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Successive defects asymmetric simple exclusion processes with particles of arbitrary size*

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This paper uses various mean-field approaches and the Monte Carlo simulation to calculate asymmetric simple exclusion processes with particles of arbitrary size in the successive defects system. In this system, the hopping probability p ($p < 1$) and the size d of particles are not constant. Through theoretical calculation and computer simulation, it obtains the exact theoretical results and finds that the theoretical results are in agreement with the computer simulation. These results are helpful in analysing the effect of traffic with different hopping probabilities p and sizes d of particle.

Keywords: successive defects, asymmetric simple exclusion processes, arbitrary size, Monte Carlo simulation, mean-field theory

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1. Introduction

Traffic jams have become a focus recently. Previously, a cellular automaton model were used to describe and calculate traffic problems.[1−4] Now, people use asymmetric simple exclusion processes to study traffic problems whose conditions are periodic or have open boundaries.[5−8] On a one-dimensional lattice, a particle can move from one site to the right with rate p, or to the left with rate q. Under the condition of open boundaries, particles can enter the system with rate α only if the first (the left boundary) site is not occupied and can exit the system with rate β if the last site is not empty.

Using matrix product and recursion methods,[9] we can exactly calculate the stationary state properties (i.e., currents, densities and phase diagrams) of particles of the size $d = 1$ for the asymmetric simple exclusion process (ASEP). The matrix product method cannot be applied to calculate the $d > 1$ ASEP. However, two particles of arbitrary size with diffusion constants and particle velocities have been calculated.[10] Dynamical exponents have also been derived by using Bethe ansatz solutions.[11] In 2003, steady-state properties of a totally ASEP (TASEP) with particles of arbitrary size have been studied by Gregory W. Lakatos and Tom Chou.[12]

Recently, only one defective site in a periodic chain has been studied,[13,14] the current and density profile of the system can be affected remarkably. Subsequently, many defects on a ring of ASEP have been studied by Tripathy and Barma.[15,16] In 1998, ASEP in an open chain was studied by Kolomeisky,[17] followed by an extension[18] in which particles of arbitrary size (the number of sites occupied by particle is more than 1) were studied. Most recently, the disordered ASEP has become the focus.[19−29] In these models, the size of particle is equal to 1 (i.e., $d = 1$).

In this paper, various mean-field approaches are used to calculate the stationary state density profiles and currents of successive defects for TASEP ($q = 0$) with particles of arbitrary size (i.e., $d \geq 1$). In this model, the hopping probabilities on the whole bulk are p which is less than 1. In the following section, simple mean-field and refined mean-field approaches will be used to study the model, respectively. In Section 3, Monte Carlo computer simulations are discussed. Section 4 gives a summary and conclusions.

2. Mean-field theories

The identical particles of diameter $d \geq 1$ move along a one-dimensional lattice which has $N \gg d$ sites. The particles interact through hard core exclusion. The density profiles and currents of stationary

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state are determined by the entrance rate α, exit rate β and hopping probability p, respectively. With each motion, the particles can move only from one site to the right.

2.1. Simple mean field

By using the simplest mean-field approach to discuss the stationary state of the system, we assume that the particles which do not have correlations are on average uniformly distributed along the lattice. The quantity P_i is defined as the probability of a particle at site i. The location of a particle's left edge determines its position in the lattice. Only the d sites ahead of the particle are empty, the left edge of a particle of size d can move forward with the hopping probability of p ($p < 1$), the stationary state current is then

$$J = P_i \left(1 - \frac{P_{i+1}}{p}\right)\left(1 - \frac{P_{i+2}}{p}\right)\cdots\left(1 - \frac{P_{i+d}}{p}\right). \quad (1)$$

Far from boundaries, the locally uniform probabilities $P_i \approx \rho$, thus

$$J = \rho\left(1 - \frac{\rho}{p}\right)^d. \quad (2)$$

When $\mathrm{d}J/\mathrm{d}\rho = 0$, the density profile and current of maximal current phase can be obtained

$$\rho_{\max} = \frac{p}{1+d}, \quad J_{\max} = \frac{pd^d}{(1+d)^{(1+d)}}. \quad (3)$$

In the following, we discuss the conditions which are entry at the left end and exit from the right end, respectively. In the system, a particle may only enter the system if the d sites are empty (Fig.1(a)). Similarly, a particle may only exit the system if the last d sites are occupied (Fig.1(b)). Since $d/n \to 0$, the differences arising from the variations in entrance and exit are finite-sized effects which vanish in the macroscopic limit.[12]

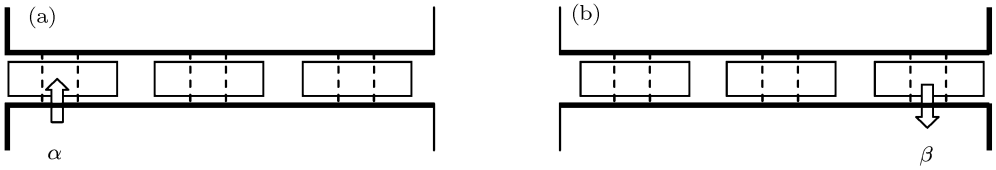


Fig.1. Entering and exiting the system. (a) Completely entering into the first d sites; (b) Completely, one-step exiting the last d sites.

For the entry-limited current, we set

$$\alpha \coprod_{j=1}^{d} (1 - P_j) = pP_1 \coprod_{j=1}^{d} (1 - P_{1+j})$$

and approximate $P_{1+j} \approx P_1$. The occupation probability near the left entry site is thus $P_1 \approx \alpha/p$ and we can obtain the stationary state current

$$J_L = \alpha\left(1 - \frac{\alpha}{p}\right)^d. \quad (4)$$

Similarly, when the current is limited by the exit rate of the system,

$$\beta P_{N-d+1} \coprod_{j=1}^{d} (1 - P_{N-d+1+j})$$
$$= pP_{N-d} \coprod_{j=1}^{d} (1 - P_{N-d+j}).$$

The occupation probability near the right exit site is thus $P_N \approx 1 - \beta/p$ and we can obtain the stationary state current

$$J_R = \frac{\beta^d (p - \beta)}{p^d}. \quad (5)$$

The additional exclusion between particles with sizes greater than a single lattice site is neglected by the simple mean-field theoretical calculation. Since, the results of the currents and density profiles can be obtained when the system is in the stationary state.

(i) When $\alpha < p/(1+d)$ and $\beta^d (p-\beta)/p^d > \alpha (1-\alpha/p)^d$ the system is in the low density (LD) phase, since $J_L = \alpha (1-\alpha/p)^d$ and $\rho_1 = \alpha/p$.

(ii) When $\beta < pd/(1+d)$ and $\alpha (1-\alpha/p)^d > \beta^d (p-\beta)/p^d$ the system is in the high density (HD) phase, since $J_R = \beta^d (p-\beta)/p^d$ and $\rho_N = 1 - \beta/p$.

(iii) When $\alpha > p/(1+d)$ and $\beta > pd/(1+d)$, the system is in the maximal current (MC) phase, since

$$J_{\max} = \frac{pd^d}{(1+d)^{(1+d)}}, \quad \rho_{\max} = \frac{p}{1+d}. \tag{6}$$

When $d = 1$ and $p = 1$, the system changes to a single lattice site TASEP and when $d > 1$ and $p = 1$, the system changes to TASEP with particles of arbitrary size.

2.2. Refined mean field

In the first place, we briefly review the conclusion of the Tonks equilibrium distribution.[12] The i-th particle entry of the system forms the left edge of the lattice and locates at site j ($X_i = j$). We can get the probability by the ratio $P(X_{i+1} > j + d|X_i = j) = Z(n, N-1)/Z(n, N)$ if the lattice site $(j + d)$ is not occupied. The steady-state current in a uniform phase is thus

$$\begin{aligned} J &= P(X_i = j)\, P(X_{i+1} > j + d/X_i = j) \\ &= \rho \frac{1 - \rho d}{1 - \rho(d-1)}. \end{aligned} \tag{7}$$

When the system is in the stationary state (i.e., $\mathrm{d}J/\mathrm{d}\rho = 0$), the maximal current (MC) and density can be obtained as follows:

$$\rho_{\max} = \frac{1}{\sqrt{d}\left(\sqrt{d}+1\right)}, \quad J_{\max} = \frac{1}{\left(\sqrt{d}+1\right)^2}. \tag{8}$$

In this paper, the hopping probability of particles is equal to p ($p < 1$), since equation (7) changes to

$$\begin{aligned} J &= pP(X_i = j)\, P(X_{i+1} > j + d/X_i = j) \\ &= \rho \frac{1 - (\rho/p)d}{1 - (\rho/p)(d-1)}, \end{aligned} \tag{9}$$

and equation (8) changes to

$$\rho_{\max} = \frac{p}{\sqrt{d}\left(\sqrt{d}+1\right)}, \quad J_{\max} = \frac{p}{\left(\sqrt{d}+1\right)^2}. \tag{10}$$

Here, a mean-field theory which accurately models the behaviour of the particles at the end of the lattice is developed. The results are comparable with the Monte Carlo computer simulation; we can find out that they are in good agreement. The entrance site at the left of the lattice is considered as shown in Fig.2(a).

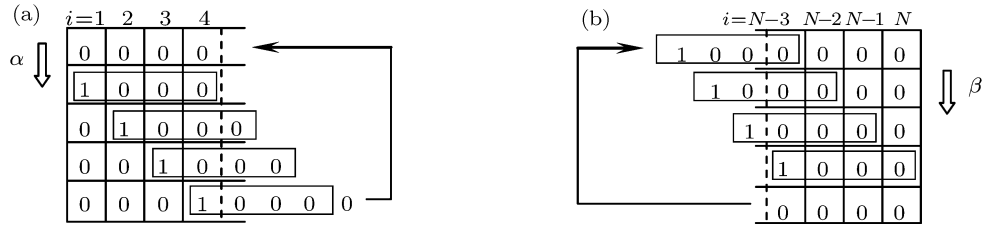


Fig.2. Mean-field scheme for particle of size $d = 4$ in the system: (a) particles enter the system from the left end; (b) particles exit the system from the right end.

$$J_L = \alpha P(X_i > d) = P(X_1 = d, X_2 > 2d). \tag{11}$$

By completely neglecting exclusion beyond one lattice site in the simple mean-field assumption, but in the refined mean-field theory, we can use the relative probabilities to limited currents. For example, if the first d sites are empty, the probability is

$$\begin{aligned} P(X_1 > d) &= \frac{(1-\theta/p)^d}{d(\theta/p)(1-\theta/p)^{d-1} + (1-\theta/p)^d} \\ &= \frac{p-\theta}{p+(d-1)\theta}, \end{aligned} \tag{12}$$

where θ/p is the relative likelihood that a site is occupied by assuming a nearly uniform particle density. The sum of the weights of all possible states of the first d lattice sites can be described by the factor $d(\theta/p)(1-\theta/p)^{d-1} + (1-\theta/p)^d$ (Fig.3(a)). When the first particle is at site d and is not blocked from moving one lattice site to the right, we can get the probability

$$\begin{aligned} &P(X_1 = d, X_2 > 2d) \\ &= \frac{(\theta/p)(1-\theta/p)^d}{d(\theta/p)(1-\theta/p)^{d-1} + (1-\theta/p)^d}. \end{aligned} \tag{13}$$

If the density varies slowly, θ/p represents a mean over the first d sites. The mean occupation of the first d sites is then

$$\rho_L \equiv \sum_{i=1}^{d} \rho_i = \frac{(\theta/p)\,(1-\theta/p)^{d-1}}{d(\theta/p)\,(1-\theta/p)^{d-1} + (1-\theta/p)^d} = \frac{\theta}{p+(d-1)\,\theta}. \quad (14)$$

Using Eq.(14), we find that $\theta = \alpha$, and

$$\rho_L = \frac{\alpha}{p+(d-1)\,\alpha}, \quad J_L = \frac{\alpha\,(p-\alpha)}{p+(d-1)\,\alpha}. \quad (15)$$

Similarly, in the high density phase, the current is limited by the exit rate β, all possible states between sites $(N-d)$ and N are considered. The exiting site at the right side of the lattice is considered as shown in Fig.2(b).

$$J_R = \beta P\,(X_n < N-d) = P\,(X_n = d, X_{n-1} < N-2d)\,. \quad (16)$$

Completely neglecting exclusion beyond one lattice site in the simple mean-field assumption, but in the refined mean-field theory, we can use the relative probabilities to limited currents. For example, if the last d sites are occupied, the probability is that

$$P\,(X_n < N-d) = \frac{(\theta/p)^d}{d\,(1-\theta/p)\,(\theta/p)^{d-1} + (\theta/p)^d}, \quad (17)$$

where θ/p is the relative likelihood that a site is occupied by assuming a nearly uniform particle density. The sum of the weights of all possible states of the last d lattice sites can be described by the factor $d\,(1-\theta/p)\,(\theta/p)^{d-1}+(\theta/p)^d$ (Fig.3(b)). When the last particle is before the site $(N-d)$ and is not blocked from moving one lattice site to the right, we can get the probability

$$P\,(X_n = d, X_{n-1} < N-2d) = \frac{(1-\theta/p)\,(\theta/p)^d}{d\,(1-\theta/p)\,(\theta/p)^{d-1} + (\theta/p)^d}. \quad (18)$$

If the density varies slowly, θ/p represents a mean over the last d sites. The mean occupation of the last d sites is then

$$P_R \equiv \sum_{i=1}^{d} \rho_i = \frac{(1-\theta/p)\,(\theta/p)^{d-1}}{d\,(1-\theta/p)\,(\theta/p)^{d-1} + (\theta/p)^d}. \quad (19)$$

Using Eq.(16), we find that $\theta = \beta$, and

$$P_R = \frac{p-\beta}{dp-(d-1)\,\beta}, \quad J_R = \frac{\beta\,(p-\beta)}{dp-(d-1)\,\beta}. \quad (20)$$

Since, the results of the currents and density profiles can be obtained when the system is in the stationary state.

i) When $\alpha < \beta$ and $\alpha < p^2/(\sqrt{d}+1)$, the system is in the low density (LD) phase, since

$$\rho_L = \frac{\alpha}{p+(d-1)\,\alpha}, \quad J_L = \frac{\alpha\,(p-\alpha)}{p+(d-1)\,\alpha}.$$

ii) When $\alpha > \beta$ and $\beta < p^2/(\sqrt{d}+1)$, the system is in the high density (HD) phase, since

$$P_R = \frac{p-\beta}{dp-(d-1)\,\beta}, \quad J_R = \frac{\beta\,(p-\beta)}{dp-(d-1)\,\beta}.$$

iii) When $\alpha > p^2/(\sqrt{d}+1)$ and $\beta > p^2/(\sqrt{d}+1)$, the system is in the MC phase, since

$$\rho_{\max} = \frac{p}{\sqrt{d}\left(\sqrt{d}+1\right)}, \quad J_{\max} = \frac{p}{\left(\sqrt{d}+1\right)^2}. \quad (21)$$

When $d = 1$ and $p = 1$, equations (15) and (20) reduce to the exact results of the standard TASEP and when $p = 1$ and $d > 1$, equations (15) and (20) change to TASEP with particles of arbitrary size. The results of the current and density profiles are still decided by entrance rate α, exit rate β and hopping probability p.

3. Monte Carlo simulation

In this section we will discuss the results of theory and simulation. The theoretical approaches which are simple mean-field and refined mean-field approaches are used to describe successive defects ASEPs with particles of arbitrary size. We use Monte Carlo computer simulations to test our theoretical results. By comparison, we can find out that the results of the refined mean-field approach are in agreement with the simulation.

Each lane in the system has been performed for different hopping probabilities with $p = 0.8$, 0.6 and 0.4. Most of our computer simulations are obtained for $L = 1000$ and we can neglect the finite-size effects (see Fig.3). The stationary current and density profiles can be obtained by the number of the running time steps 1.1×10^9 in each experiment. The first 10^8 time steps are discarded as transients.

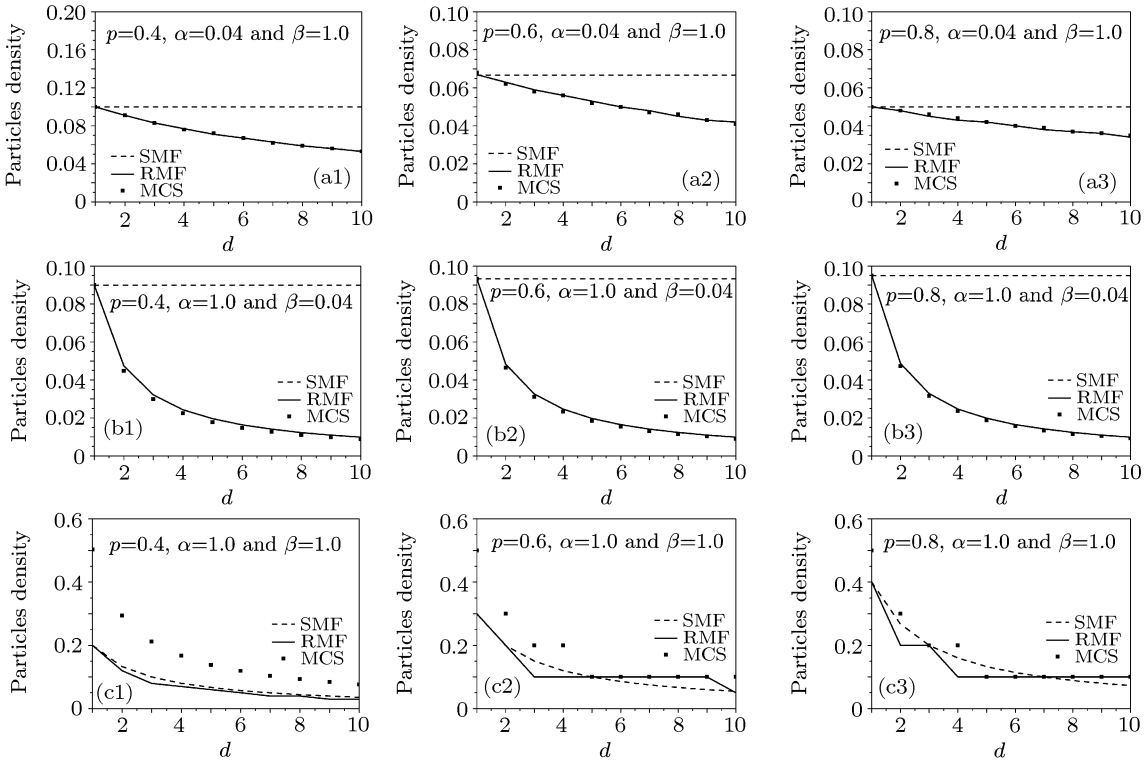


Fig.3. The particle density of the different phases (i.e., LD, HD and MC phases) as a function of particle size d. The phase diagrams show the results of the simple mean-field, refined mean-field and Monte Carlo simulations. (a1–a3) The particles density ρ_L as a function of size d and hopping probability p for $\alpha = 0.04$ and $\beta = 1.0$. (b1–b3) The particles density ρ_R as a function of size d and hopping probability p for $\alpha = 1.0$ and $\beta = 0.04$. (c1–c3) The particles density $\rho_{\max}$ as a function of size d and hopping probability p for $\alpha = 1.0$ and $\beta = 1.0$.

Theoretical density profiles and Monte Carlo computer simulations are shown in Fig.3. We can find that on the LD and HD phases, the results of the refined mean-field approach agree quite well with Monte Carlo simulations and on the MC phase, with the decreasing hopping probability p, the results of theoretical calculation are not in good agreement, in particular, when $p = 0.4$. Throughout the whole process, the results of the refined mean-field approach are given by Eq.(21) which match the Monte Carlo results within simulation error ($\approx$4%). When the hopping probability p is fixed, the MC phase will decrease with the increasing size d of particle (see Fig.4(a)) and when the size d of particle is fixed, the MC phase will increase with the decreasing hopping probability p of the particle (see Fig.4(b)). If particle density multiplies the size d of particle, we can get the average density.

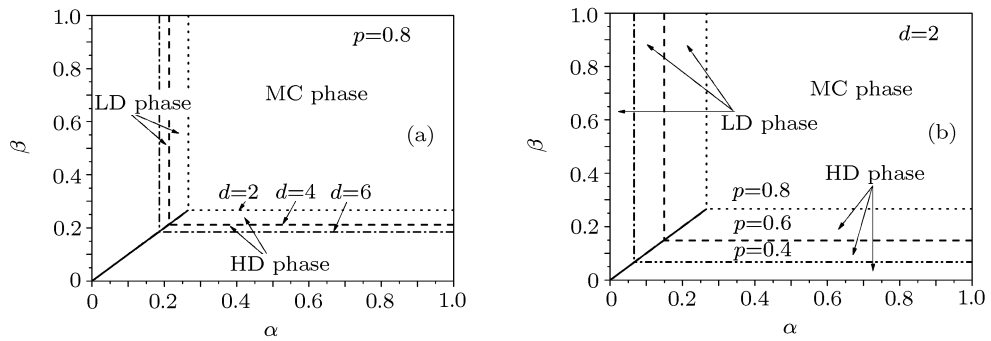


Fig.4. Phase diagram with the fixed hopping probability p or size d of particle is shown. (a) When hopping probability $p = 0.8$, $d = 2$, 4 and 6, respectively. (b) When size of particle $d = 2$, $p = 0.8$, 0.6 and 0.4, respectively.

The currents for various fixed d and $p = 0.8$ as functions of α and β are shown in Fig.5. The critical values α^* and β^* where the phase boundaries lie can be estimated from these current's curves. However, the currents

which are calculated by the simple mean-field approach are always underestimated and are shown by the dashed curves.

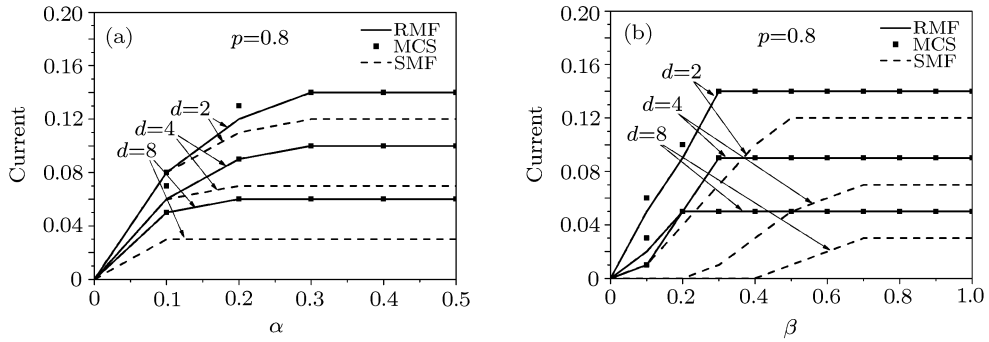


Fig.5. Currents as functions of α and β for fixed $d = 2$, 4 and 8 and $p = 0.8$. (a) Fixed $\beta = 1.0$; (b) Fixed $\alpha = 1.0$. The MC simulations are described by the squares, the dashed curves correspond to simple mean-field results and the solid curves correspond to refined mean-field results.

4. Summary and conclusions

The currents and densities of a TASEP with particles of arbitrary size in the successive defects system have been studied by mean-field theories. We use different mean-field approaches and Monte Carlo simulations to calculate the steady-state currents and densities. The results show that there are three phases (i.e., LD, HD and MC phases) like the normal TASEPs, too.

The results of the refined mean-field approach are given by Eq.(21) which matches the Monte Carlo results; we can find that the results of the refined mean-field approach agree quite well with the Monte Carlo simulations (within simulation error). When the hopping probability p is fixed, the MC phase will decrease with the size d of particle and when the size d of particle is fixed, the MC phase will increase with the decreasing hopping probability p of particle. These tell us that when hopping probability p decreases or the size d of particle increases, the system will be in the MC phase quickly.

The extension of exclusion processes to larger particles ($d > 1$) allows one to directly apply them to many traffic problems, they can help us to analyse and solve the traffic jams more successfully.

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