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Fidelity of an electron in one-dimensional determined potentials*

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We numerically study the fidelity of an electron in the one-dimensional Harper model and in the one-dimensional slowly varying potential model. Our results show that many properties of the two models can be well reflected by the fidelity: (i) the mobility edge and metal–insulator transition can be characterized by the static fidelity; (ii) the extended state and localized state can be identified by the dynamic fidelity. Therefore, it may broaden the applied areas of the fidelity.

Keywords: fidelity, extended and localized state, metal–insulator transition, mobility edge
PACC: 0367, 7155J, 7110F, 7130

1. Introduction

Fidelity[1] is an important concept and has been studied in various areas of physics, for example, statistical mechanics, quantum chaos[2,3] and quantum information.[4] It is defined as the overlap of an initial state with the state obtained after forward unperturbed evolution followed by backward perturbed evolution. Fidelity has attracted a lot of attention recently.[5−14] In general, fidelity is considered in two ways, i.e., the fidelity of independence of time and dynamic fidelity. The approach to identify quantum phase transitions (QPTs) based on the ground-state fidelity has been proposed and applied to various systems.[5−10] In addition, the fidelity of dependence of time, $F(t)$, of systems has been studied by using the Wigner Lorentzian random matrix (WLRM) model[14] and it has been shown that the fidelity may be used to characterize metal–insulator transition (MIT).

On the other hand, disorder induced MIT has been investigated extensively since Anderson published his famous paper about disorder induced localization in 1958.[15] It is well known that all eigenstates are localized and there is no mobility edge separating localized and extended states for the one-dimensional (1D) Anderson model.[16] However, it is found that specific extended states and/or mobility edges exist in several 1D determined and random models with short-range and long-range correlation.[17−22] Well-studied examples of these models are the ordinary Harper model and the slowly varying potential model. The two models have been widely used to study the MIT by many different methods. Because the fidelity approach has been examined in many systems recently, it is interesting to perform a detailed study of the fidelity in the above two models.

In this paper, we numerically investigate the fidelity for an electron moving in the Harper model and the slowly varying potential model. By using a measure of the static and dynamic fidelity, we analyse many properties about the two models. This provides us with a new quantity, i.e., fidelity, to reflect extended state, localized state, mobility edge and MIT.

The paper is organized as follows. In Section 2, the formalism and the models are described. In Section 3, the numerical results for an electron moving in the Harper model and the slowly varying potential model are presented. Section 4 is devoted to the conclusion.

2. Formalism and models

2.1. Fidelity

We consider an electron moving in a one-dimensional lattice with N sites. Generally, the tight-binding Hamiltonian for an electron can be written as

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follows:[23]

$$H = t\sum_{n=1}^{N}(C_n^+ C_{n+1} + C_{n+1}^+ C_n) + \sum_{n=1}^{N}\varepsilon_n C_n^+ C_n, \quad (1)$$

where t is a nearest-neighbour hopping integral, $C_n^+(C_n)$ is the creation (annihilation) operator of the nth site, ε_n is the on-site potential, while N is the size of the system. In our numerical studies, we take $t = 1$ without loss of generality.

Let $|n\rangle$ denote $|0, \ldots, 1_n, \ldots, 0\rangle$. The general eigenstate of an electron with eigenenergy E_β is

$$|\Psi_\beta\rangle = \sum_n \psi_n^\beta |n\rangle = \sum_n \psi_n^\beta C_n^+ |0\rangle, \quad (2)$$

where ψ_n^β is the amplitude of the βth eigenstate at the nth site[23] and $\sum_n |\psi_n^\beta|^2 = 1$; we can obtain the eigenequation

$$t(\psi_{n+1} + \psi_{n-1}) + \varepsilon_n \psi_n = E\psi_n, \quad (3)$$

where E is the eigenenergy.

At first, we consider the fidelity of independence of time. For an unperturbed eigenstate $|\Psi_\beta\rangle$ for a Hamiltonian H and the perturbed eigenstate $|\Psi'_\beta\rangle$ for a Hamiltonian $H' = H + \Delta H$ (ΔH is a small perturbation), correspondingly, we can define the fidelity of a system at the individual eigenstate, i.e.,

$$F^\beta = |\langle\Psi_\beta|\Psi'_\beta\rangle|. \quad (4)$$

From the above definitions, it can be shown that F^β represents the overlap of $|\Psi_\beta\rangle$ and $|\Psi'_\beta\rangle$, and all the values of F^β are between zero and 1. As a further gross measure we also average over all the eigenstates, i.e., the spectrum averaged fidelity

$$\langle F\rangle = \frac{1}{M}\sum_\beta F^\beta, \quad (5)$$

where M is the number of all eigenstates.

Secondly, we investigate the dynamics of the fidelity. In general, the fidelity of dependence of time, $F(t)$, is described[14] as

$$F(t) = |\langle\Psi_0|\mathrm{e}^{\mathrm{i}H't}\mathrm{e}^{-\mathrm{i}Ht}|\Psi_0\rangle|^2, \quad (6)$$

where the Hamiltonian H and $H' = H + \Delta H$ represent the reference Hamiltonian and its perturbed variance, respectively, while $|\Psi_0\rangle$ is an initial state.

From Eq.(3), the eigenvectors $|\Psi_k\rangle$ and $|\Psi'_k\rangle$ can be obtained, where $|\Psi_k\rangle$ and $|\Psi'_k\rangle$ satisfy the normalization condition $\langle\Psi_m|\Psi_n\rangle = \delta_{mn}$ and $\langle\Psi'_m|\Psi'_n\rangle = \delta_{mn}$, respectively. One can show that,

$$\mathrm{e}^{-\mathrm{i}Ht}|\Psi_k\rangle = \mathrm{e}^{-\mathrm{i}E_k t}|\Psi_k\rangle, \quad (7)$$

$$\mathrm{e}^{-\mathrm{i}H't}|\Psi'_k\rangle = \mathrm{e}^{-\mathrm{i}E'_k t}|\Psi'_k\rangle. \quad (8)$$

For an initial state $|\Psi_0\rangle$, we can write

$$|\Psi_0\rangle = \sum_{k=1}^{N} C_k|\Psi_k\rangle = \sum_{k'=1}^{N} C'_{k'}|\Psi'_{k'}\rangle. \quad (9)$$

Therefore, according to Eqs.(6)–(9), we can easily write

$$F(t) = |\sum_{k,k'=1}^{N} C'^*_{k'} C_k \mathrm{e}^{\mathrm{i}(E'_{k'} - E_k)t}\langle\Psi'_{k'}|\Psi_k\rangle|^2, \quad (10)$$

where $C_k = \langle\Psi_k|\Psi_0\rangle$ and $C'_{k'} = \langle\Psi'_{k'}|\Psi_0\rangle$.

2.2. The models

For an electron moving in the Harper model and in the slowly varying potential model, the on-site potential can be written as[18]

$$\varepsilon_n = \lambda\cos(2\pi\sigma n^\nu) \quad (11)$$

and the perturbed on-site potential is defined as

$$\varepsilon'_n = (\lambda + \Delta\lambda)\cos(2\pi\sigma n^\nu), \quad (12)$$

where the parameter λ and $\Delta\lambda$ characterize the strength of the determined potential and the perturbation; σ and $0 \le \nu \le 1$ are positive numbers which completely define the tight-binding problem.

For $\nu = 1$, it is a well known Harper model. It has been found that for $\lambda > 2$ all the eigenstates are localized.[24,25] For $\lambda < 2$ all the eigenstates are extended. For $\lambda = 2$ the situation is critical. MIT can occur at $\lambda = 2$.

For $0 < \nu < 1$, we employ the slowly varying potential model.[18] It is well known for this model that there are two mobility edges at $E_c = \pm(2.0 - \lambda)$ provided that $\lambda < 2.0$. Extended states are found in the middle of the band ($|E| < 2.0 - \lambda$), and localized states at the band edge ($2.0 - \lambda < |E| < 2.0 + \lambda$). For $\lambda > 2.0$, all states are found to be localized.

3. Numerical results

3.1. For the static fidelity

Firstly, we study the fidelity of eigenstates in the Harper model. The perturbation strength $\Delta\lambda$ is chosen as 0.0001. As a typical case, we take $\sigma = (\sqrt{5} - 1)/2$. In fact, as is customary in the context of a quasiperiodic system, the value of σ may be

approximated by a convergent series of truncated fractions: $F_{n-1}/F_n (n = 1, 2, \ldots)$, where F_n is a Fibonacci sequence. The size of the system is chosen as $N = F_n$. In this way, we can obtain the periodic approximation for the quasiperiodic potential.

In Fig.1, the fidelity F^β of the individual eigenstates as a function of eigenenergies E is shown for N=2584 at (a) $\lambda = 1$, (b) $\lambda = 2$, and (c) $\lambda = 3$, respectively. When $\lambda = 1$ and $\lambda = 3$, all F^β are close to 1.0. The situation is the same with all values for $\lambda < 2$ and $\lambda > 2$. However, when $\lambda = 2$, the distribution of F^β shows a special structure with a broad interval, which implies that all the eigenstates are critical in the model.

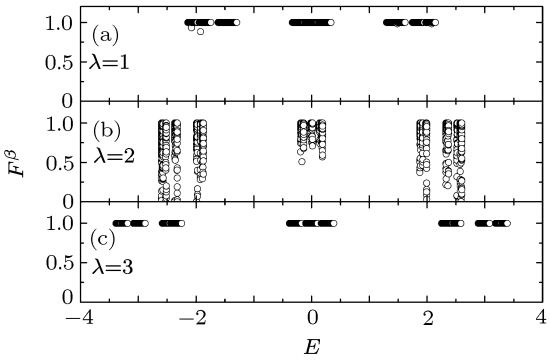


Fig.1. The fidelity F^β versus eigenenergies E for $N = 2584$ at (a) $\lambda = 1$, (b) $\lambda = 2$, and (c) $\lambda = 3$.

In Fig.2, we give the behaviours of the spectrum averaged fidelity $\langle F \rangle$ varying with λ for different size N. There is an abrupt decrease of $\langle F \rangle$ near the critical value $\lambda_c = 2$. The transition becomes sharper when N increases. In addition, we investigate the influence of different $\Delta\lambda$ and find from Fig.3 that the value of $\Delta\lambda$ mainly affects the degree of the abrupt decrease of $\langle F \rangle$ near the critical value $\lambda_c = 2$. Therefore,

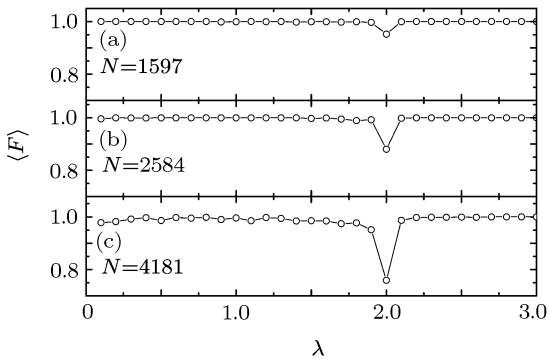


Fig.2. The spectrum averaged fidelity $\langle F \rangle$ varying with λ for three different N.

the spectrum averaged fidelity can reflect MIT in the model.

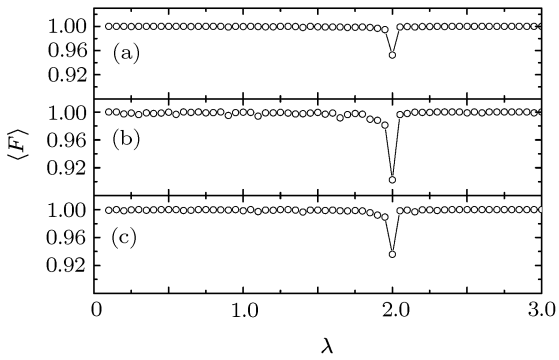


Fig.3. The spectrum averaged fidelity $\langle F \rangle$ varying with λ for $N = 1597$ at (a) $\Delta\lambda = 0.0001$, (b) $\Delta\lambda = 0.005$, and (c) $\Delta\lambda = -0.001$.

Secondly, we study the fidelity of eigenstates in the slowly varying potential model. Without loss of generality, we set $\pi\sigma = 0.1$, $\nu = 0.7$ and $\Delta\lambda = 0.01$. The fidelity F^β of the individual eigenstates versus eigenenergies E is plotted in Figs.4(a)–4(d) for $\lambda = 0.5, 1.5, 2.0$ and 2.5, respectively. The model has mobility edges separating extended from localized eigenstates when $\lambda \leq 2$. It is clearly shown that there are sharp transitions in F^β at the mobility edges $E_c = \pm(2.0 - \lambda)$ in Figs.4(a), 4(b) and 4(c). Here we take $N = 2000$, and the results are similar to those for other system sizes. The above results are consistent with those in Ref.[18]. Therefore, the mobility edges can be distinguished by the quantity F^β.

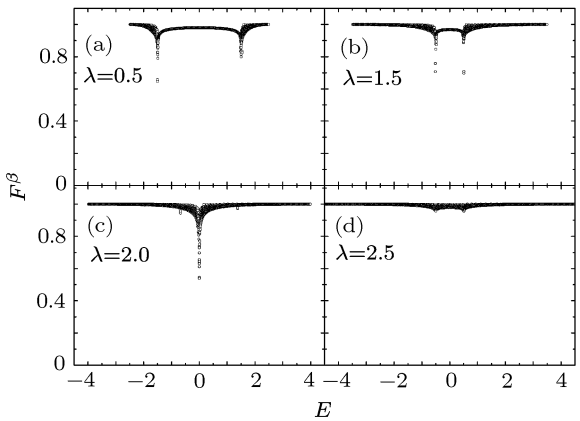


Fig.4. The fidelity F^β versus eigenenergies E for $N = 2000$ at (a) $\lambda = 0.5$, (b) $\lambda = 1.5$, (c) $\lambda = 2.0$, and (d) $\lambda = 2.5$.

3.2. For dynamic fidelity

For an arbitrary initial state $|\Psi_0\rangle$, according to Eqs.(1), (3), (10), (11) and (12), we can obtain dynamical behaviours of fidelity for the two models. Figures

5 and 6 display dynamic behaviours of the fidelity for different λ in the one-dimensional Harper model and in the one-dimensional slowly varying potential model, respectively. From Fig.5(a) for $\lambda = 1.0$, we observe that the fidelity firstly decays with time, and finally oscillates irregularly around a finite value $F(t) \neq 0$. However, when $\lambda = 3.0$ in Fig.5(c), the fidelity decays to close to zero as time increases finally. The reason for the kind of difference is that for $\lambda > 2$ all the eigenstates are localized, and all the eigenstates are extended when $\lambda < 2$ in the Harper model. The reason is similar to the decay of the fidelity in the one-dimensional slowly varying potential model in Fig.6. The localized states are always involved in the model. We can conclude that if only the localized states exist, the fidelity, $F(t)$, decays to close to zero when $t \to \infty$. Therefore, the quantity $F(t)$ can reflect the existence of the localized states.

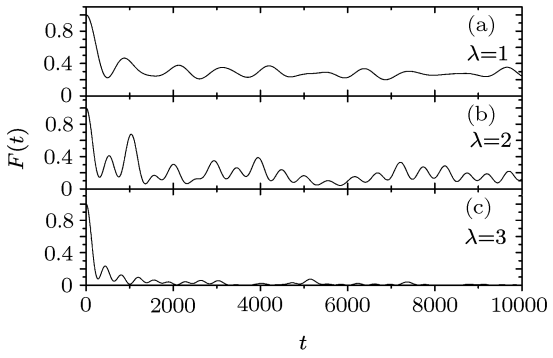


Fig.5. Fidelity varying with time for different λ in the Harper model ($N = 1597$, $\Delta\lambda = 0.01$).

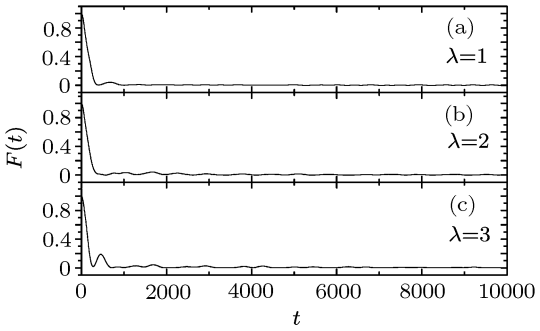


Fig.6. Fidelity varying with time for different λ in the slowly varying potential model ($N = 3000$, $\Delta\lambda = 0.01$).

4. Conclusion

In summary, we study the fidelity in the one-dimensional Harper model and in the one-dimensional slowly varying potential model, respectively. For an electron moving in the Harper model, the fidelity F^{β} of the individual eigenstates are specific when $\lambda_{\rm c} = 2.0$. For an electron moving in the slowly varying potential model, there are sharper transitions in F^{β} at the mobility edges. From the curve of the spectrum averaged fidelity $\langle F \rangle$ versus the potential parameter λ for the Harper models, we see clearly that there is a sharper transition at the critical potential parameter value $\lambda_{\rm c} = 2.0$. These results are consistent with those obtained by other methods.[23,26−28] In addition, we also find the reason for the difference in the decay of the dynamic fidelity, $F(t)$, for different λ in the two models. Therefore, fidelity is a new quantity to reflect extended state, localized state, mobility edge and MIT.

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