth}\left[\frac{-\tan\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}{1\pm\sec\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}\right], \tag{126}$$

$$u(x,t)=4\,\mathrm{arccoth}\left[\frac{\cot\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}{1\pm\csc\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)}\right], \tag{127}$$

$$u(x,t)=4\,\mathrm{arccoth}\left[\frac{A\cos\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\sqrt{A^2-B^2}}{A\sin\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{128}$$

$$u(x,t)=4\,\mathrm{arccoth}\left[\frac{-A\sin\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\mp\sqrt{A^2-B^2}}{A\cos\left(\frac{1}{2}a_1\left(x+\frac{a_1^4+16}{4a_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{129}$$

$$u(x,t)=4\,\mathrm{arctanh}\left[\pm\exp\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{130}$$

$$u(x,t)=4\,\mathrm{arccoth}\left[-\exp\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{131}$$

$$u(x,t)=2\ln\left[-\tanh\left(\frac{1}{4}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{132}$$

$$u(x,t)=2\ln\left[-\coth\left(\frac{1}{4}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{133}$$

$$u(x,t)=2\ln\left[-\tanh\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\mp\mathrm{i}\,\mathrm{sech}\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{134}$$

$$u(x,t)=2\ln\left[-\coth\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\mp\mathrm{csch}\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{135}$$

$$u(x,t)=2\ln\left[\frac{-\tanh\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)}{1\pm\mathrm{sech}\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)}\right], \tag{136}$$

$$u(x,t)=2\ln\left[\frac{-\coth\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)}{1\pm\mathrm{i}\,\mathrm{csch}\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)}\right], \tag{137}$$

$$u(x,t)=2\ln\left[\frac{-A\cosh\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\mp\sqrt{A^2+B^2}}{A\sinh\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{138}$$

$$u(x,t)=2\ln\left[\frac{-A\sinh\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\mp\sqrt{B^2-A^2}}{A\cosh\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{139}$$

$$u(x,t)=2\,\mathrm{arcsinh}\left[\pm\,\mathrm{csch}\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{140}$$

$$u(x,t)=2\,\mathrm{arcsinh}\left[\pm\,\mathrm{i}\,\mathrm{sech}\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{141}$$

$$u(x,t)=2\,\mathrm{arccosh}\left[\pm\tanh\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{142}$$

$$u(x,t)=2\,\mathrm{arccosh}\left[\pm\coth\left(\frac{1}{2}b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{143}$$

$$u(x,t)=2\,\mathrm{arccosh}\left[\pm\tanh\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\mp\,\mathrm{i}\,\mathrm{sech}\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{144}$$

$$u(x,t)=2\,\mathrm{arccosh}\left[\pm\coth\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\mp\,\mathrm{csch}\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\right], \tag{145}$$

$$u(x,t)=2\,\mathrm{arccosh}\left[\frac{\pm\tanh\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)}{1\pm\,\mathrm{sech}\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)}\right], \tag{146}$$

$$u(x,t)=2\,\mathrm{arccosh}\left[\frac{\pm\coth\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)}{1\pm\,\mathrm{i}\,\mathrm{csch}\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)}\right], \tag{147}$$

$$u(x,t)=2\,\mathrm{arccosh}\left[\frac{\pm A\cosh\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\mp\sqrt{A^2+B^2}}{A\sinh\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\pm B}\right], \tag{148}$$

$$u(x,t)=2\,\mathrm{arccosh}\left[\frac{\pm A\sinh\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\mp\sqrt{B^2-A^2}}{A\cosh\left(b_1\left(x-\frac{b_1^4-16}{4b_1^2}t\right)+\xi_0\right)\pm B}\right]. \tag{149}$$

In the above equations, $a_1$, $b_1$ and $\xi_0$ are arbitrary real constants with $a_1\neq 0$ and $b_1\neq 0$.

Evidently, the solutions (96), (97), (114), (115), (122), (123), (130) and (131) are equivalent to those given in Ref.[32]. The rest of the solutions, however, are the new solutions of the mKdV–sinh-Gordon equation which cannot be seen in the existing references to the best of our knowledge. The structures of some solutions are shown in Figs.1–6.

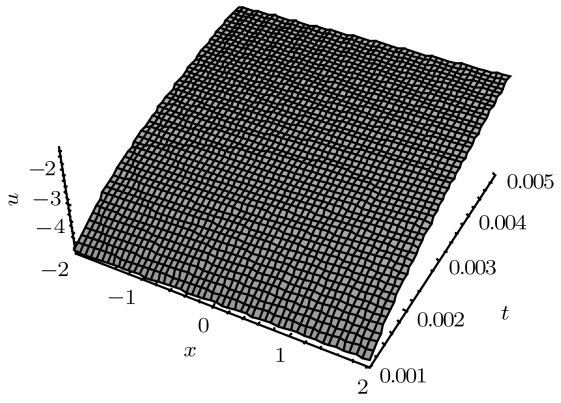



**Fig.1.** Exact solution (96) with $a_1=0.01$ and $\xi_0=0$.

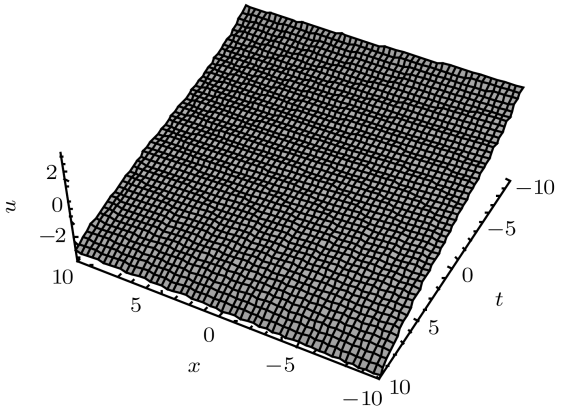



**Fig.2.** Exact solution (104) with $a_1=0.01$ and $\xi_0=0$.

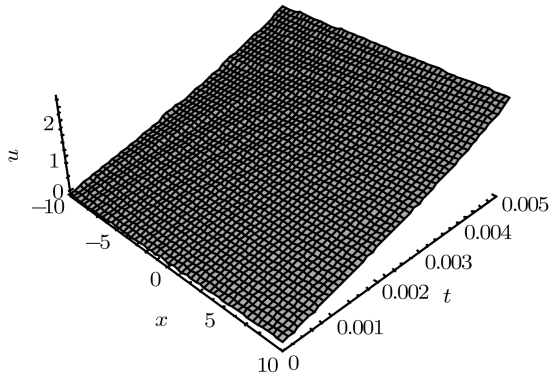



**Fig.3.** Exact solution (114) with $a_1 = 0.01$ and $\xi_0 = 0$.

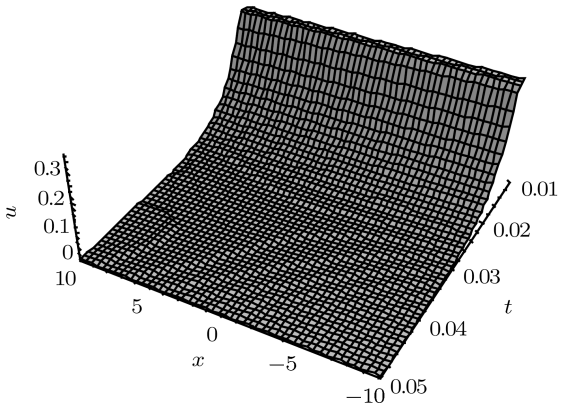



**Fig.4.** Exact solution (130) with the sign positive, $b_1 = -0.01$ and $\xi_0 = 0$.

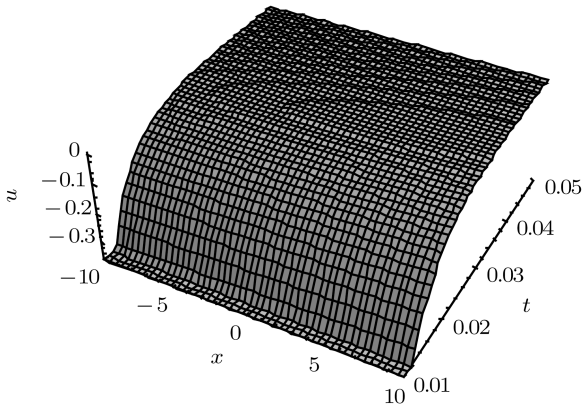



**Fig.5.** Exact solution (132) with $b_1 = -0.01$ and $\xi_0 = 0$.

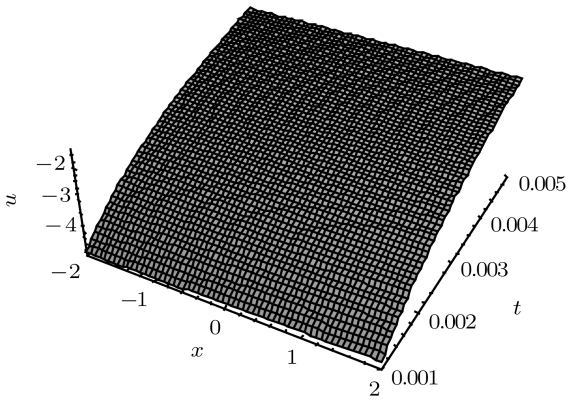



**Fig.6.** Exact solution (140) with the sign positive, $b_1 = -0.01$ and $\xi_0 = 0$.

Finally, it is worth while to mention that we have verified all solutions (especially new solutions) obtained by putting them back into the original equation with the aid of Mathematica.

## 3. Conclusions

In conclusion, we have modified the work done by Wazwaz by introducing a more general ODE, and with the aid of its rich types of solutions, we successfully recovered all solutions obtained by Wazwaz for the mKdV–sinh-Gordon equation. More importantly, we also picked up some new formal solutions of the mKdV–sinh-Gordon equation. It is readily seen that our approach is more convenient and powerful than that of Wazwaz for solving the mKdV–sinh-Gordon equation. It deserves investigating further how to extend this approach to solve other NEEs in mathematics and physics.

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