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On particles tunneling from the Taub-NUT-AdS black hole*

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This paper discusses tunneling of scalar particles and Dirac particles from the Taub-NUT-AdS black hole by the Hamilton–Jacobi equation, initially used by Angheben *et al*, and the Dirac equation, recently proposed by Kerner and Mann. This is performed in the dragging coordinate frame so as to avoid the ergosphere dragging effect. A general form is obtained for the temperature of scalar and Dirac particles tunneling from the Taub-NUT-Ads black hole, which is commensurate with other methods as expected.

Keywords: Hawking radiation, scalar particles, Dirac particles, tunneling
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1. Introduction

Since Stephen Hawking first proved Hawking radiation, considering the creation of a black hole by calculating the Bogoliubov transformations between the initial and final states of incoming and outgoing radiation in the context of collapse geometry,[1] many theoretical physicists have given their attention to this topic. Till now, one can derive Hawking radiation from black holes in many ways[2−10] such as Euclidean quantum gravity, path integrals, the Damour Ruffini method and so on. However, most of them rely on quantum theory on a fixed background, and the obtained emission spectrum is purely thermal. In 2000 Parikh and Wilczek[11] proposed a semi-classical method of modeling Hawking radiation as a tunneling effect, which was based on the work of Kraus and Wilczek.[12] This method involved calculating the imaginary part of the action for the process of s-wave emission across the horizon, which in turn is related to the Boltzmann factor for emission at the Hawking temperature. Using the WKB approximation, the tunneling probability for the classically forbidden trajectory of the s-wave coming from inside to outside the horizon is given by $\Gamma \sim \exp(-2\,\mathrm{Im}\,W/\hbar)$, where W is the classical action of the trajectory to leading order in $\hbar$. Expanding the action in terms of the particle energy, the Hawking temperature is recovered at linear order. That is, in order to obtain the emission temperature, one should first find the imaginary part of the action. This method was later extended to stationary black holes and the cases of massive and massless particles.[13−21] However, for this method, one must do a Painlevé coordinate transformation to solve the null geodesic equation, which takes different forms for the massless particles and massive particles. Soon after, Angheben *et al*,[22,23] based on the work of Srinivasan and Padmanabhan,[24] proposed another refined approach to depict particle tunneling. The key trick of this method is also to find out the imaginary part of the action of emission. It however assumes that the action satisfies the relativistic Hamilton–Jacobi equation, and solves it by picking an appropriate ansatz for the action at the near-horizon to get its explicit value. This method was shown to be not only valid for the tunneling of massless particles but also for massive particles. Recently, applications of quantum tunneling methods to the fermion case was also presented[25] to describe Hawking radiation of fermions with spin 1/2 via tunneling from the Rindler space time and a general non-rotating black hole space time. The action of radiated Dirac particles here is solved by the general Dirac equation, which requires choosing the Gamma matrixes appropriately so as to get the expected Hawking temperature. This method until now has been generalized to a series of black holes.[26−29]

In this paper we attempt to discuss quantum tunneling of scalar and Dirac particles from the Taub-NUT-AdS black hole background by the mentioned Hamilton–Jacobi equation and Dirac equation respectively. The Taub-NUT metric is a generalization of

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the Schwarzschild metric and has played an important role in the conceptual development of general relativity and in the construction of brane solutions in string theory and M-theory.[30] The NUT charge plays the role of a magnetic mass, inducing a topology in the Euclidean section at infinity that is a Hopf fibration of a circle over a 2-sphere. Taub-NUT-AdS spaces have been of particular interest in recent years because of the role they play in furthering our understanding of the AdS-CFT correspondence.[31] Along these lines, the thermodynamics of various Taub-NUT-AdS solutions has been a subject of intense study in recent years. Their entropy is not proportional to the area of the event horizon and their free energy can sometimes be negative.[32] Solutions of Einstein equations with a negative cosmological constant Λ and a nonvanishing NUT charge have a boundary metric that has closed time like curves. The behavior of quantum field theory is significantly different in such spaces, and it is of interest to understand how ADS/CFT works in these sorts of cases.

This paper is structured as follows. In Section 2 we describe the background geometry of the Taub-NUT-AdS black hole and make the dragging coordinate transformation. Then in Section 3 we introduce the Hamilton–Jacobi equation to discuss tunneling of scalar particles and in Section 4, we adopt the Dirac equation to further discuss the case of Dirac particles. Finally we give our concluding remarks in Section 5.

2. The Taub-NUT-AdS black hole

The line element of the Taub-NUT-AdS solution with cosmological constant $\Lambda = -3/l^2$ is given by[33]

$$\mathrm{d}s^2 = -H\mathrm{d}t^2 + \frac{1}{F(r)}\mathrm{d}r^2 + G\left(\mathrm{d}\varphi - \frac{F}{G}4nf_k^2\left(\frac{\theta}{2}\right)\mathrm{d}t\right)^2 + (r^2+n^2)\mathrm{d}\theta^2, \quad (1)$$

where

$$F(r) = k\frac{r^2-n^2}{r^2+n^2} + \frac{-2Mr + \frac{1}{l^2}(r^4+6n^2r^2-3n^4)}{r^2+n^2}, \quad (2)$$

$$H(r,\theta) = \left(F + F^2\frac{16n^2f_k^4(\theta/2)}{G}\right), \quad (3)$$

$$G(r,\theta) = f_k^2(\theta/2)(r^2+n^2-f_k^2(\theta/2)(4n^2F + k(r^2+n^2))), \quad (4)$$

and k is a discrete parameter that takes the values 1, 0, –1, which respectively defines the values of function $f_k(\theta)$ as $\sin\theta$, θ, $\sinh\theta$. One of the interesting properties of Taub-NUT space is the existence of closed time like curves (CTCs). The presence of closed time like curves in Taub-NUT-AdS space can be seen by considering the curve generated by the Killing vector ∂_φ and by examining

$$g_{\varphi\varphi} = f_k^2(\theta/2)(r^2+n^2-f_k^2(\theta/2)(4n^2F + k(r^2+n^2))). \quad (5)$$

For $k = 1, 0, 1$ as $4n^2/l^2 > 1$, the quantity $g_{\varphi\varphi} < 0$, yielding a time like ∂_φ; the curve $r = r_0$, $t = t_0$, and $\theta = \theta_0$ becomes a CTC. However, as $4n^2/l^2 \leq 1$, there exists a special subclass of hyperbolic Taub-NUT solutions that do not contain CTCs.

According to the null super-surface equation

$$g^{\mu\nu}\frac{\partial f}{\partial x^\mu}\frac{\partial f}{\partial x^\nu} = 0, \quad (6)$$

we obtain the event horizon from $F(r_+) = 0$ and find that it does not coincide with the infinite red-shift surface, thus there is an ergosphere dragging between them. In order to eliminate it, making the dragging coordinate transformation

$$\Omega = \frac{\mathrm{d}\varphi}{\mathrm{d}t_{kk}} = -\frac{g_{03}}{g_{33}} = \frac{4nF(r)}{G}f_k^2(\theta/2), \quad (7)$$

then we can transform the line element Eq.(1) as

$$\mathrm{d}s^2 = H(r,\theta)\mathrm{d}t^2 + F(r)^{-1}\mathrm{d}r^2 + (r^2+n^2)\mathrm{d}\theta^2, \quad (8)$$

which represents a 3-dimensional hyper-surface in 4-dimensional Taub-NUT-AdS space time. Now the event horizon and the infinite red-shift surface in the dragging coordinate system are coincident and the semi-classical WKB approximate is valid.

3. Tunneling of scalar particles

In this section, we concentrate on discussing tunneling of scalar particles from the Taub-NUT-AdS black hole by making use of the Hamilton–Jacobi equation. We assume that the action of the outgoing particle is given by the classical action I that satisfies the relativistic Hamilton–Jacobi equation

$$g^{\mu\nu}\partial_\mu I\partial_\nu I + u^2 = 0, \quad (9)$$

where u and $g^{\mu\nu}$ are the mass of the emitted particles and the inverse metric tensor derived from the metric

Eq.(8) respectively. Based on Eq.(8), we can expand Eq.(9) as

$$-\frac{1}{H}\left(\partial_t I\right)^2+F\left(\partial_r I\right)^2 +\frac{1}{(r^2+n^2)\sin^2\theta}\left(\partial_\theta I\right)^2+u^2=0. \tag{10}$$

As the time-like Killing vector $(\partial/\partial t)^a$ and the space-like Killing vector $(\partial/\partial\varphi)^a$ are taken into account, we carry out the following separation of variables

$$I=-(\omega-J\Omega)t+W\left(r,\theta\right)+K, \tag{11}$$

where Ω, J are real constants which respectively represent the emitted particle's energy and angular momentum, K is a complex constant and $\Omega-J\Omega$ is the energy of radiation measured by observers in the dragging coordinate frame. Then we find that the radial action is

$$W\left(r,\theta\right)=\int\frac{\mathrm{d}r}{\sqrt{HF}}\sqrt{\left(\Omega-J\Omega\right)^2-H\left(\frac{1}{(r^2+n^2)\sin^2\theta}W_\theta^2+u^2\right)}. \tag{12}$$

The imaginary part of the action comes only from the pole at the horizon. It is important to introduce the proper spatial distance so as to make sure of the correctness of the result. The proper spatial distance between any two points at the fixed time t is defined as

$$\mathrm{d}\sigma^2=\frac{1}{F}\mathrm{d}r^2+(r^2+n^2)\sin^2\theta\mathrm{d}\theta^2. \tag{13}$$

Note that the particle emission is treated as an ellipsoid shell to be run out and the motion is only in the radial direction, therefore we have

$$\sigma=\int\frac{1}{\sqrt{F}}\mathrm{d}r. \tag{14}$$

Near the horizon, $H(r,\theta)$ approximates to $F(r)$, therefore, both of them here can be expanded as

$$F(r)=(r-r_+)F'(r)+\ldots, \tag{15}$$

where $F'(r)=\partial_r F(r)$. Now we obtain

$$\sigma=2\frac{\sqrt{r-r_+}}{\sqrt{F'(r_+)}}. \tag{16}$$

In this case, Eq.(10) can be further reduced as

$$W\left(\sigma\right)=\frac{2}{F'}\int\frac{\mathrm{d}\sigma}{\sigma}\sqrt{\left(\Omega-J\Omega\right)^2-\frac{\sigma^2}{4}F'^2\left(\frac{1}{(r^2+n^2)\sin^2\theta}W_\theta^2+u^2\right)}. \tag{17}$$

To proceed further we need to estimate the last integral. Note that there is a pole at the horizon where $\sigma=0$. Deforming the integration contour from the real σ-axis to the lower complex σ-plane, the imaginary part of the radiated particle's action can be written as

$$\mathrm{Im}\,I=\frac{2\pi\left(\Omega-J\Omega\right)}{F'}. \tag{18}$$

According to the WKB approximation, the corresponding Hawking temperature takes the form

$$T_{\mathrm{BH}}=\frac{F'\left(r_h\right)}{4\pi}=\frac{4\pi l^2 r_+}{3(r_+^2+n^2)-l^2}, \tag{19}$$

which is the same form found by using the Wick rotation method.[34]

4. Tunneling of Dirac particles

As mentioned above, all tunneling approaches use the WKB approximation relating the tunneling probability with the action's imaginary part of emission from the classically forbidden trajectory. The major problem of the tunneling paradigm hence is how to find the classical action of radiation. As for the Dirac

particles, we restore the following Dirac equation

$$\gamma^{\mu}D_{\mu}\psi-\frac{\mathrm{i}\,m}{\hbar}\psi=0, \tag{20}$$

where the Greek indices $\mu,\nu=0,1,2,3$, m is the mass of the fermions and D_{μ} the spinor covariant derivative defined by

$$\begin{aligned} D_{\mu}&=\partial_{\mu}+\Omega_{\mu},\\ \Omega_{\mu}&=\frac{1}{2}\,\mathrm{i}\,\Gamma_{\mu}^{\alpha\beta}\Sigma_{\alpha\beta},\\ \Sigma_{\alpha\beta}&=\frac{1}{4}\,\mathrm{i}[\gamma^{\alpha},\gamma^{\beta}]. \end{aligned} \tag{21}$$

To solve the Dirac equation, we should first find the γ^{μ} matrixes. Based on the relation $\{\gamma^{\mu},\gamma^{\nu}\}=2g^{\mu\nu}I$, we pick for them as

$$\begin{aligned} \gamma^{t}&=\frac{1}{\sqrt{H}}\begin{pmatrix}\mathrm{i}&0\\0&-\mathrm{i}\end{pmatrix},\\ \gamma^{r}&=\sqrt{F}\begin{pmatrix}0&\sigma^{3}\\\sigma^{3}&0\end{pmatrix},\\ \gamma^{\theta}&=\frac{1}{\sqrt{P}}\begin{pmatrix}0&\sigma^{1}\\\sigma^{1}&0\end{pmatrix}, \end{aligned} \tag{22}$$

in which $\sigma^{i}(i=1,2,3)$ is the Pauli Sigma matrixes, $P=(r^2+n^2)$. Dirac particles taking the spin $1/2$ take the form as spin up and spin down. In the Pauli matrix σ^3 representation, they can be explicitly expressed by the eigenvectors $\xi_{\uparrow/\downarrow}$, and the corresponding eigenvalues are $1/-1$. In this paper, we are only interested in the tunneling of the spin up particles since the calculation of spin down is fully and physically similar to this case, because the number of the spin up and spin down particles is statistically the same; as a spin up particle tunnels out, its partner also should do so at the same Hawking temperature. For the spin up particles, we adopt the following ansatz of wavefunction

$$\begin{aligned} \psi_{\uparrow}(t,r,\theta,\varphi)&=\begin{bmatrix}A(t,r,\theta,\varphi)\xi_{\uparrow}\\B(t,r,\theta,\varphi)\xi_{\uparrow}\end{bmatrix}\exp\left[\frac{\mathrm{i}}{\hbar}I_{\uparrow}(t,r,\theta,\varphi)\right],\\ &=\begin{bmatrix}A(t,r,\theta,\varphi)\\0\\B(t,r,\theta,\varphi)\\0\end{bmatrix}\exp\left[\frac{\mathrm{i}}{\hbar}I_{\uparrow}(t,r,\theta,\varphi)\right], \end{aligned} \tag{23}$$

where $I_{\uparrow}$ is the action of emitted spin up particles. Substituting the above ansatz into the Dirac equation (11), after dividing by the exponential term in the wave function and multiplying by $\hbar$, we find the prominent terms

$$\frac{\mathrm{i}\,A}{\sqrt{H\,(r,\theta)}}\partial_{t}I_{\uparrow}+B\sqrt{F\,(r)}\partial_{r}I_{\uparrow}-mA=0, \tag{24}$$

$$B\frac{1}{\sqrt{P}}\partial_{\theta}I_{\uparrow}=0, \tag{25}$$

$$-\frac{\mathrm{i}\,B}{\sqrt{H\,(r,\theta)}}\partial_{t}I_{\uparrow}+A\sqrt{F\,(r)}\partial_{r}I_{\uparrow}-mB=0, \tag{26}$$

$$B\frac{1}{\sqrt{P}}\partial_{\theta}I_{\uparrow}=0. \tag{27}$$

Employing the variable separation as Eq.(11) (where we consider only the positive frequency contributions without loss of generality), we can rewrite the above equations as

$$\frac{-\,\mathrm{i}\,A}{\sqrt{H\,(r,\theta)}}(\Omega-J\Omega)+B\sqrt{F\,(r)}W\,(r,\theta)_{,r}-mA=0, \tag{28}$$

$$W\,(r,\theta)_{,\theta}=0, \tag{29}$$

$$\frac{\mathrm{i}\,B}{\sqrt{H\,(r,\theta)}}(\Omega-J\Omega)+A\sqrt{F\,(r)}W\,(r,\theta)_{,r}-mB=0, \tag{30}$$

$$W\,(r,\theta)_{,\theta}=0, \tag{31}$$

here $W\,(r,\theta)_{,r}=\partial_{r}W\,(r,\theta)$, $W\,(r,\theta)_{,\theta}=\partial_{\theta}W\,(r,\theta)$. Now we carry out an explicit analysis on the above equations. For Eqs.(29) and (31), we find that $W\,(r,\theta)$ is independent of θ, which means that the θ part has no contribution to the imaginary part of the action; this will be shown in our next calculation. For the radial equations (28) and (30), we should consider the mass effect. If $m=0$, they decouple and describe the motion of massless fermions; there exist two possible solutions as follows:

$$A=\mathrm{i}\,B, W\,(r,\theta)_{,r}=-\frac{\Omega-J\Omega}{\sqrt{F\,(r)\,H\,(r,\theta)}}, \tag{32}$$

$$A=-\,\mathrm{i}\,B, W\,(r,\theta)_{,r}=\frac{\Omega-J\Omega}{\sqrt{F\,(r)\,H\,(r,\theta)}}. \tag{33}$$

When $m\neq 0$, they describe the motion of massive fermion coupling. After eliminating $W\,(r,\theta)$ and taking the near-horizon approximation, we find the same results as Eq.(32) and Eq.(33). Therefore the action of massless and massive Dirac particles is

$$W_{\pm}=\pm\int\frac{\Omega-J\Omega}{\sqrt{F\,(r)\,H\,(r,\theta)}}\mathrm{d}r, \tag{34}$$

where $+/-$ sign corresponds to the outgoing/ingoing modes. To obtain the explicit value, we first expand $F\,(r)\,,H\,(r,\theta)$ at the near horizon region as Eq.(15), then, Eq.(34) can be integrated as

$$W_{\pm}=\pm\pi\,\mathrm{i}\,\frac{\Omega-\Omega_{0}}{F'\,(r)}, \tag{35}$$

where

$$\Omega_0 = \frac{4nF'(r_+)}{(r_+^2 + n^2 - f_k^2(\theta/2)k(r_+^2 + n^2))} \tag{36}$$

is the chemical potential. The total tunneling rate hence can be expressed as

$$\Gamma \sim \frac{P(\text{out})}{P(\text{in})} = \frac{\exp(-2\,\text{Im}\,W_+)}{\exp(+2\,\text{Im}\,W_-)} = \exp\left(-4\pi\frac{\Omega - \Omega_0}{F'(r)}\right), \tag{37}$$

giving the expected Hawking temperature

$$T_{\text{BH}} = \frac{F'(r_{\text{h}})}{4\pi} = \frac{4\pi l^2 r_+}{3(r_+^2 + n^2) - l^2} \tag{38}$$

of Dirac particles tunneling from the Taub-NUT-AdS black hole, which is the same as that of scalar particles.

5. Concluding remarks

We have discussed tunneling of scalar and Dirac particles from the Taub-NUT-AdS black hole by the Hamilton–Jacobi equation and Dirac equation. We also obtained the corresponding emission temperature despite existing CTCs by requiring that the action of emission satisfies the corresponding equation of motion. Note that however the emission spectrum here is purely thermal. In fact as the particle, here, a scalar and Dirac particle, with energy Ω tunnels out the horizon, the black hole mass would reduce due to universal energy conservation while the total ADM mass is fixed. The action thus should be modified by integrating energy Ω from M to $M - \Omega$, which results in the corrected emission temperature.

Note that for Dirac particles with spin 1/2, when measuring spin along the radial direction, there would be two cases. Thus we also should consider the case of spin down. When the wave function of spin down particles is inserted into the Dirac equation (20), we can obtain the same emission Hawking temperature, which in turn confirms the statistical explanation mentioned in Section 4.

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