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Wigner function and density operator of the photon-subtracted squeezed thermal state*

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By using the technique of integration within an ordered product of operators, the normal ordered density operator of the photon-subtracted squeezed thermal state (PSSTS) is derived. Then the corresponding Wigner function is presented by using the coherent state representation of the Wigner operator. The nonclassical properties of the PSSTS are discussed based on the negativity of the Wigner function.

Keywords: Wigner function, photon-subtraction, squeezed thermal state, integration with in an ordered product technique

PACC: 0365, 0367, 4250

1. Introduction

The Wigner function[1] was first introduced by Wigner in 1932 to calculate quantum corrections to a classical distribution function of a quantum-mechanical system. It has now become a very popular tool to study the nonclassical properties of quantum states. Some very good reading material about the Wigner function in quantum optics theory is Schleich's book 'Quantum Optics in Phase Space'.[2] For a density operator ρ of a single-mode system, the Wigner function is defined as[1−3]

$$\begin{aligned} & W(p,q) \\ &= \mathrm{tr}\left[\rho \Delta(p,q)\right] \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \mathrm{d}q \left\langle q+\frac{v}{2}\right| \rho \left|q-\frac{v}{2}\right\rangle \mathrm{e}^{-\mathrm{i}pv}, \end{aligned} \quad (1)$$

where $\Delta(p,q)$ is the single-mode Wigner operator, $|q\rangle$ is the eigenvector of the coordinate operator $Q|q\rangle = q|q\rangle$,

$$|q\rangle = \pi^{-1/4}\exp\left[-\frac{1}{2}q^2+\sqrt{2}qa^\dagger-\frac{1}{2}a^{\dagger 2}\right]|0\rangle, \quad (2)$$

$a^\dagger, a$ are Bose creation and annihilation operators, respectively, satisfying $[a,a^\dagger]=1$; they are related to Q and P by $Q=(a+a^\dagger)/\sqrt{2}$ and $P=(a-a^\dagger)/(\mathrm{i}\sqrt{2})$. Using the technique of integration within an ordered product (IWOP) of operators[4,5] and the vacuum projector $|0\rangle\langle 0| =: \exp\left(-a^\dagger a\right):$ (: : denotes normal ordering), we have obtained the explicit normally ordered form of $\Delta(p,q)$

$$\Delta(p,q) = \Delta(\alpha,\alpha^*) = \frac{1}{\pi}: \mathrm{e}^{-(q-Q)^2-(p-P)^2}:, \quad (3)$$

where $\alpha = (q+\mathrm{i}p)/\sqrt{2}$. Respectively performing integrations over $\mathrm{d}q$ and $\mathrm{d}p$ leads to

$$\begin{aligned} \int_{-\infty}^{\infty}\mathrm{d}q\Delta(p,q) &= \frac{1}{\sqrt{\pi}}: \mathrm{e}^{-(p-P)^2}: = |p\rangle\langle p|, \\ \int_{-\infty}^{\infty}\mathrm{d}p\Delta(p,q) &= \frac{1}{\sqrt{\pi}}: \mathrm{e}^{-(q-Q)^2}: = |q\rangle\langle q|. \end{aligned} \quad (4)$$

So $W(p,q)$ in phase space has its physical meaning as its marginal distribution, which gives the probability of measurement in coordinate space and momentum space, respectively.

It is well known that Wigner functions are quasiprobability distributions because they may be negative in phase space. Nevertheless, the partial negativity of the Wigner function is indeed a good indication of the highly nonclassical character of the state.[6] Therefore it is worth obtaining the Wigner function for any states and to measure their negative values.

On the other hand, it is possible to generate and manipulate various nonclassical optical fields in a real laboratory. Photon subtraction is just a useful way to conditionally manipulate the nonclassical state of an optical field. Recently, the single-photon subtracted-squeezed vacuum state has been experimentally prepared with a pulsed and continuous wave squeezed vacuum.[7−9] In addition, photon

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subtraction is also applied to improve the entanglement between Gaussian states,[10] loophole-free tests of Bell's inequality,[11] and quantum computing.[12] The nonclassicality of a single-photon subtracted from squeezed vacuum state has also been theoretically investigated by demonstrating its Wigner function.[13] For other examples of calculating the Wigner functions, refer to Refs.[14]–[16].

In this paper, we shall introduce a kind of new quantum state—the photon-subtracted squeezed thermal state, then derive its normal ordered form of density operator and the Wigner function. Our work is arranged as follows. In Section 2, we briefly review the displaced-squeezed thermal state. By using the IWOP technique, the anti-normal ordering form of the density operator of the photon-subtracted squeezed thermal state (PSSTS) is derived in Section 3 after introducing the PSSTS. Then, in Section 4, the normal ordered density operator of the PSSTS is obtained by using the P-representation theory in the coherent state basis. Section 5 is devoted to calculating the Wigner function and the nonclassical properties of the PSSTS are discussed based on the negativity of the Wigner function.

2. Displaced-squeezed thermal state

For a displaced-squeezed thermal field, the density operator is

$$\rho_{\rm s} = D\left(\alpha'\right) S\left(r\right) \rho_{\rm c} S^{\dagger}\left(r\right) D^{\dagger}\left(\alpha'\right), \tag{5}$$

where $D\left(\alpha'\right) = \exp(\alpha' a^{\dagger} - \alpha'^{*} a)$ and $S\left(r\right) = \exp[{\rm i}r(QP + PQ)/2]$ are the displacement operator and squeezing operator,[17] respectively, $\rho_{\rm c} = (1 - {\rm e}^{\lambda})\,{\rm e}^{\lambda a^{\dagger} a}$, $({\rm tr}\rho_{\rm c} = 1)$, is qualified to be a density operator of thermal (chaotic) field when $\lambda = -\omega\hbar/kT$, k is the Boltzmann constant. In Ref.[18] by using the Weyl ordering invariance under similarity transformations and the IWOP technique, we have converted $\rho_{\rm s}$ to the normally ordered Gaussian form

$$\rho_{\rm s} = \frac{1}{\tau_1 \tau_2} {:}\ \exp\left[-\frac{\left(q' - Q\right)^2}{2\tau_1^2} - \frac{\left(p' - P\right)^2}{2\tau_2^2}\right] {:}\,, \tag{6}$$

where

$$\begin{aligned} 2\tau_1^2 &= (2n+1)\ {\rm e}^{2r} + 1, \\ 2\tau_2^2 &= (2n+1)\ {\rm e}^{-2r} + 1, \end{aligned} \tag{7}$$

n is the average photon number for $\rho_{\rm c}$,[19]

$$n = (\,{\rm e}^{-\lambda} - 1)^{-1} = (\,{\rm e}^{\omega\hbar/kT} - 1)^{-1}. \tag{8}$$

The form in Eq.(6) is similar to the bivariate normal distribution in statistics.

3. Anti-normal ordering form of the density operator of PSSTS

In this section, we first introduce the photon-subtracted squeezed thermal state. Theoretically, it can be obtained by repeatedly operating the photon annihilation operator a on a displacement squeezed thermal state, so its density operator is defined as

$$\rho = C^{-1} a^m \rho_{\rm s} a^{\dagger m}, \tag{9}$$

where $C = {\rm tr}(a^m \rho_{\rm s} a^{\dagger m})$ is a normalization constant. For simplicity, in this paper, we examine Eq.(6) in the special case $\alpha' = (g' + {\rm i}p')/\sqrt{2} = 0$, i.e.,

$$\begin{aligned} \rho_{\rm s} &= 2\sqrt{fg}{:}\ \exp\left[-fQ^2 - gP^2\right] {:}\,, \\ f &= \frac{1}{2\tau_1^2} < \frac{1}{2}, \quad g = \frac{1}{2\tau_2^2} < 1. \end{aligned} \tag{10}$$

To derive the normal ordering form of ρ in Eq.(9) we first derive the anti-normal ordering form of $\rho_{\rm s}$ by using the formula[20]

$$\begin{aligned} \rho_{\rm s} = \int \frac{{\rm d}^2\beta}{\pi} {:}\, \left\langle -\beta\right| \rho_{\rm s} \left|\beta\right\rangle \\ \times \exp\left(\left|\beta\right|^2 + \beta^{*} a - \beta a^{\dagger} + a^{\dagger} a\right) {:}\,, \end{aligned} \tag{11}$$

where $:\ :$ stands for the anti-normal ordering, $|\beta\rangle$ is also a coherent state. Substituting Eq.(10) into Eq.(11) and using the integration formula[3]

$$\begin{aligned} &\int \frac{{\rm d}^2 z}{\pi} \exp\left(\zeta \left|z\right|^2 + \xi z + \eta z^{*} + f z^2 + g z^{*2}\right) \\ &= \frac{1}{\sqrt{\zeta^2 - 4fg}} \exp\left[\frac{-\zeta\xi\eta + \xi^2 g + \eta^2 f}{\zeta^2 - 4fg}\right], \end{aligned} \tag{12}$$

whose convergent condition is

$$\begin{aligned} &{\rm Re}\left(\zeta + f + g\right) < 0,\ {\rm Re}\left(\frac{\zeta^2 - 4fg}{\zeta + f + g}\right) < 0, \\ &{\rm Re}\left(\zeta - f - g\right) < 0,\ {\rm Re}\left(\frac{\zeta^2 - 4fg}{\zeta - f - g}\right) < 0. \end{aligned} \tag{13, 14}$$

We have the anti-normal ordering form of $\rho_{\rm s}$,

$$\rho_{\rm s} = 2\sqrt{fg} \int \frac{{\rm d}^2\beta}{\pi} {:} \exp\left[-\left|\beta\right|^2 - f\left(\frac{\beta - \beta^{*}}{\sqrt{2}}\right)^2\right.$$

$$
\begin{aligned}
&- g\left(\frac{\beta+\beta^*}{\sqrt{2}\mathrm{i}}\right)^2+\beta^* a-\beta a^\dagger+a^\dagger a\Big]\ \vdots \\
&=2\sqrt{fg}\int\frac{\mathrm{d}^2\beta}{\pi}\ \vdots\ \exp\Big[-(1-f-g)\,|\beta|^2-a^\dagger\beta \\
&\quad+a\beta^*-\frac{1}{2}(f-g)\left(\beta^2+\beta^{*2}\right)+a^\dagger a\Big]\ \vdots \\
&=\frac{2\sqrt{fg}}{\sqrt{D}}\ \vdots\ \exp\left[-a^\dagger a\,(1-f-g)/D\right] \\
&\quad\times\exp\left[-\frac{(f-g)}{2D}\left(a^{\dagger2}+a^2\right)+a^\dagger a\right]\ \vdots\,, \qquad (15)
\end{aligned}
$$

in which

$$D=(2f-1)(2g-1)\,. \qquad (16)$$

4. Normal ordering form of the density operator of PSSTS

Now, we derive the normal ordering form of Eq.(9). Since we have obtained the anti-normal ordering form of ρ_{s}, the anti-normal ordering form of ρ is

$$
\begin{aligned}
\rho=&\frac{2\sqrt{fg}}{C\sqrt{D}}\ \vdots\ a^m\exp\left[-a^\dagger a\,(1-f-g)/D\right] \\
&\times\exp\left[-\frac{(f-g)}{2D}\left(a^{\dagger2}+a^2\right)+a^\dagger a\right]a^{\dagger m}\ \vdots\,. \qquad (17)
\end{aligned}
$$

Then using the P-representation theory in the coherent state basis,[17] and the vacuum projector $|0\rangle\langle 0| =: \exp(-a^\dagger a):$, we have

$$
\begin{aligned}
\rho=&\int\frac{\mathrm{d}^2 z}{\pi}\rho\,|z\rangle\langle z| \\
=&\,2C^{-1}\frac{\sqrt{fg}}{\sqrt{D}}\int\frac{\mathrm{d}^2 z}{\pi}\,|z|^{2m}\,|z\rangle\langle z| \\
&\times\exp\left[\frac{g-f}{2D}\left(z^{*2}+z^2\right)-\frac{(f+g-4fg)}{D}\,|z|^2\right] \\
=&\,2C^{-1}(-1)^m\frac{\sqrt{fg}}{\sqrt{D}}\frac{\partial^m}{\partial\mu^m}\int\frac{\mathrm{d}^2 z}{\pi}\colon\exp\Big[-\mu\,|z|^2 \\
&+za^\dagger+z^*a-\frac{(f-g)}{2D}\left(z^{*2}+z^2\right)-a^\dagger a\Big]\colon, \qquad (18)
\end{aligned}
$$

where $\mu=(1-f-g)/D>0$. Further, using the integral formula Eq.(12) again, we have the normally ordered form of the density operator of the PSSTS

$$\rho=2(-1)^m C^{-1}\sqrt{fgD}\frac{\partial^m}{\partial\mu^m}\left\{\frac{1}{\sqrt{F_\mu}}G\left(\mu,a^\dagger,a\right)\right\}, \qquad (19)$$

where

$$
\begin{aligned}
&G\left(\mu,a^\dagger,a\right) \\
&=:\ \exp\left[\frac{\mu D^2a^\dagger a-E\left(a^{\dagger2}+a^2\right)}{F_\mu}-a^\dagger a\right]\colon, \qquad (20)
\end{aligned}
$$

and

$$F_\mu=D^2\mu^2-(f-g)^2\,,\quad E=\frac{D}{2}(f-g)\,. \qquad (21)$$

5. The Wigner function of the PSSTS

From Eq.(19) we evaluate the Wigner function of the PSSTS. For a single-mode system, the Wigner function is given by

$$W(\alpha,\alpha^*)=\mathrm{tr}[\rho\Delta(\alpha,\alpha^*)],\quad \alpha=\frac{1}{\sqrt{2}}(q+\mathrm{i}p)\,, \qquad (22)$$

where $\Delta(\alpha,\alpha^*)$ is the Wigner operator, whose coherent state representation is

$$\Delta(\alpha,\alpha^*)=\int\frac{\mathrm{d}^2\beta}{\pi^2}\,|\alpha+\beta\rangle\langle\alpha-\beta|\,\mathrm{e}^{\alpha\beta^*-\alpha^*\beta}, \qquad (23)$$

where $|\beta\rangle=\exp(-|\beta|^2/2+\beta a^\dagger)\,|0\rangle$ is the coherent state.[21]

Substituting Eq.(20) into Eq.(22) and using Eq.(12), we obtain

$$
\begin{aligned}
&\mathrm{tr}[G\left(\mu,a^\dagger,a\right)\Delta(\alpha,\alpha^*)] \\
=&\int\frac{\mathrm{d}^2\beta}{\pi^2}\,\langle\alpha-\beta|\,G\left(\mu,a^\dagger,a\right)|\alpha+\beta\rangle\,\mathrm{e}^{\alpha\beta^*-\alpha^*\beta} \\
=&\int\frac{\mathrm{d}^2\beta}{\pi}\exp\left[-\frac{\mu D^2+F_\mu}{F_\mu}\,|\beta|^2-2\beta\alpha^*\right. \\
&\left.+2\beta^*\alpha-\frac{E}{F_\mu}\left(\beta^{*2}+\beta^2\right)\right] \\
=&\frac{F_\mu}{\sqrt{\left(\mu D^2+F_\mu\right)^2-4E^2}} \\
&\times\exp\left[4F_\mu\frac{-|\alpha|^2\left(\mu D^2+F_\mu\right)-E\left(\alpha^{*2}+\alpha^2\right)}{\left(\mu D^2+F_\mu\right)^2-4E^2}\right]. \qquad (24)
\end{aligned}
$$

Then substituting Eq.(24) into Eq.(19) we obtain the Wigner function

$$
\begin{aligned}
&W(\alpha,\alpha^*) \\
=&\frac{2C^{-1}}{(-)^m\pi}\sqrt{fgD}\,\mathrm{e}^{2|\alpha|^2}\frac{\partial^m}{\partial\mu^m}\left\{\frac{\sqrt{F_\mu}}{\sqrt{\left(\mu D^2+F_\mu\right)^2-4E^2}}\right. \\
&\left.\times\exp\left[-4F_\mu\frac{\left(\mu D^2+F_\mu\right)|\alpha|^2+E\left(\alpha^{*2}+\alpha^2\right)}{\left(\mu D^2+F_\mu\right)^2-4E^2}\right]\right\}. \qquad (25)
\end{aligned}
$$

In particular, in the case of the single-photon-subtracted thermal state (SPSSTS), $m = 1$, from Eq.(25) we obtain

$$\begin{aligned} & W_{m=1}(\alpha, \alpha^*) \\ &= 2\frac{C^{-1}}{\pi}\frac{\sqrt{fg}}{R_1^{5/2}}\exp\left\{-\frac{fq^2}{1-f}-\frac{gp^2}{1-g}\right\} \\ &\times\left(\frac{D}{R_2}\right)^{3/2}\left[R_1R_3+2D\left(\frac{R_1}{R_2}q^2+p^2\right)\right], \quad (26) \end{aligned}$$

where

$$\begin{aligned} R_1 &= 1-2f+D = 2(1-2f)(1-g) > 0, \\ R_2 &= 1-2g+D = 2(1-2g)(1-f), \\ R_3 &= 1-f-g+D = 4fg-3g-3f+2. \quad (27) \end{aligned}$$

Noting when $D < 0$, then $R_2 < 0$, i.e., $D/R_2 > 0$, thus from Eq.(26) one can see that the Wigner function of the SPSSTS is always real. When the factor $R_1R_3 + 2D(R_1q^2/R_2 + p^2) < 0$, the Wigner function of the SPSSTS has its negative distribution in phase space. This indicates that the Wigner function of the SPSSTS always has negative values under the condition $R_3 < 0$ (i.e., $g > (2-3f)/(3-4f)$) at the phase space centre $q = p = 0$, which is similar to the case of the single-photon-subtracted squeezed vacuum.[13] Combining $R_3 < 0$ with Eqs.(7) and (10) we obtain the condition of the Wigner function exhibiting negative values,

$$\frac{\mathrm{e}^{2r}+\mathrm{e}^{-2r}}{2} > 2n+1. \quad (28)$$

Using Eq.(26), we depict the Wigner functions of the SPSSTS in Figs.1 and 2 for several different values

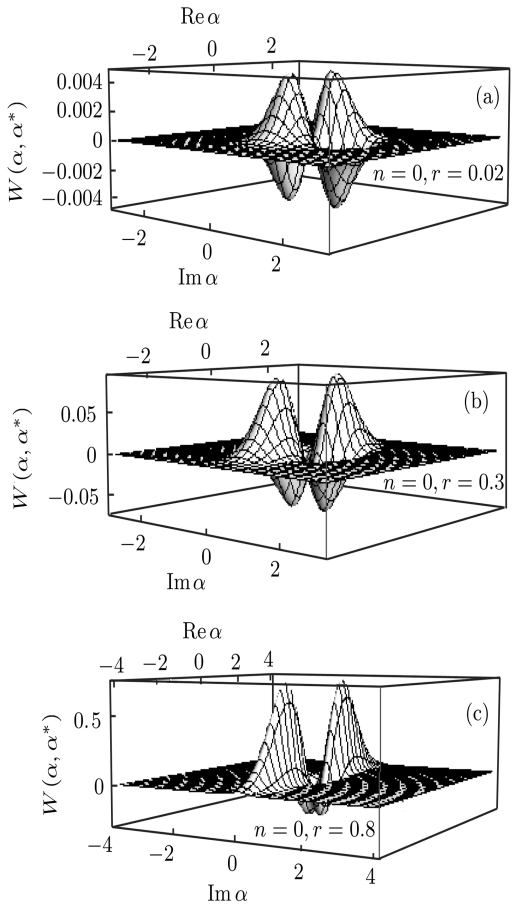


Fig.1. Wigner function of single-photon-subtracted squeezed thermal mixed state for different values of n and r: (a) $n = 0, r = 0.02$, (b) $n = 0, r = 0.3$, (c) $n = 0, r = 0.8$.

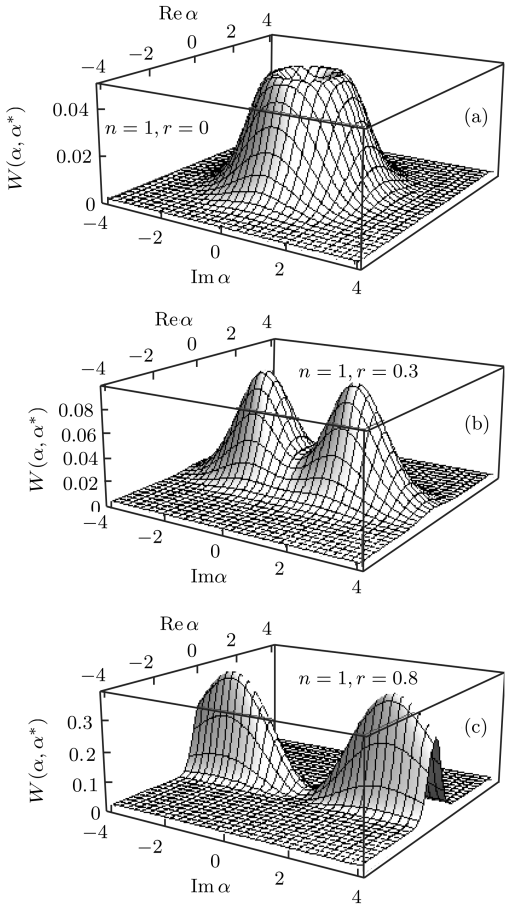


Fig.2. Wigner function of single-photon-subtracted squeezed thermal mixed state for different values of n and r: (a) $n = 1, r = 0$, (b) $n = 1, r = 0.3$, (c) $n = 1, r = 0.8$.

of n and r in phase space. It is easy to see that the negative region of the Wigner function gradually disappears as the n (or the temperature) increases for a given r (see Figs.1(c) and 2(c)). In particular, for $n = 0$ and the large squeezing case (see Fig.1), the Wigner function is similar to that of a Schrödinger cat state. There are two positive peaks and two negative peaks. From Fig.2, one can see that, for the zero-squeezing case, the Wigner function is not Gaussian. The Wigner function is split into two peaks and the distance between them becomes larger as the squeezing parameter r increases.

In summary, by using the IWOP technique of integration and the coherent state representation of the Wigner operator, we have derived the Wigner function and density operator of the PSSTS and demonstrated the nonclassical properties of the PSSTS based on the negativity of the Wigner function.

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