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Photon-added and photon-subtracted displaced Fock states and their non-classical properties*

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A new kind of quantum optical state, photon-added and -subtracted displaced Fock states, is introduced by applying the inverse of bosonic creation and annihilation operators to displaced Fock states. The quantum statistical properties of these states are investigated by numerical methods. Numerical results indicate that these states reveal some interesting non-classical properties, such as anti-bunching effects, sub-Poisson distributions and negativities of their Wigner functions.

Keywords: Boson inverse operators, photon-added and -subtracted states, non-classical property

PACC: 4250, 0365

1. Introduction

The non-classical properties of optical fields have been extensively studied both theoretically and experimentally in recent decades.[1−6] These properties include so-called anti-bunching effects, sub-Poisson distributions of the photons and the negative Wigner function, etc.. Typically, the anti-bunching character is a core criterion[7−9] of the desirable single-photon source, which plays an important role in quantum secure communications.[10] The sub-Poisson distribution of photons relates to the so-called Mandel factor Q: $Q < 0$ corresponds to the sub-Poisson distribution, and $Q = 0$ (or $Q > 0$) implies that the distribution of photons is Poisson (or super-Poisson). The former is one of the typical non-classical properties and the latter is just the classical character. Also, the quasi-probability distribution function, i.e., the so-called Wigner function, is usually utilized to determine a quantum state which is either non-classical or classical. Basically, if the Wigner function of a quantum state is negative, then such a quantum state should be non-classical and thus certain non-classical effects can be revealed. Indeed, a series of non-classical quantum states has been verified by showing the negativities of their Wigner functions.[11−16]

Both coherent states and displaced Fock states are typical quantum states in quantum optics. By applying the inverse of bosonic operators, $\hat{a}^{-1}$ and $(\hat{a}^{+})^{-1}$, to these basic quantum states various non-classical states can be generated.[17,18] Originally, the inverses of bosonic operators were introduced by Dirac and Fock to investigate the phase operators of quantum optical fields.[19,20] In 1997, Wei *et al* introduced a new kind of non-classical quantum state[18,21] by applying these inverse operators to the well-known coherent states. Consequently, Yang *et al* also defined some quantum states, called photon-added and -subtracted odd and even coherent states[22] by applying these operators to the odd and even coherent states, respectively.

In this work, we introduce another kind of new quantum state by applying the generalized inverses of bosonic operators to the displaced Fock states. Their non-classical properties are investigated by a numerical method. The rest of this paper is organized as follows. In Section 2, we define the photon-added and photon-subtracted displaced Fock states. The second-order correlation functions and Mandel parameters of these new quantum states are discussed in detail in Section 3. The negativities of Wigner functions for these states are shown in Section 4. Finally, we summarize our conclusions.

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2. Photon-added and photon-subtracted displaced Fock states

It is well-known that an inverse of the number operator, $\hat{N}^{-1}$, is defined by[21]

$$\hat{N}^{-1}|n\rangle = \hat{a}^{-1}(\hat{a}^{+})^{-1}|n\rangle = \begin{cases} n^{-1}|n\rangle, & n \neq 0, \\ 0, & n = 0. \end{cases} \quad (1)$$

Also, the generalized inverses of the creation and annihilation operators, namely $\hat{a}^{-1}$ and $(\hat{a}^{+})^{-1}$, are introduced as (see, e.g.,[19])

$$\hat{a}^{-1}|n\rangle = (n+1)^{-1/2}|n+1\rangle, \quad (2)$$

and

$$(\hat{a}^{+})^{-1}|n\rangle = \begin{cases} n^{-1/2}|n-1\rangle, & n \neq 0, \\ 0, & n = 0, \end{cases} \quad (3)$$

respectively. As a consequence, we can formally define the L-power of the inverse operators

$$\hat{a}^{-L}|n\rangle = \left[\frac{n!}{(n+L)!}\right]^{1/2}|n+L\rangle, \quad (4)$$

$$(\hat{a}^{+})^{-L}|n\rangle = \begin{cases} \left[\frac{(n-L)!}{n!}\right]^{1/2}|n-L\rangle, & n \geq L, \\ 0, & n < L. \end{cases} \quad (5)$$

On the other hand, the displaced Fock states are defined by

$$D(\alpha)|M\rangle = A\exp(-|\alpha|^2/2)|M-n+j\rangle \quad (6)$$

with

$$A = \sum_{n=0}^{M}\sum_{j=0}^{\infty}\frac{(-\alpha)^n\sqrt{M!}}{n!\sqrt{(M-n)!}}\frac{\alpha^j\sqrt{(M-n+j)!}}{j!\sqrt{(M-n)!}}. \quad (7)$$

Above, $D(\alpha) = \exp(\alpha a^{+} - \alpha^{*}a)$ is the so-called displaced operator.

From Eq.(2), we can see that the generalized inverse of the bosonic annihilation operator $\hat{a}^{-1}$ behaves as an excited operator; to add a photon to the quantum state. Therefore, the state $\hat{a}^{-1}|M\rangle$ is a single photon-added state of the number state $|M\rangle$, and the state $\hat{a}^{-1}D(\alpha)|M\rangle$ is consequently defined as the single photon-added displaced Fock state. Certainly, double photon-added displaced Fork states can be introduced as $\hat{a}^{-2}D(\alpha)|M\rangle$, and L photon-added displaced Fock states, namely, $|+L,\alpha,n\rangle$ can be represented by

$$\begin{aligned} &|+L,\alpha,n\rangle \\ &= \hat{a}^{-L}D(\alpha)|M\rangle \\ &= H_a B\exp(-|\alpha|^2/2)|M-n+j+L\rangle \end{aligned} \quad (8)$$

with

$$B = \sum_{n=0}^{M}\sum_{j=0}^{\infty}\frac{(-\alpha)^n\sqrt{M!}}{n!\sqrt{(M-n)!}}\frac{\alpha^j\sqrt{(M-n+j)!}}{j!\sqrt{(M-n)!}} \times\left[\frac{(M-n+j)!}{(M-n+j+L)!}\right]^{1/2}, \quad (9)$$

$$|H_a|^2 = 1/[\exp(-|\alpha|^2)\cdot|B|^2]. \quad (10)$$

Similarly, the L photon-subtracted displaced Fock states, namely, $|-L,\alpha,n\rangle$, can be defined as

$$\begin{aligned} &|-L,\alpha,n\rangle \\ &= (\hat{a}^{+})^{-L}D(\alpha)|M\rangle \\ &= H_{a^{+}}C\exp(-|\alpha|^2/2)|M-n+j-L\rangle \end{aligned} \quad (11)$$

with

$$C = \sum_{n=0}^{M}\sum_{j=L+n-M}^{\infty}\frac{(-\alpha)^n\sqrt{M!}}{n!\sqrt{(M-n)!}}\frac{\alpha^j\sqrt{(M-n+j)!}}{j!\sqrt{(M-n)!}} \times\left[\frac{(M-n+j-L)!}{(M-n+j)!}\right]^{1/2} \quad (12)$$

and

$$|H_{a^{+}}|^2 = 1/[\exp(-|\alpha|^2)\cdot|C|^2]. \quad (13)$$

3. The properties of photon-added and photon-subtracted displaced Fock states

3.1. The second-order correlation function of photon-added and -subtracted displaced Fock states

The second-order correlation function, defined by

$$g^2 = \frac{\langle\hat{a}^{+2}\hat{a}^2\rangle}{(\langle\hat{a}^{+}\hat{a}\rangle)^2} \quad (14)$$

can be utilized to determine either the bunching or anti-bunching effects of the optical field: $g^2 < 1$ indicates that a photon is anti-bunching and thus the optical state is non-classical, while $g^2 \geq 1$ defines the classical (or random) optical field.

The second-order correlation functions of the L photon-added and -subtracted states introduced above can be easily calculated as

$$g_a^2 = \frac{{}_a\langle\hat{a}^{+2}\hat{a}^2\rangle_a}{({}_a\langle\hat{a}^{+}\hat{a}\rangle_a)^2} = \frac{|H_a|^2|B'|^2}{\exp(-|\alpha|^2)(|H_a|^2|B''|^2)^2} \quad (15)$$

with

$$B' = \sum_{n=0}^{M}\sum_{j=0}^{\infty} \frac{(-\alpha)^n\sqrt{M!}}{n!\sqrt{(M-n)!}} \frac{\alpha^j\sqrt{(M-n+j)!}}{j!\sqrt{(M-n)!}} \times \left[\frac{(M-n+j)!}{(M-n+j+L)!}\right]^{1/2} [(M-n+j+L) \times(M-n+j+L-1)]^{1/2}, \quad (16)$$

$$B'' = \sum_{n=0}^{M}\sum_{j=0}^{\infty} \frac{(-\alpha)^n\sqrt{M!}}{n!\sqrt{(M-n)!}} \frac{\alpha^j\sqrt{(M-n+j)!}}{j!\sqrt{(M-n)!}} \times \left[\frac{(M-n+j)!}{(M-n+j+L)!}\right]^{1/2} \times(M-n+j+L)^{1/2} \quad (17)$$

and

$$g^2_{a^+} = \frac{{}_{a^+}\langle \hat{a}^{+2}\hat{a}^2\rangle_{a^+}}{({}_{a^+}\langle \hat{a}^{+}\hat{a}\rangle_{a^+})^2} = \frac{|H_{a^+}|^2|C'|^2}{\exp(-|\alpha|^2)(|H_{a^+}|^2|C''|^2)^2} \quad (18)$$

with

$$C' = \sum_{n=0}^{M}\sum_{j=L+n-M}^{\infty} \frac{(-\alpha)^n\sqrt{M!}}{n!\sqrt{(M-n)!}} \cdot \frac{\alpha^j\sqrt{(M-n+j)!}}{j!\sqrt{(M-n)!}} \times \left[\frac{(M-n+j-L)!}{(M-n+j)!}\right]^{1/2} \cdot [(M-n+j-L) \times(M-n+j-L-1)]^{1/2}, \quad (19)$$

$$C'' = \sum_{n=0}^{M}\sum_{j=L+n-M}^{\infty} \frac{(-\alpha)^n\sqrt{M!}}{n!\sqrt{(M-n)!}} \times\frac{\alpha^j\sqrt{(M-n+j)!}}{j!\sqrt{(M-n)!}} \cdot \left[\frac{(M-n+j-L)!}{(M-n+j)!}\right]^{1/2} \times(M-n+j-L)^{1/2}, \quad (20)$$

respectively.

Figures 1 and 2 show how the second-order correlation functions of L photon-added and -subtracted states depend on the added and subtracted photons L, respectively. It is seen that $g^2 < 1$ for the relatively small $|\alpha|^2$ (e.g., $|\alpha|^2 < 0.7$ in Fig.1 for $L = 0, M = 1$), and $g^2 > 1$ for larger values of $|\alpha|^2$ (e.g., $|\alpha|^2 > 0.7$ in Fig.1 for $L = 0, M = 1$).

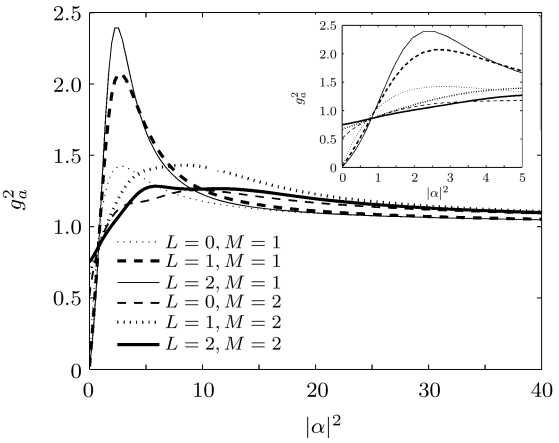


Fig.1. The second-order correlation function of L photon-added states $|+L,\alpha,n\rangle$ versus the parameter $|\alpha|^2$.

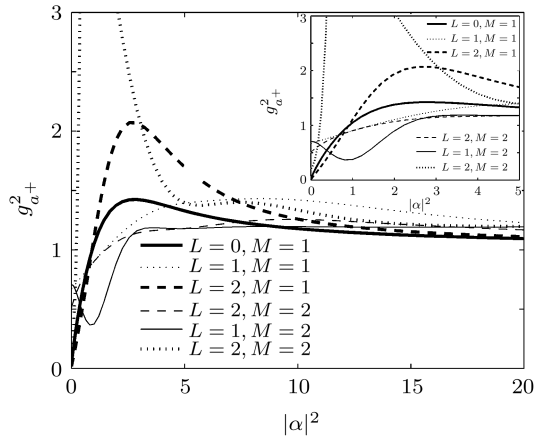


Fig.2. The second-order correlation function of L photon-subtracted states $|-L,\alpha,n\rangle$ versus the parameter $|\alpha|^2$.

3.2. The analysis of Mandel factor Q

Another non-classical property of optical fields is the sub-Poisson distribution of photons. The sub-Poisson distribution of photons relates to the so-called Mandel factor Q

$$Q = \frac{\langle(\hat{a}^+\hat{a})^2\rangle - (\langle\hat{a}^+\hat{a}\rangle)^2 - \langle\hat{a}^+\hat{a}\rangle}{\langle\hat{a}^+\hat{a}\rangle}. \quad (21)$$

$Q = 0$ corresponds to the Poisson distribution, and $Q < 0$ (or $Q > 0$) implies that the distribution of photons is sub-Poisson (or super-Poisson). The former is one of the typical non-classical properties and the latter is just the classical character.

The Mandel factor for the present L photon-added and -subtracted states can be calculated as:

$$Q_a = \frac{{}_a\langle\hat{a}^{+2}\hat{a}^2\rangle_a - ({}_a\langle\hat{a}^+\hat{a}\rangle_a)^2}{{}_a\langle\hat{a}^+\hat{a}\rangle_a} - 1$$

$$= \frac{|H_a|^2|B'|^2 - \exp(-|\alpha|^2)(|H_a|^2|B''|^2)^2}{|H_a|^2|B''|^2} - 1, \tag{22}$$

and

$$Q_{a^+} = \frac{{}_{a^+}\langle \hat{a}^{+2}\hat{a}^2\rangle_{a^+} - ({}_{a^+}\langle \hat{a}^{+}\hat{a}\rangle_{a^+})^2}{{}_{a^+}\langle \hat{a}^{+}\hat{a}\rangle_{a^+}} - 1$$
$$= \frac{|H_{a^+}|^2|C'|^2 - \exp(-|\alpha|^2)(|H_{a^+}|^2|C''|^2)^2}{|H_{a^+}|^2|C''|^2} - 1 \tag{23}$$

respectively.

Figures 3 and 4 display the statistical distributions of photons for the introduced L photon-added and -subtracted states. We can see from these numerical results that the value of the Mandel factor is less than zero when the $|\alpha|^2$ is sufficiently small (e.g., $|\alpha|^2 < 4.8$ in Fig.3 for $L = 2, M = 2$). Under these parameters the photons of L photon-added and -subtracted states are of sub-Poisson distribution. When $|\alpha|^2$ increases, the value of the Mandel factor can be larger than zero (e.g., $|\alpha|^2 > 4.8$ in Fig.3 for $L = 2, M = 2$), implying that the photons in these cases are of super-Poisson distributions.

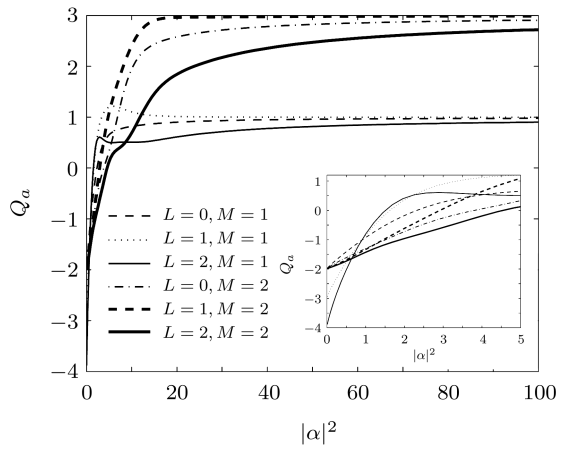


Fig.3. The Mandel factor Q of L photon-added states $|+L, \alpha, n\rangle$ versus the parameter $|\alpha|^2$.

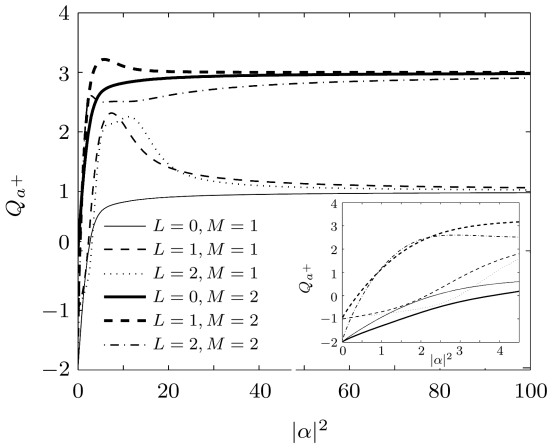


Fig.4. The Mandel factor Q of L photon-subtracted states $|-L, \alpha, n\rangle$ versus the parameter $|\alpha|^2$.

4. Negativity of the Wigner function

The most obvious property of the Wigner function is that it may take a negative value for certain quantum states, as a signature of their non-classical nature. Therefore, calculation of the Wigner function of a quantum state might determine if such a quantum state is classical or non-classical.

For the simplest quantum state, i.e., the number state $|n\rangle$ with the density operator

$$\rho_n = |n\rangle\langle n|, \tag{24}$$

the Wigner function is

$$W_{\rho n}(\alpha, \alpha^*) = \frac{2}{\pi}(-1)^n \exp(-2|\alpha|^2) L_n^{(0)}(4|\alpha|^2). \tag{25}$$

Here,

$$L_n^{(j)}(x) = \Sigma_{k=0}^{n}(-1)^k \frac{(n+j)!}{(n-k)!(j+k)!}\frac{x^k}{k!} \tag{26}$$

is the Laguerre polynomial. For the photon-added and -subtracted displaced Fock states introduced in this paper, the density operator take the forms

$$\rho_a = |H_a|^2|B|^2 \exp(-|\alpha|^2) \times |M-n+j+L\rangle\langle M-n+j+L|, \tag{27}$$

and

$$\rho_{a^+} = |H_{a^+}|^2|C|^2 \exp(-|\alpha|^2) \times |M-n+j-L\rangle\langle M-n+j-L|, \tag{28}$$

respectively.

Using the Wigner function of the number states, i.e., Eq.(25), the Wigner functions of the present

photon-added and -subtracted displaced Fock states can be easily expressed as

$$W_a = |H_a|^2|B|^2\exp(-|\alpha|^2)\frac{2}{\pi}(-1)^{M-n+j+L} \times \exp(-2|\alpha|^2)L^{(0)}_{M-n+j+L}(4|\alpha|^2), \quad (29)$$

and

$$W_{a^+} = |H_{a^+}|^2|C|^2\exp(-|\alpha|^2)\frac{2}{\pi}(-1)^{M-n+j-L} \times \exp(-2|\alpha|^2)L^{(0)}_{M-n+j-L}(4|\alpha|^2), \quad (30)$$

respectively.

Figures 5 and 6 display the values of the Wigner function W_a for different L and M. Clearly, the Wigner function Eq.(29) is non-Gaussian in phase space $\Re = \{\mathrm{Re}(\alpha), \mathrm{Im}(\alpha)\}$ and has some negative regions, which is more evidence of non-classicality.

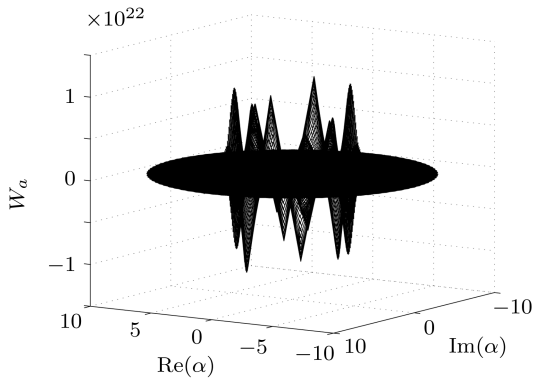


Fig.5. The Wigner function W_a of L photon-added states $|+L,\alpha,n\rangle$ versus $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ for $L=1$ and $M=1$.

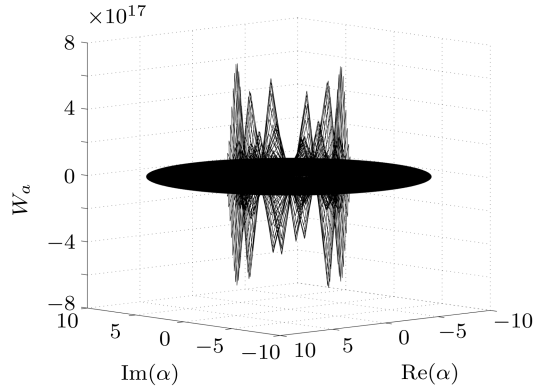


Fig.6. The Wigner function W_a of L photon-added states $|+L,\alpha,n\rangle$ versus $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ for $L=2$ and $M=2$.

Figures 7 and 8 show how the Wigner function W_{a^+} changes with the complex parameter α for different L and M. Obviously, such a function is also non-Gaussian in the phase space $\Re$, and also has some negative regions. This indicates that the L photon-subtracted states are also non-classical quantum states.

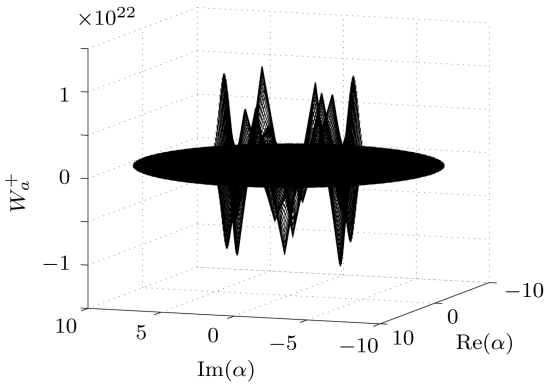


Fig.7. The Wigner function W_{a^+} of L photon-subtracted states $|-L,\alpha,n\rangle$ versus $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ for $L=1$ and $M=1$.

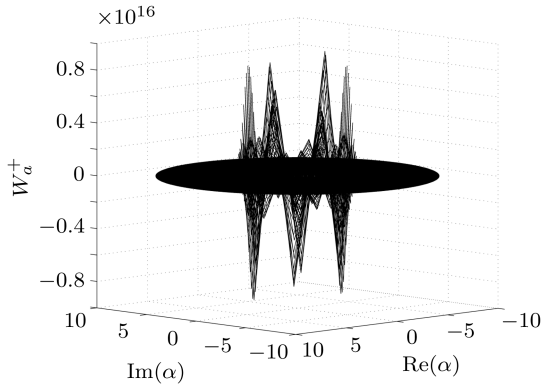


Fig.8. The Wigner function W_{a^+} of L photon-subtracted states $|-L,\alpha,n\rangle$ versus $\mathrm{Re}(\alpha)$ and $\mathrm{Im}(\alpha)$ for $L=2$ and $M=2$.

5. Conclusion

In this paper, we introduce a series of new quantum states, the photon-added and -subtracted displaced Fock states, by applying generalized inverses of bosonic operators to the displaced Fock states. We have calculated the second-order correction function and the Mandel factor Q of these new quantum states. These results indicate that the photons of the new quantum states have anti-bunching effects and sub-Poisson distributions in the case of relatively small values of $|\alpha|^2$. Both the sub-Poisson distributions and the anti-bunching effects reflect the non-classical properties of these introduced optical states.

To further confirm the non-classical natures of these quantum states, we also investigated their Wigner functions. The results indicate that negative values obviously exist for certain parameters. Therefore, the new quantum states introduced in this paper are clearly non-classical.

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