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Influence of velocity spatiotemporal correlations on the anomalous scaling exponents of passive scalars*

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In this paper, we consider spatial-temporal correlation functions of the turbulent velocities. With numerical simulations on the Gledzer–Ohkitani–Yamada (GOY) shell model, we show that the correlation function decays exponentially. The advecting velocity field is regarded as a colored noise field, which is spatially and temporally correlative. For comparison, we are also given the scaling exponents of passive scalars obtained by the Gaussian random velocity field, the multi-dimensional normal velocity field and the She–Leveque velocity field, introduced by She, *et al.* We observe that extended self-similarity scaling exponents $H(p)/H(2)$ of passive scalar obtained by the colored noise field are more anomalous than those obtained by the other three velocity fields.

Keywords: shell models, passive scalar, anomalous scaling exponents

PACC: 0340G, 4725C

1. Introduction

A most striking property of a passive scalar quantity advected by a fully developed turbulent flow is its spatiotemporal intermittent behaviour. Intermittency in passive scalars advected by turbulent flows is thought to be connected to the existence of self-similar processes transferring fluctuations from large to small scales. There has been much recent interest in the problem of passive scalar advection in turbulence fluid. In particular, much interest has been focused on the anomalous scaling exponents of the passive scalar.[1−5] Statistics of the passive scalar must be strongly related to the properties of the advecting velocity fields. Unfortunately, due to the lack of knowledge about the statistical properties of a fully developed turbulent flow the problem of intermittency in the passive scalar has not been solved. Recently, much attention has been paid in this direction by studying the intermittency properties of a passive scalar advected by a velocity field which is Gaussian, δ-correlated in time, and has self-similar spatial correlations.[6−9] Such a choice is far from being realistic because of the assumed fast-time decorrelation and Gaussianity. Whereas, the big advantage of this model is that, due to the δ correlation in time, all equations for equal-time n-point correlations are formally closed, that is, the n-th order correlation depends only on lower-order correlations. Nevertheless, many interesting analytical and phenomenological results for this Gaussian model have been obtained, giving the impression that some additional new insight into the intermittency problem of passive scalars has been gained.[10,11]

The problem of defining a shell model for the advection of a passive scalar by a deterministic and chaotic velocity field has already been investigated in Ref.[12]. Here we are going to present our numerical and analytical results obtained by a weakly modified version of the shell model defined in Ref.[12], but using a coloured noise surrogate for the advecting velocity field. The major advantage of using a coloured noise velocity field is that, now, we can solely study the influence of velocity-field correlations in Fourier space. For qualitatively analysing the statistical phasic-spatial characteristic of velocity field, the GOY shell model is chosen.

The paper is organized as follows: in Section 2 we introduce and discuss the spatiotemporal correlation of velocity field in the GOY shell model. In Section 3 we introduce the coloured noise field and calculate its scaling exponents. The evolution equation of passive scalars is numerically solved in Section 4 to obtain the scaling exponents of passive scalar. Concluding remarks and suggestions for further work are given

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in Section 5. In Appendix A we explicitly derive the correlation function of coloured noise field, which is numerically solved by the Newtonian method.

2. The spatiotemporal correlation of velocity field

The evolution equation of the GOY shell model[13,14] with N shell can be written as[15]

$$\left(\frac{\mathrm{d}}{\mathrm{d}t}+\nu k_n^2\right)u_n = \mathrm{i}k_n\left(u_{n+1}^* u_{n+2}^* - \frac{\delta}{r}u_{n-1}^* u_{n+1}^* - \frac{1-\delta}{r^2}u_{n-1}^* u_{n-2}^*\right) + \bar{f}\delta_{n,4}, \quad (1)$$

where $*$ denotes the complex conjugate, and $n = 1, ..., N$, $r = k_{n+1}/k_n = 2$. The parameter δ (here $\delta = 0.5$) plays an important role in both the static and the dynamical properties of the model since it is related to a second quadratic invariant of the equations. The forcing term $\bar{f}$ acts only on the fourth shell. In the following, we are interested in the statistical behaviour of the spatiotemporal correlation of velocity:

$$F_{nm}(\tau) = \mathrm{Re}\left(f_{nm}^u(\tau)\right), \quad (2)$$

where

$$f_{nm}^u(\tau) \equiv \left[1/\sqrt{S_2^u(k_n)\, S_2^u(k_m)}\right]\langle(u_n(t)\cdot u_m^*(t+\tau))\rangle,$$

with $S_2^u(k_n) = \langle u_n(t)\cdot u_n^*(t)\rangle \cdot \tau$ being a time interval. We use the four-order Runge–Kutta method to solve the stochastic functions (1), and then associate with formula (2) to obtain the curve of spatiotemporal correlation, as shown in Fig.1. In fact, $F_{nm}^{(\tau)}$ is an exponential decay function. Via numerical compution, the fit function of spatiotemporal correlation reads

$$F_{nm}(\tau) = \frac{\bar{D}_{nm}}{\alpha_{nm}}\exp\left(-\alpha_{nm}\tau\right), \quad (3)$$

where $\bar{D}_{nm}$ and α_{nm} are shell coefficient and decay coefficient respectively. Figure 1 shows the spatiotemporal correlation of velocity $F_{nm}(\tau)$ versus τ at $m = 4$, 7, 10, 13. From Fig.1, we can see that the curve of $F_{47}(\tau)$ shows obvious fluctuations. It indicates that there is 3-periodic relevancy in the GOY shell model. The correlation function $F_{n,n+3}(\tau)$ between n-shell and $n+3$ shell has a bigger relevancy than others. It just needs some time to transmit energy from the n-th shell to the $(n+3)$-th shell, so the fitting curve behaves with a temporal delay.

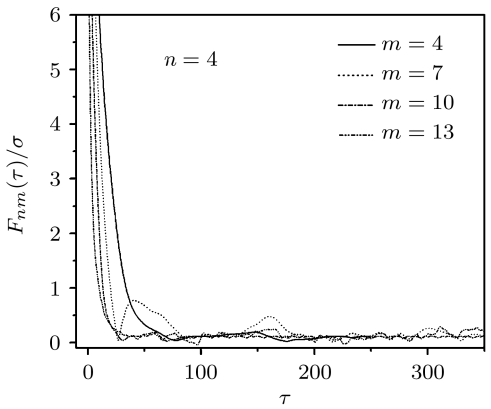


Fig.1. The evolution of spatiotemporal correlation of velocity with τ for $m = 4$, 7, 10 and 13.

In Fig.1, $\sigma = [1/S_2^u(k_n)\, S_2^u(k_m)]\,\langle(u_n(t)\cdot u_m^*(t))\rangle$. The curves of $F_{nm}(\tau)$ show obvious exponential decay.

3. Anomalous scaling exponents of colour noise velocity field

According to the character of spatiotemporal correlation function of velocity field, one can find that the correlation function $F_{nm}(\tau)$ exponentially decays along with the temporal interval τ. For the sake of easily researching the influence of $F_{nm}(\tau)$ on anomalous scaling exponents of passive scalars, the exponential Gaussian coloured noise field is chosen as the advecting velocity field. Such a velocity field satisfies the Langevin function:

$$\frac{\mathrm{d}u_n(t)}{\mathrm{d}t} = -\sum_m a_{nm}u_m(t) + \sum_k b_{nk}\Gamma_k(t) \quad (4)$$

with $\langle\Gamma_n(t)\rangle = 0$, $\langle\Gamma_n(t)\Gamma_m(t')\rangle = 2C_n\delta_{nm}\delta(t'-t)$.

Using equation (4) to compute spatiotemporal correlation function of $u_n(t)$, one can obtain

$$\sum_{n,m}^{19} t_{in}t_{jm}\mathrm{Re}\,\langle u_n(t_1)u_m^*(t_2)\rangle = \frac{g_{ij}}{\bar{a}_{ii}+\bar{a}_{jj}}\exp\left(-\bar{a}_{ii}\,(t_1-t_2)\right) \quad (5)$$

with $t_1 - t_2 = \tau(t_1 > t_2)$. The details of which may be found in Appendix. The coefficients t_{in}, t_{jm}, $\bar{a}_{ij}$, g_{ij} are the member of matrix $T = \{t_{ij}\}$, $A = \{a_{ij}\}$, $G = \{g_{ij}\}$ and $TAT^{-1} = \bar{A} = \{\bar{a}_{ii}\}$, respectively.

In the following, we firstly study the scaling exponents of coloured noise velocity field, shown in Eq.(4). The scaling laws of $u_n(t)$ are given by

$$\langle|u_n|^p\rangle \propto k_n^{\zeta(p)}, \quad (6)$$

where $\zeta(p)$ is the scaling exponents of $u_n(t)$. Let us now discuss the results obtained from Eqs.(4) and (6) with numerical simulation. The scaling $\zeta_{\rm TS}(p)$, the scaling exponent of coloured noise velocity field, does not follow the Kolmogorov (1941) prediction $\zeta(p) = p/3$,[16] even for small values of p, as shown in Fig.2. In fact, $\zeta_{\rm TS}(p)$ is a strongly nonlinear function of p. Figure 2 also shows the scaling exponent $\zeta_{\rm SL}(p)$ of the hierarchical structure model, which is inspired by She and Leveque (SL model).[17−19] Both $\zeta_{\rm TS}(p)$ and $\zeta_{\rm SL}(p)$ are all anomalous scaling exponents, showing the fundamentality of the spatiotemporal correlation of the velocity field. We think that the difference between $\zeta_{\rm TS}(p)$ and $\zeta_{\rm SL}(p)$ may statistically come from the Gaussian character of the coloured noise field. Whereas, the SL model introduces non-Gassian log-Possion distribution function, which fits the experimental data very well. Obviously, the scaling exponent of SL model is better than that of the coloured noise velocity field.

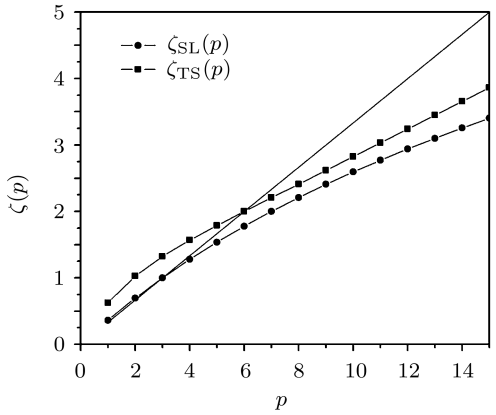


Fig.2. $\zeta(p)$ versus p. The full line indicates the Kolmogorov (1941) prediction $\zeta(p) = p/3$. The squares are the scaling exponents of coloured noise velocity field, $\zeta_{\rm TS}(p)$. The circles are the scaling exponents of the hierarchical structure model, $\zeta_{\rm SL}(p)$.

4. Anomalous scaling exponents of passive scalars in coloured noise velocity field

Our passive scalar shell model, introduced by Jensen, *et al*,[12] is defined by the following equations

$$\begin{aligned}\frac{{\rm d}\theta_n}{{\rm d}t} &= -Dk_n^2\theta_n + {\rm i}h_n\theta^*_{n+2}u^*_{n+1} + {\rm i}h_n\theta^*_{n+1}u^*_{n+2} \\ &+ {\rm i}e_n\theta^*_{n+1}u^*_{n-1} - {\rm i}e_n\theta^*_{n-1}u^*_{n+1} \\ &+ {\rm i}g_n\theta^*_{n-1}u^*_{n-2} + {\rm i}g_n\theta^*_{n-2}u^*_{n-1} + \bar{f}\delta_{n,4}, \quad (7)\end{aligned}$$

where $*$ denotes the complex conjugate, and $e_n = k_n/2$, $g_n = -k_{n-1}/2$, $h_n = k_{n+1}/2$, with $u_{-1} = u_0 = \theta_{-1} = \theta_0 = 0$ as boundary conditions. The forcing term $\bar{f}\delta_{n,4}$ acts only on the fourth shell. We assume that $f(t)$ is an independent complex of the Gaussian variable and δ-correlated in time, with $\langle f(t)f^*(t')\rangle = C_0\delta(t-t')$. In numerical simulation, the model is truncated to a finite number of shells N (here $N = 19$) with the additional boundary conditions $u_{N+1} = u_{N+2} = \theta_{N+1} = \theta_{N+2} = 0$. A shell of radius $k_n = r^n k_0$, with $r > 1$ and the standard choice for the radius between shells is $r = 2$.

In the following, we prescribe several kinds of random velocity field $u_n(t)$ to advect the passive scalar. The first one is Gaussian random velocity,[11] in which velocity $u_i(t)$ of different shells is independent; the second one is log-Poisson random velocity, inspired by She and Leveque (SL velocity);[18] the third one is multi-dimensional normal random velocity introduced in Ref.[20]; the fourth random velocity is coloured noise field introduced in the above section. We use $H_{\rm G}(p)$ to denote the scaling exponents advected by independent Gaussian random velocity field. $H_{\rm MN}(p)$, $H_{\rm SL}(p)$ and $H_{\rm TS}(p)$ denote the scaling exponents of passive scalar advected by multi-dimensional normal random velocity field, SL velocity, and coloured noise field introduced in the above section, respectively. $H_{\rm GOY}(p)$ represents the scaling exponents of passive scalars advected by GOY shell model. $H_{\rm X}(p)$ represents the scaling exponents of passive scalars modified by the difference between $H_{\rm G}(p)$ and $H_{\rm SL}(p)$. We are interested in the scaling behaviour of the p-th order structure functions, and the scaling laws of the p-th order structure functions are given by

$$\langle|\theta_n|^p\rangle \propto k_n^{H(p)}, \quad (8)$$

where $H(p)$ is the scaling exponents of $\theta_n(t)$. To investigate the nature properties, the p-order scaling exponents $H(p)$ are normalized by the second-order scaling exponents $H(2)$, using the standard, extended self-similarity (ESS) procedure.

We have performed the calculation of the scaling exponent of passive scalars for six cases. Figure 3 shows the scaling exponents $H(p)$ using ESS up to order $p = 15$ for different advecting velocity fields. Apparently, the scaling exponent of passive scalars is anomalous in all the cases. We find that the curve of $H_{\rm TS}(p)$ is closer to that of $H_{\rm GOY}(p)$ than others, which means $H_{\rm TS}(p)$ is more anomalous than others. It indicates that the coloured noise field can be relatively easy to drive an intermittent passive scalar field,

and the spatiotemporal correlations of velocity seriously affect the intermittent of passive scalar. However, they can never reach the curves of $H_{\rm GOY}(p)$. The difference may come from the dynamics relationship. The velocity functions of the GOY model have a finite correlation time, the artificially prescribed model velocity field, like our coloured noise field, might have a white-noise correlation structure in time, which is one of the reasons for the observed difference in the scaling exponents. On the other hand, the high order moment may statistically have some impacts on the scaling exponents of passive scalars, which needs further research.

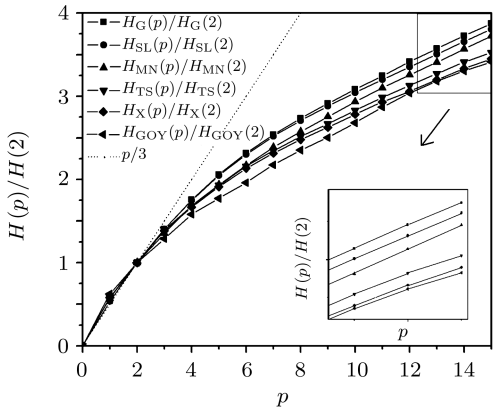


Fig.3. Scaling exponents $H(p)$ normalized by $H(2)$ using ESS up to order $p = 15$. The dash line represents normal scaling $p/3$ predicted by Obukhov.[21] Different subscripts represent different advecting velocity fields.

5. Conclusion and discussion

We have proposed and studied a shell model for a randomly passive scalar advected by a coloured noise field. We also introduced a multi-dimensional normal velocity field and a SL non-Gaussian velocity field, which all have typical anomalous scaling exponents. At the same time independent Gaussian random velocity was proposed to advect the passive scalar. We found several properties of passive scalar. (i) Intermittence is an intrinsic property of the passive scalar function. So intermittent passive scalar can be obtained with independent Gaussian random velocity; (ii) The spatiotemporal correlation of the advecting velocity can excite additional intermittence of the passive scalar. So $H_{\rm TS}(p)$ is more anomalous than others; (iii) The message of the dynamics relationship of the real velocity is also important. It determines the energy cascade and synergism between different scales. The spatiotemporal correlation we considered is just 2-order correlations, where the high-order correlations are neglected, so $H_{\rm MN}(p)$ can never reach the experimental data.

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Appendix: Calculation of equation (5)

The equation (4) reads

$$\frac{{\rm d}u_n(t)}{{\rm d}t} = -\sum_m a_{nm}u_m(t) + \sum_k b_{nk}\Gamma_k(t), \qquad \text{(A1)}$$

namely,

$$\frac{{\rm d}U}{{\rm d}t} = -AU + B\Gamma, \qquad \text{(A2)}$$

where $U = (u_1, \ldots, u_n)^{\rm T}$, $A = \{a_{ij}\}$, $B = \{b_{ij}\}$, $\Gamma = (\Gamma_1, \ldots, \Gamma_n)^{\rm T}$.

The equation (A2) has a formal solution

$$U(t) = {\rm e}^{-At}U(0) + \int_0^t {\rm e}^{-A(t-t')}B\Gamma(t'){\rm d}t'. \qquad \text{(A3)}$$

Then the temporal correlation function reads

$$\begin{aligned}
&\langle U(t_1)U^+(t_2)\rangle \\
&= \left\langle \left[{\rm e}^{-At_1}U(0) + \int_0^{t_1} {\rm e}^{-A(t_1-t')}B\Gamma(t'){\rm d}t'\right]\left[U^+(0){\rm e}^{-A^+t_2} + \int_0^{t_2}\Gamma^+(t'')B^+{\rm e}^{-A^+(t_2-t'')}{\rm d}t''\right]\right\rangle \\
&= \left\langle {\rm e}^{-At_1}U(0)U^+(0){\rm e}^{-A^+t_2}\right\rangle + \left\langle \int_0^{t_1} {\rm e}^{-A(t_1-t')}B\Gamma(t'){\rm d}t' \int_0^{t_2}\Gamma^+(t'')B^+{\rm e}^{-A^+(t_2-t'')}{\rm d}t''\right\rangle \\
&\quad + \left\langle {\rm e}^{-At_1}U(0)\int_0^{t_2}\Gamma^+(t'')B^+{\rm e}^{-A^+(t_2-t'')}{\rm d}t''\right\rangle + \left\langle \int_0^{t_1} {\rm e}^{-A(t_1-t')}B\Gamma(t'){\rm d}t'U^+(0){\rm e}^{-A^+t_2}\right\rangle. \qquad \text{(A4)}
\end{aligned}$$

The matrix $A = \{a_{ij}\}$ can be diagonalized by the matrix $T = \{t_{ij}\}$, then,

$$TAT^{-1} = \bar{A} = \{\bar{a}_{ii}\} = \left\{\sum_{n,m}^{19} t_{in}a_{nm}c_{mi}\right\}, \tag{A5}$$

Then we obtain some equations as follows:

$$TU = \bar{U} = \{\bar{u}_i\} = \left\{\sum_{n}^{19} t_{in}u_n(t)\right\}, \tag{A6a}$$

$$TBT^{-1} = \bar{B} = \left\{\bar{b}_{ij}\right\} = \left\{\sum_{n,m}^{19} t_{in}b_{nm}c_{mj}\right\}, \tag{A6b}$$

$$T\Gamma = \bar{\Gamma} = \left\{\bar{\Gamma}_i\right\} = \left\{\sum_{n}^{19} t_{in}\Gamma_n(t)\right\}. \tag{A6c}$$

Left-half multiplying T to Eq.(A3),

$$TU(t) = T\mathrm{e}^{-At}U(0) + T\int_0^t \mathrm{e}^{-A(t-t')}B\Gamma(t')\,\mathrm{d}t' = T\mathrm{e}^{-At}T^{-1}TU(0) + \int_0^t T\mathrm{e}^{-A(t-t')}T^{-1}TBT^{-1}T\Gamma(t')\,\mathrm{d}t',$$

namely,

$$\bar{U}(t) = \mathrm{e}^{-\bar{A}t}\bar{U}(0) + \int_0^t \mathrm{e}^{-\bar{A}(t-t')}\bar{B}\bar{\Gamma}(t')\,\mathrm{d}t',$$

then,

$$\begin{aligned}
&\left\langle \bar{U}(t_1)\bar{U}^+(t_2)\right\rangle \\
=\ &\left\langle \mathrm{e}^{-\bar{A}t_1}\bar{U}(0)\bar{U}^+(0)\mathrm{e}^{-\bar{A}^+t_2}\right\rangle + \left\langle \int_0^{t_2}\int_0^{t_1} \mathrm{e}^{-\bar{A}(t_1-t')}\bar{B}\bar{\Gamma}(t')\bar{\Gamma}^+(t'')\bar{B}^+\mathrm{e}^{-\bar{A}^+(t_2-t'')}\,\mathrm{d}t''\mathrm{d}t'\right\rangle \\
=\ &\left\langle \mathrm{e}^{-\bar{A}t_1}\bar{U}(0)\bar{U}^+(0)\mathrm{e}^{-\bar{A}^+t_2}\right\rangle + \left\langle \int_0^{t_2}\int_0^{t_1} \mathrm{e}^{-\bar{A}(t_1-t')}G\mathrm{e}^{-\bar{A}^+(t_2-t'')}\,\mathrm{d}t''\mathrm{d}t'\right\rangle,
\end{aligned} \tag{A7}$$

where

$$\begin{aligned}
\langle G\rangle =\ &\left\langle \bar{B}\bar{\Gamma}(t')\bar{\Gamma}^+(t'')\bar{B}^+\right\rangle = \{g_{ij}\}\,\delta(t'-t'') = \left\{\sum_{k,q}^{19}\bar{b}_{ik}m_{kq}\bar{b}^+_{qj}\delta(t'-t'')\right\} \\
=\ &\left\{\sum_{n_1m_1knqn_3m_3}^{19} t_{in_1}b_{n_1m_1}c_{m_1k}t_{kn}t_{qn}C_nt_{jn_3}b_{n_3m_3}c_{m_3q}\delta(t'-t'')\right\},
\end{aligned} \tag{A8}$$

and

$$\{m_{ij}\} = \langle M\rangle = \left\langle \bar{\Gamma}(t')\bar{\Gamma}^+(t'')\right\rangle = \left\{\sum_{n}^{19} t_{in}t_{jm}C_n\delta(t'-t'')\right\}. \tag{A9}$$

If $a_{im}t_1$, $a_{jn}t_2 \gg 1$ and $t_1 > t_2$, the system goes to equilibrium state, one can obtain,

$$\left\langle \bar{U}(t_1)\bar{U}^+(t_2)\right\rangle = \left\{\frac{g_{ij}}{\bar{a}_{ii}+\bar{a}_{jj}}\exp\left(-\bar{a}_{ii}\left|t_1-t_2\right|\right)\right\}. \tag{A10}$$

Based on Eq.(A6a), we obtain

$$\left\langle \bar{U}(t_1)\bar{U}^+(t_2)\right\rangle = \left\{\sum_{n,m}^{19} t_{in}t_{jm}\left\langle u_n(t_1)u_m^*(t_2)\right\rangle\right\}. \tag{A11}$$

Substituting Eq.(A10) into Eq.(A11) yields

$$\sum_{n,m}^{19} t_{in}t_{jm}\mathrm{Re}\left\langle u_n(t_1)u_m^*(t_2)\right\rangle = \frac{g_{ij}}{\bar{a}_{ii}+\bar{a}_{jj}}\exp\left(-\bar{a}_{ii}\left|t_1-t_2\right|\right). \tag{A12}$$

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