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Amplitude control of limit cycle in a van der Pol–Duffing system*

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This paper applies washout filter technology to amplitude control of limit cycles emerging from Hopf bifurcation of the van der Pol–Duffing system. The controlling parameters for the appearance of Hopf bifurcation are given by the Routh–Hurwitz criteria. Noticeably, numerical simulation indicates that the controllers control the amplitude of limit cycles not only of the weakly nonlinear van der Pol–Duffing system but also of the strongly nonlinear van der Pol–Duffing system. In particular, the emergence of Hopf bifurcation can be controlled by a suitable choice of controlling parameters. Gain-amplitude curves of controlled systems are also drawn.

Keywords: bifurcation control, limit cycle, Hopf bifurcation, van der Pol–Duffing system

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1. Introduction

Bifurcation control is one of the topics attracting rapidly growing interest in the control of nonlinear dynamical systems.[1−5] Bifurcation control generally means to design a controller that is capable of modifying the bifurcation characteristics of a bifurcating nonlinear system; a typical objective is to monitor the amplitude of limit cycles of a given nonlinear system. An overview of recent progress in the theories, methods and application of bifurcation control is given in Ref.[1]. Up to now, the objectives of bifurcation control have ranged from delaying the appearance of Hopf bifurcation and/or changing the stabilities of some bifurcating periodic solutions[6−9] to managing the relocation and/or suppression of chaos. In particular, some nontraditional control objectives have been formulated by taking into account the consequence of the system dynamics near regular and degenerate Hopf bifurcation.[10] These studies have motivated several recent investigations concerning amplitude control of oscillations in nonlinear systems,[11−14] for instance, amplitude suppression of a van der Pol oscillator can be realized by a time delay state feedback.[15] Tang and Chen investigated amplitude control of a van der Pol system by applying a state feedback controller;[16] the van der Pol–Duffing system has also attracted researchers' attention,[17,18] and resonance behaviour and the jump phenomenon in the coupled van der Pol–Duffing system have been studied.[19] Maccari investigated the vibration amplitude of a van der Pol–Duffing system under time-delayed position and velocity feedbacks and the method shows that the feedback gains can enhance the control performance and obtain the desired vibration amplitude.[20] Chen and Yu controlled the Hopf bifurcation by a time-delayed feed-back controller using a polynomial function.[21]

Washout filters are commonly used in control systems for power systems[22] and aircraft.[23] Wang and Abed developed washout filter technology and applied the washout filter-aided feedback controller to bifurcation control of a chaotic system.[10] Zhou *et al* controlled the Hopf bifurcation of a neural equation with discrete time delay based on washout filter technology.[24] Wang *et al* applied an extended washout filter to control the Hopf bifurcation in Hodgkin–Huxley equations.[25] In this paper, a washout filter was added to control the amplitude of limit cycles in a van der Pol–Duffing system.

The equation of motion of a free vibration van der Pol–Duffing system is

$$\ddot{x} + x - \varepsilon(\mu - \beta\, x^2)\dot{x} + \varepsilon\alpha x^3 = 0. \tag{1}$$

If ε is a small positive parameter ($\varepsilon \ll 1$), we can find the first-order approximate solution of Eq.(1) as

$$x = a\cos(\ t + \varphi). \tag{2}$$

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The amplitude a and phase φ of the response are governed by the following equations:

$$a' = \frac{1}{2}\mu a - \frac{1}{8}\beta a^3, \qquad \varphi' = 0. \tag{3}$$

Consequently, the amplitude of the limit cycle of Eq.(1) is given by

$$a = 2\sqrt{\frac{\mu}{\beta}} \quad \text{or} \quad a = 0. \tag{4}$$

Note that if $\mu = \beta = 1, \alpha = 0.5$, then $a = 2$; this is a familiar result. Mathematical analysis shows that a van der Pol–Duffing system has a unique, stable limit cycle toward which all other solutions approach as $t \to \infty$. The time–displacement (t–x) curve and the phase space (x–$\dot{x}$) trajectory of the van der Pol–Duffing equation (1) using the numerical method are shown in Fig.1.

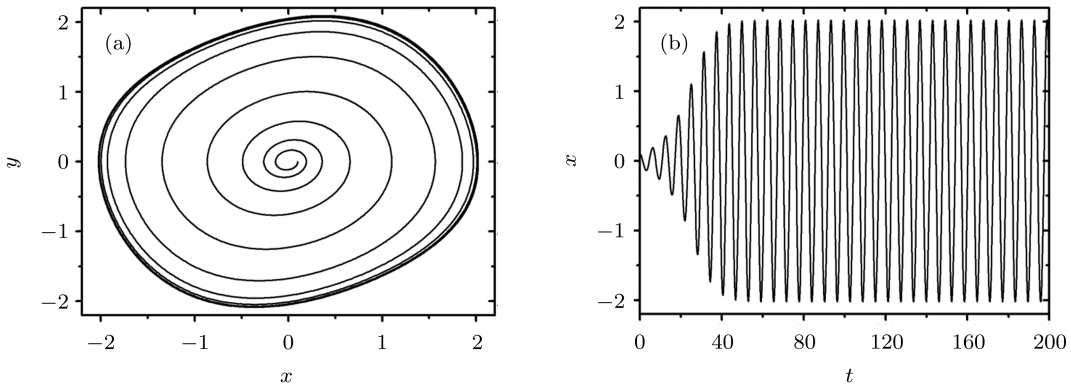


Fig.1. Numerical solution of the van der Pol–Duffing oscillator Eq.(1) with $\varepsilon = 0.2$, $\mu = \beta = 1, \alpha = 0.5$: (a) phase space ($x$–$\dot{x}$) trajectory; (b) time–displacement (t–x) curve.

If ε is not a small parameter, the van der Pol–Duffing oscillator Eq.(1) is a strongly nonlinear system. From numerical analysis, the uncontrolled strongly nonlinear system is given by Eq.(1); when $\varepsilon = 2$, it has a limit cycle with amplitude $a = 1.9635$ as presented in Fig.2.

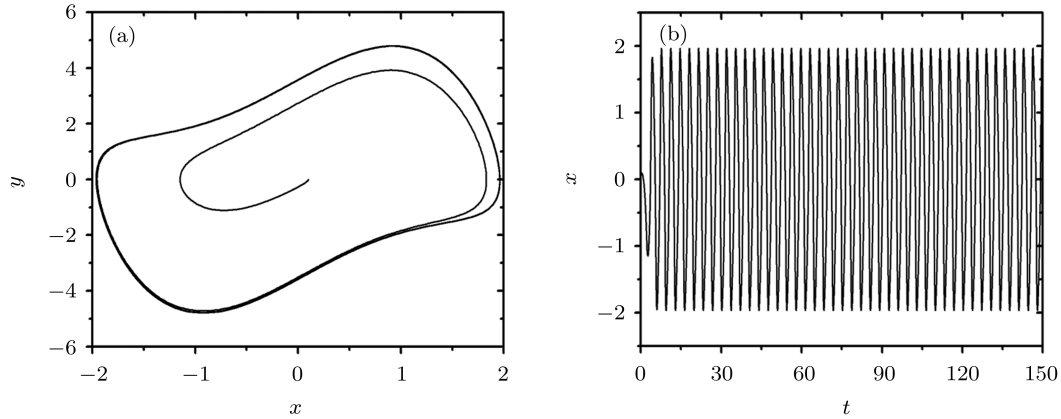


Fig.2. Numerical solution of the van der Pol–Duffing oscillator Eq.(1) with $\varepsilon = 2, \mu = \beta = 1, \alpha = 0.5$: (a) phase space ($x$–$\dot{x}$) trajectory; (b) time–displacement (t–x) curve.

2. Washout filter feedback controller

The washout filter-aided controller assumes the following structure:

$$\dot{\omega} = x - d \times \omega \triangleq \nu, \tag{5}$$

$$u = g(\nu; \boldsymbol{K}), \tag{6}$$

where $\boldsymbol{K}$ is the control gain vector, considered as a parameter vector of the nonlinear control function g, and d is the washout filter time constant. The equation $g(0;\boldsymbol{K}) = 0$ preserves the original equilibrium points. Due to the nature of the washout filter, $u = 0$ when $\dot{\omega} = y = 0$, so all the equilibrium points remain unchanged when control actions are applied.

A washout filter is a stable high-pass filter with a transfer function:

$$G(s) = \frac{y(s)}{x(s)} = \frac{s}{s+d}, \tag{7}$$

where d is a time constant. The washout filter is stable when d are positive parameters, otherwise, the washout filter is unstable.[10]

The van der Pol–Duffing system Eq.(1) can be cast into the following simultaneous equations:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \varepsilon(\mu - \beta x^2)y - \varepsilon\alpha x^3, \end{aligned} \tag{8}$$

where a washout filter in y is used, so the controlled system which adds a washout filter-aided controller is as follows:

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \varepsilon(\mu - \beta x^2)y - \varepsilon\alpha x^3 + u, \\ \dot{\omega} &= x - d\omega. \end{aligned} \tag{9}$$

For the system (9), in addition to the creation of a Hopf bifurcation, the controller can control the amplitude of limit cycles. For this purpose, consider a specialized nonlinear control function of the form

$$g(\nu;\boldsymbol{K}) = k_1\upsilon + k_3\upsilon^3, \tag{10}$$

where $\boldsymbol{K} = (k_1, k_3)^{\mathrm{T}}$ is the gain vector; the aim of the linear term in the controller is the creation of a Hopf bifurcation, and the cubic term controls the amplitude of the limit cycle. $d = 0.2$ was taken, so Eq.(9) becomes

$$\begin{aligned} \dot{x} &= y, \\ \dot{y} &= -x + \varepsilon(\mu - \beta x^2)y - \varepsilon\alpha x^3 \\ &\quad + k_1(y - 0.2\omega) + k_3(y - 0.2\omega)^3, \\ \dot{\omega} &= y - 0.2\omega. \end{aligned} \tag{11}$$

2.1. Controlling the weakly nonlinear van der Pol–Duffing oscillator

When $\varepsilon = 0.2$, the van der Pol–Duffing oscillator Eq.(1) is a weakly nonlinear system. Letting $\mu = \beta = 1,\ \alpha = 0.5$, the Jacobian matrix of the linearized system Eq.(11) is

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & k_1 + 0.2 & -0.2k_1 \\ 0 & 1 & -0.2 \end{bmatrix}. \tag{12}$$

The characteristic equation of the Jacobian matrix Eq.(12) has the form

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{13}$$

where $c_1 = -k_1$, $c_2 = 0.96$, $c_3 = 0.2$. According to the Routh–Hurwitz criteria we know that when $c_1c_2 = c_3$ Eq.(11) will give birth to Hopf bifurcation, so we can obtain $k_1 = -5/24$, and Eq.(11) produces a limit cycle of Hopf bifurcation. Thus we can adjust the nonlinear parameter k_3 to obtain stable bifurcated periodic orbits (limit cycles).[6]

2.2. Controlling the strongly nonlinear van der Pol–Duffing oscillator

If ε is not a small parameter, the van der Pol–Duffing oscillator Eq.(1) is a strongly nonlinear system. As for the weakly nonlinear system, we let $\mu = \beta = 1, \alpha = 0.5$, and the Jacobian matrix of the linearized system Eq.(11) is

$$\boldsymbol{J} = \begin{bmatrix} 0 & 1 & 0 \\ -1 & k_1 + 2 & -0.2k_1 \\ 0 & 1 & -0.2 \end{bmatrix}. \tag{14}$$

The characteristic equation of the Jacobian matrix $\boldsymbol{J}$ has the form

$$\lambda^3 + c_1\lambda^2 + c_2\lambda + c_3 = 0, \tag{15}$$

where $c_1 = -(1.8 + k_1)$, $c_2 = 0.6$, $c_3 = 0.2$. From the Routh–Hurwitz criteria we also know that when $c_1c_2 = c_3$ Eq.(11) will give birth to Hopf bifurcation, so we can get $k_1 = -32/15$, and Eq.(11) produces a limit cycle of Hopf bifurcation. We can adjust the nonlinear parameter k_3 to obtain stable bifurcated periodic orbits (limit cycles).[6]

From Figs.3 and 4 we know that the washout filter-aided feedback controller can control the amplitude effectively. When $k_3 = -0.2$, the amplitude of the limit cycle $a = 0.5132$, when $k_3 = 0.02$, the amplitude of the limit cycle $a = 1.6172$. Figure 5 tells us that the washout filter-aided feedback controller can also monitor the amplitude of the strongly nonlinear system. When $k_3 = -0.5$, the amplitude of the limit cycle $a = 0.1985$.

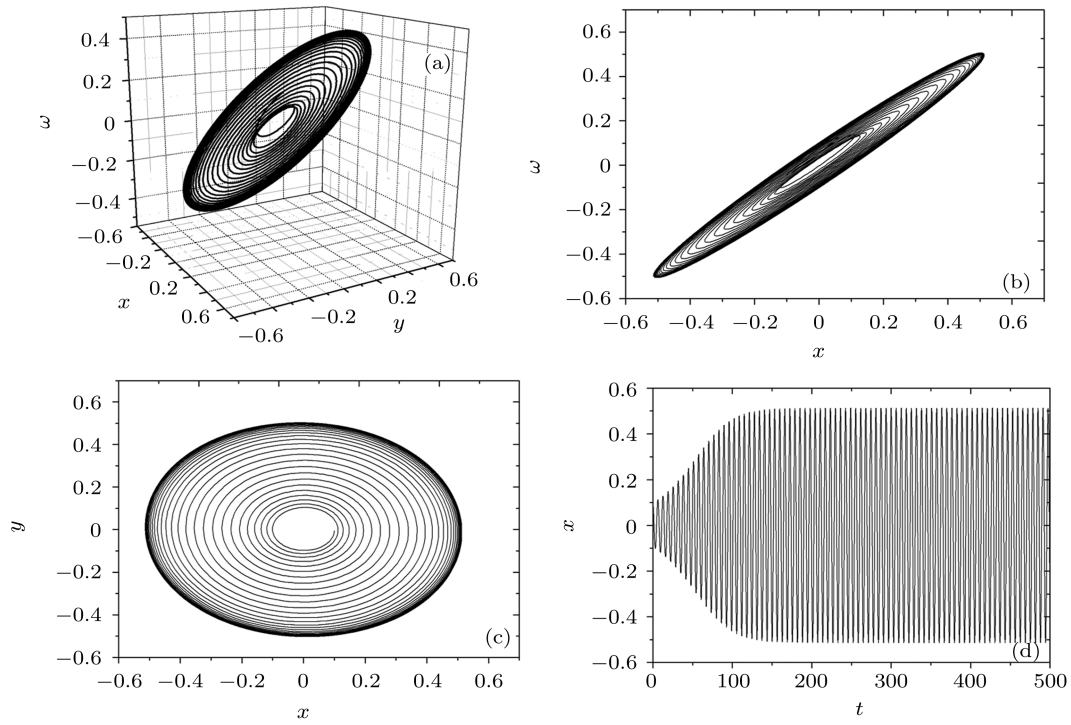


Fig.3. (a) Three dimensional state variable curve. (b) Phase space (x–ω) trajectory. (c) Phase space (x–y) trajectory. (d) Time–displacement (t–x) curve for the controlled system (11). The other parameters are fixed as $\varepsilon = 0.2, \mu = \beta = 1, \alpha = 0.5$, and $k_1 = -5/24$, $k_3 = -0.2$.

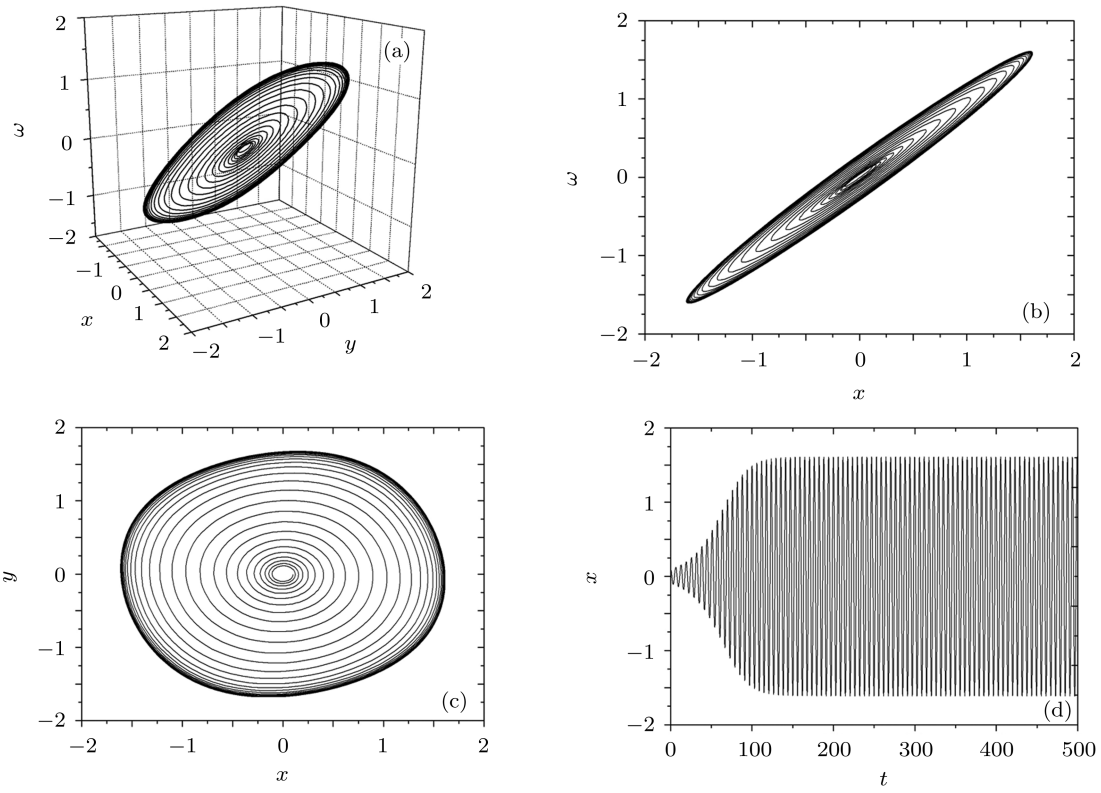


Fig.4. (a) Three dimensional state variable curve. (b) Phase space (x–ω) trajectory. (c) Phase space (x–y) trajectory. (d) Time–displacement (t–x) curve for the controlled system (11). The other parameters are fixed as $\varepsilon = 0.2$, $\mu = \beta = 1, \alpha = 0.5$ and $k_1 = -5/24$, $k_3 = 0.02$.

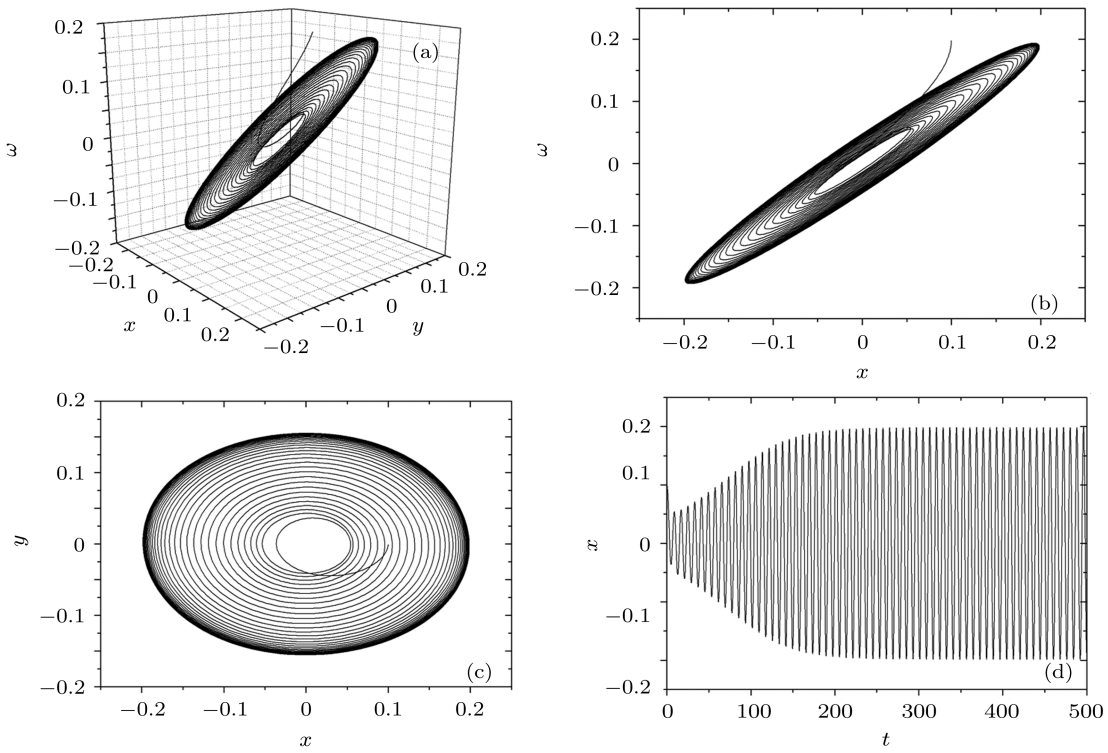


Fig.5. (a) Three dimensional state variable curve, (b) phase space (x–ω) trajectory, (c) phase space (x–y) trajectory, (d) time–displacement (t–x) curve for the controlled system (11). The other parameters are fixed as $\varepsilon = 2$, $\mu = \beta = 1, \alpha = 0.5$ and $k_1 = -32/15$, $k_3 = 0.5$.

Comparing Fig.6 with Fig.7, we find that in both figures the amplitude values $x_{\max}$ of limit cycles decrease when the control gains become smaller and approach zero, so the emergence of Hopf bifurcation can be controlled by a suitable choice of controlling parameters. Among the range of values of control gains shown in Figs.6 and 7, the limit cycles are stable.

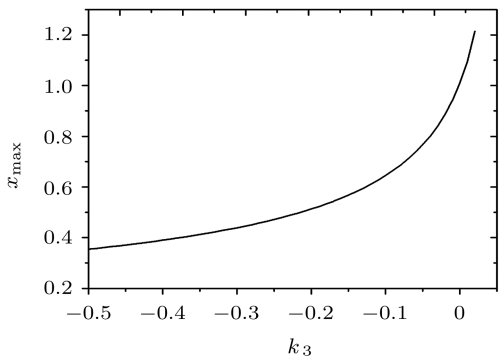


Fig.6. Gain amplitude (k_3–$x_{\max}$) curve for the weakly nonlinear controlled system.

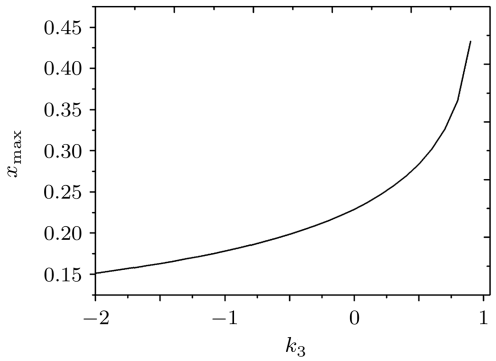


Fig.7. Gain amplitude (k_3–$x_{\max}$) curve for the strongly nonlinear controlled system.

3. Conclusion

This work addresses controlling the amplitude of Hopf bifurcation of a van der Pol–Duffing system by using a washout filter feedback controller; controlling parameters are achieved for the creation of Hopf bifurcation by the Routh–Hurwitz criteria. The amplitude of the limit cycle is obtained by numerical study. In particular, the controllers are capable of monitoring the amplitude of the limit cycle of the strong nonlinear van der Pol–Duffing oscillator.

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