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Quantum backreaction of quantum fluid in Bose–Einstein condensates*

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In this paper, with the full field operator $\hat{\psi}$ expressed in terms of a particle-number-conserving mean-field ansatz, we investigate the dynamical behaviour of Bose–Einstein condensates from microscopic physics. Including the first-order term correction from single-particle excitation and the remaining higher-order term correction from collective excitations simultaneously, we obtain the formulation for a closed local expression of quantum backreaction Q, and discuss the influence on static Bose–Einstein condensates. Even though the quantum backreaction is small, it still has some influence on its dynamics.

Keywords: Bose–Einstein condensate, quantum fluctuation, mean-field theory

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1. Introduction

Bose–Einstein condensates (BECs) in dilute systems of trapped neutral atoms open a new field for studying the quantum mechanics of macroscopic and mecroscopic systems. It is of some importance that BECs belong to a special class of quantum perfect fluids, so-called superfluids.[1] Various kinds of dynamic characters of these systems have been researched extensively[2−5] including spinor Bose–Einstein condensates[6,7] since they were realized experimentally in 1995.

In low temperature BECs, a macroscopic number of particles occupy the same zero-momentum state, therefore these systems can be approximately described by classical quantities in the mean-field theory which has made great achievements.[8,9] However, since there are zero-point motion, interaction between particles, and external infinitesimal disturbance, etc, there exists macroscopic quantum fluctuation such as single-particle excitation and collective excitation.[10] Obviously they will give rise to corresponding physical effects, and one of these effects is quantum backreaction.[11] The correct treatment of this quantum backreaction is one of the main problems in quantum gravity and is related to fundamental questions such as the initial singularity (Big Bang) or the cosmological constant. The quantum backreaction originating from quantum fluctuations has attracted many people to research its relevant areas.[12,13] For instance, research on quantum backreaction in black holes and cosmic inflation has acquired many meaningful results.[14−16]

Recently, in order to find the dynamical relation between BECs and cosmic inflation, researchers[16,17] have also investigated quantum backreaction in BECs on the basis of mean field theory (MFT). The authors have considered the low-order quantum fluctuation of the system (single-particle excitations), and have given an expression for quantum backreaction. It is found that the classical (macroscopic) equation of motion may be sufficient to derive the first-order dynamics of the linear quantum fluctuations (phonons), but the quantum backreaction as a second-order effect cannot be obtained without further knowledge of the microscopic structure (e.g. operator ordering). However, in their calculations, the authors have omitted the contribution of the higher-order term (from collective excitations). Here, we retain the higher-order correction from collective excitations, and try to derive the corresponding quantum backreaction expression.

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2. Quantum backreaction in dilute Bose–Einstein condensates

According to the scattering theory, in BECs at low temperature the s-wave scattering approximation may be adopted, and two-body interaction may be expressed with the s-wave scattering length a and the coupling strength g, which are related by $g = 4\pi a$ (we set $\hbar = m = 1$). In the Heisenberg picture, the many-particle system may be described by the field operator as follows:

$$\mathrm{i}\frac{\partial\hat{\psi}}{\partial t} = \left(-\frac{\nabla^2}{2} + V + g\hat{\psi}^{+}\hat{\psi}\right)\hat{\psi}, \tag{1}$$

where V denotes the one-particle trapping potential.

For a many-particle system trapped in a finite potential at low temperature, the majority of these particles condense in the same single-particle ground state. The field operator in the particle-number-conserving mean-field approximation may be expressed as[18]

$$\hat{\psi} = (\psi_0 + \hat{\chi} + \hat{\zeta})\hat{A}/\sqrt{\hat{N}}. \tag{2}$$

The order parameter $\psi_0 = o(\sqrt{N})$ is the one-particle ground function, $\hat{\chi} = o(N^0)$ corresponds to one-particle excitation and $\hat{\zeta} = o(1/\sqrt{N})$ is the higher-order term. $\hat{A} = \sqrt{N}a_0/\sqrt{n_0}$, $\hat{N} = \hat{A}^{+}\hat{A}$ is the total particle number operator. $\hat{n}_0 = \hat{a}_0^{+}\hat{a}_0$ is the particle number operator of the occupying macro-state. After inserting expression (2) into Eq.(1), we can obtain the Gross–Pitaevskii equation for the order parameter,

$$\mathrm{i}\frac{\partial\psi_0}{\partial t} = \left(-\frac{\nabla^2}{2} + V + g\left(|\psi_0|^2 + 2\left\langle\left(\hat{\chi}+\hat{\zeta}\right)^{+}\left(\hat{\chi}+\hat{\zeta}\right)\right\rangle\right)\right)\psi_0 + g\left\langle\left(\hat{\chi}+\hat{\zeta}\right)^2\right\rangle\psi_0^*, \tag{3}$$

the Bogoliubov–de Gennes equation for single-particle excitation

$$\mathrm{i}\frac{\partial\hat{\chi}}{\partial t} = \left(-\frac{\nabla^2}{2} + V + 2g\left(|\psi_0|^2 + \left\langle\left(\hat{\chi}+\hat{\zeta}\right)^{+}\left(\hat{\chi}+\hat{\zeta}\right)\right\rangle\right)\right)\hat{\chi} + g\left(\psi_0^2 + \left\langle\left(\hat{\chi}+\hat{\zeta}\right)^2\right\rangle\right)\hat{\chi}^{+}, \tag{4}$$

and the equation describing the remaining higher-order correction

$$\begin{aligned}\mathrm{i}\frac{\partial\hat{\zeta}}{\partial t} = {} & \left(-\frac{\nabla^2}{2} + V + 2g\left(|\psi_0|^2 + \left\langle\left(\hat{\chi}+\hat{\zeta}\right)^{+}\left(\hat{\chi}+\hat{\zeta}\right)\right\rangle\right)\right)\hat{\zeta} + g\left(\psi_0^2 + \left\langle\left(\hat{\chi}+\hat{\zeta}\right)^2\right\rangle\right)\hat{\zeta}^{+} \\ & + g\psi_0^*\left[\left(\hat{\chi}+\hat{\zeta}\right)^2 - \left\langle\left(\hat{\chi}+\hat{\zeta}\right)^2\right\rangle\right] + 2g\psi_0\left(\left(\hat{\chi}+\hat{\zeta}\right)^{+}\left(\hat{\chi}+\hat{\zeta}\right) - \left\langle\left(\hat{\chi}+\hat{\zeta}\right)^{+}\left(\hat{\chi}+\hat{\zeta}\right)\right\rangle\right).\end{aligned} \tag{5}$$

Denoting $\rho_0 = |\psi_0|^2$, $\hat{\phi} = \hat{\chi} + \hat{\zeta}$, $\rho_\phi = \langle\phi^{+}\phi\rangle$, and merging Eqs.(4) and (5) into one equation, we may obtain

$$\mathrm{i}\frac{\partial\psi_0}{\partial t} = \left(-\frac{\nabla^2}{2} + V + g\left(\rho_0 + 2\rho_\phi\right)\right)\psi_0 + g\left\langle\hat{\phi}^2\right\rangle\psi_0^*, \tag{6}$$

$$\begin{aligned}\mathrm{i}\frac{\partial\hat{\phi}}{\partial t} = {} & \left(-\frac{\nabla^2}{2} + V + 2g\left(\rho_0 + \rho_\phi\right)\right)\hat{\phi} \\ & + g\left(\psi_0^2 + \left\langle\hat{\phi}^2\right\rangle\right)\hat{\phi}^{+} + g\psi_0^*\left(\hat{\phi}^2 - \left\langle\hat{\phi}^2\right\rangle\right) \\ & + 2g\psi_0\left(\hat{\phi}^{+}\hat{\phi} - \rho_\phi\right).\end{aligned} \tag{7}$$

Equation (6) describes the dynamic behaviour of the condensed part, while equation (7) depicts the dynamic behaviours of fluctuations resulting in single-particle and many-particle excitations and the non-condensed part.

In order to calculate quantum backreaction which corresponds to the next-to-leading order term directly, we should make some additional assumptions. Here, we take another way from the macroscopic perspective. Define the total particle density as $\rho = \left\langle\hat{\psi}^{+}\hat{\psi}\right\rangle$, and the total particle current vector as

$$\boldsymbol{j} = \frac{1}{2\mathrm{i}}\left\langle\hat{\psi}^{+}\nabla\hat{\psi} - \nabla\hat{\psi}^{+}\hat{\psi}\right\rangle = \rho\boldsymbol{v}, \tag{8}$$

where $\boldsymbol{v}$ is the average velocity field of a quantum fluid. From Eq.(1) we may find that the equation of continuity for the quantum fluid is not changed by the quantum fluctuations and reads as

$$\partial_t\rho + \nabla\cdot\boldsymbol{j} = 0. \tag{9}$$

Define the particle current density vector of condensates as

$$\boldsymbol{j}_{\mathrm{o}}=\frac{1}{2\mathrm{i}}\left\langle\hat{\psi}_0^*\nabla\hat{\psi}_0-\left(\nabla\hat{\psi}_0^*\right)\hat{\psi}_0\right\rangle=\rho_0\boldsymbol{v}_0, \tag{10}$$

where $\boldsymbol{v}_0$ is the velocity field of an irrotational fluid. From Eq.(6) we can obtain

$$\partial_t\rho_0+\nabla\cdot\boldsymbol{j}_0=\mathrm{i}g\left\langle\psi_0^2\left(\hat{\phi}^+\right)^2-\left(\psi_0^*\right)^2\hat{\phi}^2\right\rangle. \tag{11}$$

Define the particle current density vector of fluctuations including the non-condensed part and quasiparticle excitations as

$$\boldsymbol{j}_\phi=\frac{1}{2\mathrm{i}}\left\langle\hat{\phi}^+\nabla\hat{\phi}-\nabla\hat{\phi}^+\hat{\phi}\right\rangle. \tag{12}$$

From Eq.(7) we can have

$$\partial_t\rho_\phi+\nabla\cdot\boldsymbol{j}_\phi=-\mathrm{i}g\left\langle\psi_0^2\left(\hat{\phi}^+\right)^2-\left(\psi_0^*\right)^2\hat{\phi}^2\right\rangle. \tag{13}$$

Equation (11) plus Eq.(13) is Eq.(9). Moreover, there are

$$\rho=\rho_0+\rho_\phi,\quad \boldsymbol{j}=\boldsymbol{j}_{\mathrm{o}}+\boldsymbol{j}_\phi,\quad \boldsymbol{v}=(\rho_0\boldsymbol{v}_0+\boldsymbol{j}_\phi)/\rho. \tag{14}$$

From Eq.(1) and expression (8), we can obtain

$$\begin{aligned}\frac{\partial\boldsymbol{j}}{\partial t}=&\frac{1}{4}\left\langle\hat{\psi}^+\nabla^3\hat{\psi}-\left(\nabla^2\hat{\psi}^+\right)\nabla\hat{\psi}+\mathrm{H.c.}\right\rangle\\&-\left\langle\hat{\psi}^+\hat{\psi}\right\rangle\nabla V-\frac{1}{2g}\nabla\left\langle g^2\left(\hat{\psi}^+\right)^2\hat{\psi}^2\right\rangle\\=&\frac{1}{4}\left\langle\psi_0^*\nabla^3\psi_0-\left(\nabla^2\psi_0^*\right)\nabla\psi_0+\mathrm{c.c.}\right\rangle\\&+\frac{1}{4}\left\langle\hat{\phi}^+\nabla^3\hat{\phi}-\left(\nabla^2\hat{\phi}^+\right)\nabla\hat{\phi}+\mathrm{H.c.}\right\rangle\\&-\rho\nabla V-\frac{1}{2g}\nabla\Big[g^2\Big\langle\rho^2+2\rho\rho_\phi+\left(\psi_0^*\right)^2\hat{\phi}^2\\&+\psi_0^2\left(\hat{\phi}^+\right)^2\Big\rangle\Big].\end{aligned} \tag{15}$$

For the condensed part, by using the Madelung representation $\psi_0=\sqrt{\rho_0}\,\mathrm{e}^{\mathrm{i}\theta}$, $\boldsymbol{v}_0=\nabla\theta$, the first term on the right-hand side of the above formula can be written as

$$\begin{aligned}&\frac{1}{4}\left\langle\psi_0^*\nabla^3\psi_0-\left(\nabla^2\psi_0^*\right)\nabla\psi_0+\mathrm{c.c.}\right\rangle\\=&\frac{\rho_0}{2}\nabla\left(\frac{\nabla^2\sqrt{\rho_0}}{\sqrt{\rho_0}}\right)-\rho_0\left(\boldsymbol{v}_0\cdot\nabla\right)\boldsymbol{v}_0-\boldsymbol{v}_0\nabla\cdot\left(\rho_0\boldsymbol{v}_0\right).\end{aligned} \tag{16}$$

Combining it with expression (14), after tedious calculations, we can have the equation describing the particle current density vector, which is analogous to the Euler equation as follows:

$$\frac{\partial\boldsymbol{j}}{\partial t}=f\left(\boldsymbol{j},\rho\right)+Q+o\left(1/\sqrt{N}\right), \tag{17}$$

$$\begin{aligned}f\left(\boldsymbol{j},\rho\right)=&\left(\nabla\cdot\boldsymbol{j}-\boldsymbol{j}\cdot\nabla\right)\left(\frac{\boldsymbol{j}}{\rho}\right)\\&+\rho\nabla\left[\frac{1}{2}\left(\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)-V-\rho g\right],\end{aligned} \tag{18}$$

$$\begin{aligned}Q=&\nabla\left(\boldsymbol{v}\otimes\boldsymbol{j}_\phi+\boldsymbol{j}_\phi\otimes\boldsymbol{v}-\rho_\phi\boldsymbol{v}\otimes\boldsymbol{v}\right)\\&-\frac{1}{2g}\nabla\left[g^2\left\langle2\rho\rho_\phi+\psi_0^2\left(\phi^+\right)^2+\left(\psi_0^*\right)^2\phi^2\right\rangle\right]\\&-\frac{1}{2}\nabla\cdot\left\langle\left(\nabla\phi^+\right)\otimes\nabla\phi+\mathrm{H.c.}\right\rangle+\frac{\nabla\rho}{4\rho}\nabla^2\rho_\phi,\end{aligned} \tag{19}$$

where $f\left(\boldsymbol{j},\rho\right)$ is the usual classics force, for it contains no explicit quantum fluctuation term, while Q is the contribution of quantum backreaction force resulting from quantum fluctuation.

In the following, we discuss a special example to show a prominent effect of the quantum backreaction. For a static condensate, $\boldsymbol{v}=0$, from expression (19) we can obtain

$$\begin{aligned}Q=&-\nabla\left\langle\left(\nabla\phi^+\right)\nabla\phi\right\rangle-\frac{1}{2g}\nabla\left(g^2\rho\left\langle2\phi^+\phi+\left(\phi^+\right)^2+\phi^2\right\rangle\right)+\frac{\nabla\rho}{4\rho}\nabla^2\rho_\phi\\=&-\nabla\left(\frac{1}{3\pi}\left(g\rho\right)^{3/2}\right)+\frac{1}{2\pi g}\nabla\left(g^{5/2}\rho^{3/2}\right)+\frac{\nabla\rho}{4\rho}\nabla^2\rho_\phi+o\left(1/\sqrt{N^3}\right)\\=&\frac{\rho}{2\pi}\nabla\sqrt{g^3\rho}+\frac{\nabla\rho}{4\rho}\nabla^2\rho_\phi+o\left(1/\sqrt{N^3}\right).\end{aligned} \tag{20}$$

Notice that the sign of Q is positive, while the contribution of the pure photon density $2\left\langle\phi^\dagger\phi\right\rangle$ is negative. We can clearly see the importance of the term $\left\langle\left(\phi^\dagger\right)^2+\phi^2\right\rangle$.

For static BECs, formulation in terms of the conventional Euler equation yields

$$\left(\frac{\partial}{\partial t}+v\cdot\nabla\right)v=-\nabla\left(V+g\rho-\frac{1}{2}\frac{\nabla^2\sqrt{\rho}}{\sqrt{\rho}}\right)+Q/\rho+o\left(1/\sqrt{N^3}\right)=0. \tag{21}$$

Substituting expression (20) into Eq.(21), and omitting the high-order correction, we can have the density distribution[15,17]

$$\rho = \frac{\mu - V}{g} + \frac{\sqrt{\mu - V}}{2\pi} + o\left(1/\sqrt{N}\right), \qquad (22)$$

where μ is the single particle chemical potential, $\rho_{\rm cl} = (\mu - V)/g$ the classical density profile, and $\rho_Q = \sqrt{\mu - V}/2\pi$ the quantum fluctuation correction. In the quasi-one-dimensional condition,[19,20] $g_{\rm 1D} = 2a_{\rm s}\omega_\perp$.[21] Integrating ρ_Q yields a change ΔR in the Thomas–Fermi size of a quasi-1D Bose–Einstein condensate induced by quantum backreaction, $\Delta R = -2^{-5/2}\left(\omega_\perp/\omega_{\rm z}\right)a_{\rm s}$. Integrating $\rho_{\rm cl} = \dfrac{\mu - V}{g_{\rm 1D}}$ gives the classical radius as $R_{\rm cl} = \left(3a_{\rm s}N\omega_\perp/\omega_{\rm z}^2\right)^{1/3}$. Therefore, $\dfrac{\Delta R}{R_{\rm cl}} = -\dfrac{1}{4\sqrt{2}}\left(\dfrac{1}{3N}\right)^{1/3}\left(\dfrac{\omega_\perp a_{\rm s}}{\omega_{\rm z}a_{\rm z}}\right)^{2/3}$. For $N = 2000$, $\omega_\perp/\omega_{\rm z} = 1000$, and $a_{\rm s}/a_{\rm z} = 1/1000$, we achieve $|\Delta R/R_{\rm cl}| \simeq 1\%$. Here we can see clearly that quantum backreaction causes the radius to decrease in the quasi-1D condition but increase in the three-dimensional case.[18,22]

3. Conclusion

Using the full field operator expressed in terms of a particle-number-conserving mean-field ansatz, we theoretically study the quantum backreaction of quantum fluctuations in dilute Bose–Einstein condensates. Even though the high-order correction is much smaller when contrasting it with the first-order term, we retain it and obtain the corresponding quantum backreaction expression.

Quantum fluctuation is a rather complicated problem. It should contain abundant physical contents in relevant areas. We will further explore them to gain a great insight into its intrinsic characteristics.

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