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Conformal invariance and conserved quantity of third-order Lagrange equations for non-conserved mechanical systems*

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This paper studies conformal invariance and conserved quantity of third-order Lagrange equations for non-conserved mechanical systems. Third-order Lagrange equations, the definition and a determining equation of conformal invariance of the system are presented. The conformal factor expression is deduced from conformal invariance and Lie symmetry. The necessary and sufficient condition that conformal invariance of the system would have Lie symmetry under single-parameter infinitesimal transformations is obtained. The corresponding conserved quantity of conformal invariance is derived with the aid of a structure equation. Lastly, an example is given to illustrate the application of the results.

Keywords: conformal invariance, conserved quantity, third-order Lagrange equation, non-conserved mechanical system

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1. Introduction

The symmetries of mechanical systems have a close connection with the associated conserved quantities. Modern methods of finding conserved quantities use the symmetries, which are mainly Noether symmetry,[1] Lie symmetry[2] and Mei symmetry.[3,4] Recently, conformal invariance (or conformal symmetry) has been introduced into symmetrical theories of mechanical systems,[5] which can also lead to conserved quantities. Research on symmetries and conserved quantities of mechanical systems not only has important mathematical significance, but also has a profound physical background. Even though differential equations of motion of a mechanical system cannot be integrated, the existence of some conserved quantities can enable us to obtain some knowledge about the local physical state of the system. A great deal of attention has been paid to this popular research field, and many important results have been obtained.[6−19] Seeking the symmetries of mechanical systems is actually seeking the mapping ensemble of independent or dependent variables corresponding to the conformal invariance of dynamical equations of mechanical systems. Galiullin *et al* studied conformal invariance of Birkhoff systems under special infinitesimal transformations.[20] Robert and Matthew researched conformal Hamiltonian systems by geometric methods, in which the authors discussed the geometric structure of conformal invariance and pointed out the difference of conformal invariance from arbitrary symmetry of dynamical systems.[21] References [5] and [22] studied respectively the conformal invariance and conformal invariance of Mei symmetry of general holonomic systems. At present, research on conformal invariance and conserved quantities of mechanical systems has already made some important achievements.[5,22−28] But all these studies are based on the first-order or second-order differential equations of motion.

With the further recognition on jerk and acceleration energy, third-order differential equations of motion are attracting a lot of attention again. Third-order differential equations of motion play an important role in the research on dynamics of varying acceleration. They not only provide a new method to solve some mechanical problems concerning the time rate of change of force and acceleration energy, but also pick up some information of mechanical problems lost by Newtonian dynamics and traditional analytical mechanics. For example, we can accurately obtain the change of acceleration state of a free falling

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body at the instant of the initial release stage by using a jumpulse equation, while the detail of this initial abrupt change has often been neglected in past research.[29]

In the 1990s, third-order Lagrange equations were first given by Mei *et al.*[30] Third-order Lagrange equations reveal the relationship between the change of motion state of the system and the time rate of change of force, which cannot be obtained by using Newtonian kinetic equations and the traditional analytical mechanical equations. Therefore, third-order Lagrange equations are an appropriate complement to second-order differential equations of motion including Newtonian kinetic equations and the traditional analytical mechanical equations etc. Currently, third-order Lagrange equations are gradually playing a role in the real world. Some differential equations of motion of systems such as the differential equations of motion of a one-dimensional oscillator in the system of rectangular coordinates can be also obtained directly by using third-order Lagrange equations, which are the same as the ones obtained by the traditional Lagrange equations. In addition, it is easier to solve the jerk of variable accelerated motion by using third-order Lagrange equations–the jerk of the terminal velocity problem, for example.[29] Now, the jerk has already found new applications in the theory of chaos etc.[31,32] To some extent third-order Lagrange equations reveal more details of the motion of the systems. In recent years, some work on the study of third-order Lagrange equations has been done.[33−36] Research into symmetries and conserved quantities of third-order Lagrange equations for mechanical systems has also gained some important results.[37−39] However, conformal invariance of third-order Lagrange equations for mechanical systems has not been reported yet.

In this paper, we study the conformal invariance and conserved quantity of third-order Lagrange equations for non-conserved mechanical systems under the single-parameter infinitesimal transformations. Meanwhile, an example is given to illustrate the application of the results.

2. Third-order Lagrange equations of the systems

For a non-conserved mechanical system composed of N particles, third-order Lagrange equations of the system can be written as[30,33]

$$\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \dddot{q}_\alpha}\right)-\frac{1}{2}\frac{\partial S}{\partial \dot{q}_\alpha}=Q_\alpha^*,\quad(\alpha=1,2,\ldots,n),\tag{1}$$

where $S=S(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ is the acceleration energy. In analytical mechanics, acceleration energy is introduced when Appell equations are established.[30] Acceleration energy can also be called the Appell function or Gibbs function, which is always the focus of establishing Appell equations. It is a very important physical quantity, and Ref.[40] discussed its physical significance. The build-up of the variable force in acceleration space causes the change in acceleration energy of the system, or the build-up of the time rate of change of force in velocity space causes the change in acceleration energy of the system. Research on acceleration energy by experiment is essential and worthwhile. However, physical experiments on acceleration energy have never been reported. $Q_\alpha^*=Q_\alpha^*(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ in Eq.(1) are the time rate of change of generalized forces. The concrete expressions of S and Q_α^* are respectively

$$S=\frac{1}{2}\sum_{i=1}^{N}m_i\ddot{\boldsymbol{r}}_i\cdot\ddot{\boldsymbol{r}}_i,\quad Q_\alpha^*=\sum_{i=1}^{N}\dot{\boldsymbol{F}}_i\cdot\frac{\partial\boldsymbol{r}_i}{\partial q_\alpha}.\tag{2}$$

Equation (1) can be described by

$$F_\alpha\equiv C_{\alpha k}(t,\boldsymbol{q},\dot{\boldsymbol{q}})\dddot{q}_k+B_\alpha(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})=0,\tag{3}$$

where

$$\begin{aligned}C_{\alpha k}&=\frac{\partial^2 S}{\partial\ddot{q}_\alpha\partial\ddot{q}_k},\\ B_\alpha&=\frac{\partial^2 S}{\partial\ddot{q}_\alpha\partial t}+\frac{\partial^2 S}{\partial\ddot{q}_\alpha\partial q_k}\dot{q}_k\\&\quad+\frac{\partial^2 S}{\partial\ddot{q}_\alpha\partial\dot{q}_k}\ddot{q}_k-\frac{1}{2}\frac{\partial S}{\partial\dot{q}_\alpha}-Q_\alpha^*.\end{aligned}\tag{4}$$

Supposing the system is nonsingular, i.e.

$$\det\left(\frac{\partial^2 S}{\partial\ddot{q}_\alpha\partial\ddot{q}_k}\right)\neq 0,\tag{5}$$

we can obtain from Eq.(1)

$$\dddot{q}_\alpha=A_\alpha(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}),\quad(\alpha=1,2,\ldots,n).\tag{6}$$

3. Conformal invariance of the systems

We introduce single-parameter infinitesimal transformations with respect to time t and generalized coordinates q_α as

$$t^*=t+\Delta t,\quad q_\alpha^*(t^*)=q_\alpha(t)+\Delta q_\alpha,\tag{7}$$

or their expansion formulas

$$t^* = t + \varepsilon\xi_0(t, \boldsymbol{q}), \quad q_\alpha^*(t^*) = q_\alpha(t) + \varepsilon\xi_\alpha(t, \boldsymbol{q}), \quad (8)$$

where ε is an infinitesimal parameter, and ξ_0, ξ_α are infinitesimal generators.

Meanwhile, we introduce an operator

$$\frac{\mathrm{d}}{\mathrm{d}t} = \frac{\partial}{\partial t} + \dot{q}_\alpha \frac{\partial}{\partial q_\alpha} + \ddot{q}_\alpha \frac{\partial}{\partial \dot{q}_\alpha} + \dddot{q}_\alpha \frac{\partial}{\partial \ddot{q}_\alpha}, \quad (9)$$

and an infinitesimal transformation vector of the generator

$$X^{(0)} = \xi_0 \frac{\partial}{\partial t} + \xi_k \frac{\partial}{\partial q_k}, \quad (k = 1, 2, \ldots, n), \quad (10)$$

whose first extension, second extension, and third extension are

$$X^{(1)} = X^{(0)} + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_k}, \quad (11)$$

$$\begin{aligned} X^{(2)} &= X^{(1)} + \left[\frac{\mathrm{d}}{\mathrm{d}t}(\dot{\xi}_k - \dot{q}_k\dot{\xi}_0) - \ddot{q}_k\dot{\xi}_0\right]\frac{\partial}{\partial \ddot{q}_k} \\ &= X^{(1)} + (\ddot{\xi}_k - 2\ddot{q}_k\dot{\xi}_0 - \dot{q}_k\ddot{\xi}_0)\frac{\partial}{\partial \ddot{q}_k}, \end{aligned} \quad (12)$$

$$\begin{aligned} X^{(3)} &= X^{(2)} + \left[\frac{\mathrm{d}}{\mathrm{d}t}(\ddot{\xi}_k - 2\ddot{q}_k\dot{\xi}_0 - \dot{q}_k\ddot{\xi}_0) - \dddot{q}_k\dot{\xi}_0\right]\frac{\partial}{\partial \dddot{q}_k} \\ &= X^{(2)} + (\dddot{\xi}_k - \dot{q}_k\dddot{\xi}_0 - 3\ddot{q}_k\ddot{\xi}_0 - 3\dddot{q}_k\dot{\xi}_0)\frac{\partial}{\partial \dddot{q}_k}. \end{aligned} \quad (13)$$

Definition If there is a nonsingular matrix $\boldsymbol{l}_\alpha^l$ satisfying the following condition

$$X^{(3)}(F_\alpha) = \boldsymbol{l}_\alpha^l(F_l), \quad (14)$$

this kind of invariance is called the conformal invariance of the non-conserved mechanical system Eq.(3) under the infinitesimal transformations Eq.(8).

Equation (14) is the determining equation of conformal invariance of the non-conserved mechanical system Eq.(3), and $\boldsymbol{l}_\alpha^l$ is called conformal factor. If the generators ξ_0 and ξ_α of the infinitesimal transformations Eq.(8) satisfy the determining equation (14), the transformations are called conformal transformations.

4. Conformal factor of conformal invariance and Lie symmetry

In order to obtain the conformal factor of conformal invariance for the non-conserved mechanical system, we calculate the difference value of the following expression:

$$X^{(3)}(F_\alpha) - X^{(3)}(F_\alpha)\big|_{F_\alpha=0}. \quad (15)$$

Using

$$\dot{\xi}_\alpha = \frac{\partial \xi_\alpha}{\partial t} + \frac{\partial \xi_\alpha}{\partial q_k}\dot{q}_k, \quad (\alpha = 0, 1, \ldots, n; \quad k = 1, 2, \ldots, n), \quad (16)$$

$$\ddot{\xi}_\alpha = \frac{\partial^2 \xi_\alpha}{\partial t^2} + 2\frac{\partial^2 \xi_\alpha}{\partial q_k \partial t}\dot{q}_k + \frac{\partial^2 \xi_\alpha}{\partial q_k \partial q_j}\dot{q}_k\dot{q}_j + \frac{\partial \xi_\alpha}{\partial q_k}\ddot{q}_k, \quad (\alpha = 0, 1, \ldots, n; \quad k, j = 1, 2, \ldots, n), \quad (17)$$

$$\begin{aligned} \dddot{\xi}_\alpha &= \frac{\partial^3 \xi_\alpha}{\partial t^3} + 3\frac{\partial^3 \xi_\alpha}{\partial q_k \partial t^2}\dot{q}_k + 3\frac{\partial^3 \xi_\alpha}{\partial q_k \partial q_j \partial t}\dot{q}_k\dot{q}_j + 3\frac{\partial^2 \xi_\alpha}{\partial q_k \partial t}\ddot{q}_k + 3\frac{\partial^2 \xi_\alpha}{\partial q_k \partial q_j}\ddot{q}_k\dot{q}_j + \frac{\partial^3 \xi_\alpha}{\partial q_k \partial q_j \partial q_\gamma}\dot{q}_k\dot{q}_j\dot{q}_\gamma + \frac{\partial \xi_\alpha}{\partial q_k}\dddot{q}_k, \\ &\quad (\alpha = 0, 1, \ldots, n; \quad k, j, \gamma = 1, 2, \ldots, n), \end{aligned} \quad (18)$$

we have

$$\begin{aligned} &X^{(3)}(F_\alpha) \\ &= C_{\alpha k}(\dddot{\xi}_k - \dot{q}_k\dddot{\xi}_0 - 3\ddot{q}_k\ddot{\xi}_0 - 3\dddot{q}_k\dot{\xi}_0) + X^{(2)}(C_{\alpha k})\dddot{q}_k + X^{(2)}(B_\alpha) \\ &= C_{\alpha k}(\dddot{\xi}_k - \dot{q}_k\dddot{\xi}_0 - 3\ddot{q}_k\ddot{\xi}_0 - 3\dddot{q}_k\dot{\xi}_0) + X^{(1)}(C_{\alpha k})\dddot{q}_k + X^{(1)}(B_\alpha) + (\ddot{\xi}_k - 2\ddot{q}_k\dot{\xi}_0 - \dot{q}_k\ddot{\xi}_0)\frac{\partial B_\alpha}{\partial \ddot{q}_k} \\ &= C_{\alpha k}\left(\frac{\partial^3 \xi_k}{\partial t^3} + 3\frac{\partial^3 \xi_k}{\partial q_m \partial t^2}\dot{q}_m + 3\frac{\partial^3 \xi_k}{\partial q_m \partial q_j \partial t}\dot{q}_m\dot{q}_j + 3\frac{\partial^2 \xi_k}{\partial q_m \partial t}\ddot{q}_m + 3\frac{\partial^2 \xi_k}{\partial q_m \partial q_j}\ddot{q}_m\dot{q}_j + \frac{\partial^3 \xi_k}{\partial q_m \partial q_j \partial q_\gamma}\dot{q}_m\dot{q}_j\dot{q}_\gamma + \frac{\partial \xi_k}{\partial q_m}\dddot{q}_m\right) \\ &\quad - C_{\alpha k}\dot{q}_k\left(\frac{\partial^3 \xi_0}{\partial t^3} + 3\frac{\partial^3 \xi_0}{\partial q_m \partial t^2}\dot{q}_m + 3\frac{\partial^3 \xi_0}{\partial q_m \partial q_j \partial t}\dot{q}_m\dot{q}_j + 3\frac{\partial^2 \xi_0}{\partial q_m \partial t}\ddot{q}_m + 3\frac{\partial^2 \xi_0}{\partial q_m \partial q_j}\ddot{q}_m\dot{q}_j + \frac{\partial^3 \xi_0}{\partial q_m \partial q_j \partial q_\gamma}\dot{q}_m\dot{q}_j\dot{q}_\gamma\right. \\ &\quad \left. + \frac{\partial \xi_0}{\partial q_m}\dddot{q}_m\right) - 3C_{\alpha k}\ddot{q}_k\left(\frac{\partial^2 \xi_0}{\partial t^2} + 2\frac{\partial^2 \xi_0}{\partial q_m \partial t}\dot{q}_m + \frac{\partial^2 \xi_0}{\partial q_m \partial q_j}\dot{q}_m\dot{q}_j + \frac{\partial \xi_0}{\partial q_m}\ddot{q}_m\right) - 3C_{\alpha k}\dddot{q}_k\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_m}\dot{q}_m\right) \\ &\quad + X^{(1)}(C_{\alpha k})\ddot{q}_k + X^{(1)}(B_\alpha) + \left[\frac{\partial^2 \xi_k}{\partial t^2} + 2\frac{\partial^2 \xi_k}{\partial q_m \partial t}\dot{q}_m + \frac{\partial^2 \xi_k}{\partial q_m \partial q_j}\dot{q}_m\dot{q}_j + \frac{\partial \xi_k}{\partial q_m}\ddot{q}_m\right. \end{aligned}$$

$$-2\ddot{q}_k\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_m}\dot{q}_m\right)-\dot{q}_k\left(\frac{\partial^2\xi_0}{\partial t^2}+2\frac{\partial^2\xi_0}{\partial q_m\partial t}\dot{q}_m+\frac{\partial^2\xi_0}{\partial q_m\partial q_j}\dot{q}_m\dot{q}_j+\frac{\partial\xi_0}{\partial q_m}\ddot{q}_m\right)\Bigg]\frac{\partial B_\alpha}{\partial\ddot{q}_k}$$
$$=C_{\alpha k}\left(\frac{\partial\xi_k}{\partial q_m}-\dot{q}_k\frac{\partial\xi_0}{\partial q_m}\right)\dddot{q}_m-3C_{\alpha k}\ddot{q}_k\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_m}\dot{q}_m\right)+X^{(1)}(C_{\alpha k})\ddot{q}_k+D(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}),$$
$$(\alpha,k,j,\gamma,m=1,2,\ldots,n), \tag{19}$$

where $D(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}})$ is the summation of items in which $\dddot{q}_\alpha$ are not included. Equivalently, we have

$$X^{(3)}(F_\alpha)\Big|_{F_\alpha=0}=C_{\alpha k}\left(\frac{\partial\xi_k}{\partial q_m}-\dot{q}_k\frac{\partial\xi_0}{\partial q_m}\right)A_m-3C_{\alpha k}A_k\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_m}\dot{q}_m\right)+X^{(1)}(C_{\alpha k})A_k$$
$$+D(t,\boldsymbol{q},\dot{\boldsymbol{q}},\ddot{\boldsymbol{q}}),\quad(\alpha,k,m=1,2,\ldots,n), \tag{20}$$

where $A_\alpha=-C^{\alpha k}B_k$. Substituting Eqs.(19) and (20) into Eq.(15), we obtain

$$X^{(3)}(F_\alpha)-X^{(3)}(F_\alpha)\Big|_{F_\alpha=0}$$
$$=C_{\alpha k}\left(\frac{\partial\xi_k}{\partial q_m}-\dot{q}_k\frac{\partial\xi_0}{\partial q_m}\right)(\dddot{q}_m-A_m)-3C_{\alpha k}(\ddot{q}_k-A_k)\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_m}\dot{q}_m\right)$$
$$+X^{(1)}(C_{\alpha k})(\ddot{q}_k-A_k),\quad(\alpha,k,m=1,2,\ldots,n). \tag{21}$$

Due to

$$\ddot{q}_k-A_k=\ddot{q}_k+C^{kl}B_l=C^{kl}(C_{ls}\ddot{q}_s+B_l)=C^{kl}F_l,\quad(k,l,s=1,2,\ldots,n), \tag{22}$$

we have

$$X^{(3)}(F_\alpha)-X^{(3)}(F_\alpha)\Big|_{F_\alpha=0}$$
$$=C_{\alpha k}\left(\frac{\partial\xi_k}{\partial q_m}-\dot{q}_k\frac{\partial\xi_0}{\partial q_m}\right)C^{ml}F_l-3C_{\alpha k}C^{kl}F_l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_m}\dot{q}_m\right)+X^{(1)}(C_{\alpha k})C^{kl}F_l$$
$$=\left[C_{\alpha k}\left(\frac{\partial\xi_k}{\partial q_m}-\dot{q}_k\frac{\partial\xi_0}{\partial q_m}\right)C^{ml}-3\delta^l_\alpha\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_m}\dot{q}_m\right)+X^{(1)}(C_{\alpha k})C^{kl}\right]F_l,\quad(\alpha,k,m,l=1,2,\ldots,n). \tag{23}$$

Let

$$\beta^l_\alpha=C_{\alpha k}\left(\frac{\partial\xi_k}{\partial q_m}-\dot{q}_k\frac{\partial\xi_0}{\partial q_m}\right)C^{ml}-3\delta^l_\alpha\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_m}\dot{q}_m\right)+X^{(1)}(C_{\alpha k})C^{kl}, \tag{24}$$

so that Eq.(23) is changed into

$$X^{(3)}(F_\alpha)-X^{(3)}(F_\alpha)\Big|_{F_\alpha=0}=\beta^l_\alpha F_l, \tag{25}$$

when Eq.(3) is Lie symmetrical under the infinitesimal transformations Eq.(8), then

$$X^{(3)}(F_\alpha)\Big|_{F_\alpha=0}=0, \tag{26}$$

from Eqs.(14) and (25), we have $\beta^l_\alpha=l^l_\alpha$. When Eq.(3) has conformal invariance under the infinitesimal transformations Eq.(8), from Eq.(25), we can obtain

$$(l^l_\alpha-\beta^l_\alpha)(F_l)=X^{(3)}(F_\alpha)\Big|_{F_\alpha=0},\quad(\alpha,l=1,2,\ldots,n). \tag{27}$$

So the necessary and sufficient condition that conformal invariance of the system Eq.(3) would have Lie symmetry under the infinitesimal transformations Eq.(8) is

$$l^l_\alpha=\beta^l_\alpha. \tag{28}$$

Thus, from the above discussion we have the following proposition.

Proposition 1 For the non-conserved mechanical system Eq.(3), the necessary and sufficient condition that conformal invariance of the system would have Lie symmetry simultaneously under the infinitesimal transformations Eq.(8) is that there exists the conformal factor

$$l^l_\alpha=C_{\alpha k}\left(\frac{\partial\xi_k}{\partial q_m}-\dot{q}_k\frac{\partial\xi_0}{\partial q_m}\right)C^{ml}$$
$$-3\delta^l_\alpha\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_m}\dot{q}_m\right)+X^{(1)}(C_{\alpha k})C^{kl},$$
$$(\alpha,k,m,l=1,2,\ldots,n). \tag{29}$$

Inference If third-order Lagrange equations (1) can be expressed in the standard form

$$F_\alpha \equiv \dddot{q}_\alpha - A_\alpha(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = 0, \quad (\alpha = 1, 2, \ldots, n), \tag{30}$$

the necessary and sufficient condition that conformal invariance of the system would have Lie symmetry simultaneously under the infinitesimal transformations Eq.(8) is the conformal factor

$$l_\alpha^l = \frac{\partial \xi_\alpha}{\partial q_l} - 3\dot{\xi}_0 - \dot{q}_\alpha \frac{\partial \xi_0}{\partial q_l}. \tag{31}$$

Actually, comparing Eq.(3) with Eq.(30), we have $C_{\alpha k} = \delta_{\alpha k}, B_\alpha(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}}) = -A_\alpha(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$. Using Eq.(29), we can easily obtain Eq.(31).

5. Conserved quantity deduced from conformal invariance

Under certain conditions, conformal invariance of the non-conserved mechanical system can also lead to conserved quantities.

Proposition 2 For the non-conserved mechanical system Eq.(3), if the infinitesimal generators ξ_0 and ξ_α satisfy the conformal factor Eq.(29) or Eq.(31), and there is a gauge function $G = G(t, \boldsymbol{q}, \dot{\boldsymbol{q}}, \ddot{\boldsymbol{q}})$ satisfying the following Lie symmetrical structure equation

$$\begin{aligned} &S\dot{\xi}_0 + X^{(2)}(S) + Q_\alpha^*(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) \\ &- \left[\frac{\partial S}{\partial q_\alpha}(\xi_\alpha - \dot{q}_\alpha \xi_0) + \frac{1}{2}\frac{\partial S}{\partial \dot{q}_\alpha}(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0 - 2\dot{q}_\alpha \dot{\xi}_0) - \frac{\partial S}{\partial \ddot{q}_\alpha}\dot{q}_\alpha \ddot{\xi}_0 - \frac{\partial S}{\partial \ddot{q}_\alpha}\ddot{q}_\alpha \dot{\xi}_0\right] + \dot{G} = 0, \end{aligned} \tag{32}$$

conformal invariance of the system can lead to the following conserved quantity:

$$I = S\xi_0 + \frac{\partial S}{\partial \ddot{q}_\alpha}(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) + G = \text{const}, \quad (\alpha = 1, \ldots, n). \tag{33}$$

Proof Taking the derivative of I with respect to t, and using Eqs.(1) and (32), we have

$$\begin{aligned} \frac{\mathrm{d}I}{\mathrm{d}t} &= \dot{S}\xi_0 + S\dot{\xi}_0 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\alpha}\right)(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) + \frac{\partial S}{\partial \ddot{q}_\alpha}(\ddot{\xi}_\alpha - \ddot{q}_\alpha \dot{\xi}_0 - \dddot{q}_\alpha \xi_0) + \dot{G} \\ &= \dot{S}\xi_0 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\alpha}\right)(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) + \frac{\partial S}{\partial \ddot{q}_\alpha}(\ddot{\xi}_\alpha - \ddot{q}_\alpha \dot{\xi}_0 - \dddot{q}_\alpha \xi_0) - X^{(2)}(S) - Q_\alpha^*(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) \\ &\quad + \left[\frac{\partial S}{\partial q_\alpha}(\xi_\alpha - \dot{q}_\alpha \xi_0) + \frac{1}{2}\frac{\partial S}{\partial \dot{q}_\alpha}(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0 - 2\dot{q}_\alpha \dot{\xi}_0) - \frac{\partial S}{\partial \ddot{q}_\alpha}\dot{q}_\alpha \ddot{\xi}_0 - \frac{\partial S}{\partial \ddot{q}_\alpha}\ddot{q}_\alpha \dot{\xi}_0\right] \\ &= \left(\frac{\partial S}{\partial t} + \frac{\partial S}{\partial q_\alpha}\dot{q}_\alpha + \frac{\partial S}{\partial \dot{q}_\alpha}\ddot{q}_\alpha + \frac{\partial S}{\partial \ddot{q}_\alpha}\dddot{q}_\alpha\right)\xi_0 + \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\alpha}\right)(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) + \frac{\partial S}{\partial \ddot{q}_\alpha}(\ddot{\xi}_\alpha - \ddot{q}_\alpha \dot{\xi}_0 - \ddot{q}_\alpha \xi_0) \\ &\quad - \frac{\partial S}{\partial t}\xi_0 - \frac{\partial S}{\partial q_\alpha}\xi_\alpha - \frac{\partial S}{\partial \dot{q}_\alpha}(\dot{\xi}_\alpha - \dot{q}_\alpha \dot{\xi}_0) - \frac{\partial S}{\partial \ddot{q}_\alpha}(\ddot{\xi}_\alpha - \dot{q}_\alpha \ddot{\xi}_0 - 2\ddot{q}_\alpha \dot{\xi}_0) - Q_\alpha^*(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) \\ &\quad + \left[\frac{\partial S}{\partial q_\alpha}(\xi_\alpha - \dot{q}_\alpha \xi_0) + \frac{1}{2}\frac{\partial S}{\partial \dot{q}_\alpha}(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0 - 2\dot{q}_\alpha \dot{\xi}_0) - \frac{\partial S}{\partial \ddot{q}_\alpha}\dot{q}_\alpha \ddot{\xi}_0 - \frac{\partial S}{\partial \ddot{q}_\alpha}\ddot{q}_\alpha \dot{\xi}_0\right] \\ &= \frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\alpha}\right)(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) - \frac{1}{2}\frac{\partial S}{\partial \dot{q}_\alpha}(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) - Q_\alpha^*(\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) \\ &= \left[\frac{\mathrm{d}}{\mathrm{d}t}\left(\frac{\partial S}{\partial \ddot{q}_\alpha}\right) - \frac{1}{2}\frac{\partial S}{\partial \dot{q}_\alpha} - Q_\alpha^*\right](\dot{\xi}_\alpha - \ddot{q}_\alpha \xi_0) \\ &= 0. \end{aligned} \tag{34}$$

6. An example

Consider that a projectile is launched with initial speed v_0 at an angle α_0 with the horizontal, the mass of the body is m_0, the dissipative force acting on the body is directly proportional to the velocity of the body, i.e. $f(v) = kv$. The coordinates at any time can be expressed as

$$\boldsymbol{r} = x\boldsymbol{i} + y\boldsymbol{j}. \tag{35}$$

Considering the forces $\boldsymbol{G} = -m_0 g\boldsymbol{j}$, $\boldsymbol{f}(\boldsymbol{v}) = -k\boldsymbol{v} = -k\dot{\boldsymbol{r}} = -k(\dot{x}\boldsymbol{i} + \dot{y}\boldsymbol{j})$, we have

$$m_0\ddot{\boldsymbol{r}} = \boldsymbol{F}, \quad S = \frac{1}{2}m_0\ddot{\boldsymbol{r}}\cdot\ddot{\boldsymbol{r}} = \frac{1}{2}m_0(\ddot{x}^2+\ddot{y}^2),$$
$$Q_1^* = -k\ddot{x}, \quad Q_2^* = -k\ddot{y}. \tag{36}$$

Substituting Eq.(36) into the third-order Lagrange equations Eq.(1), we obtain

$$F_1 = \dddot{x} + \omega\ddot{x} = 0, \quad F_2 = \dddot{y} + \omega\ddot{y} = 0, \tag{37}$$

where $\omega = k/m_0$. Actually, by using the eigenvalues of Eq.(37) and the initial condition, we can obtain the following results of the problem:[41]

$$x = \frac{v_0}{\omega}\left(1 - \mathrm{e}^{-\omega t}\right)\cos\alpha_0,$$

$$y = \left(\frac{g}{\omega^2} + \frac{v_0}{\omega}\sin\alpha_0\right)(1 - \mathrm{e}^{-\omega t}) - \frac{g}{\omega}t, \tag{38}$$

which are the same as the results from Newton's law of motion. It is easy to obtain the solutions of the problem if we use the third-order Lagrange equations of the system.

Now let us study the conformal invariance and conserved quantity of the system Eq.(37). Choosing the infinitesimal generators as

$$\xi_0 = 1, \quad \xi_1 = x, \quad \xi_2 = y, \tag{39}$$

we have

$$\begin{aligned} X^{(3)} &= \xi_0\frac{\partial}{\partial t} + \xi_\alpha\frac{\partial}{\partial q_\alpha} + (\dot{\xi}_\alpha - \dot{q}_\alpha\dot{\xi}_0)\frac{\partial}{\partial\dot{q}_\alpha} + (\ddot{\xi}_\alpha - 2\ddot{q}_\alpha\dot{\xi}_0 - \dot{q}_\alpha\ddot{\xi}_0)\frac{\partial}{\partial\ddot{q}_\alpha} + (\dddot{\xi}_\alpha - \dot{q}_\alpha\dddot{\xi}_0 - 3\ddot{q}_\alpha\ddot{\xi}_0 - 3\dddot{q}_\alpha\dot{\xi}_0)\frac{\partial}{\partial\dddot{q}_\alpha} \\ &= \frac{\partial}{\partial t} + x\frac{\partial}{\partial x} + y\frac{\partial}{\partial y} + \dot{x}\frac{\partial}{\partial\dot{x}} + \dot{y}\frac{\partial}{\partial\dot{y}} + \ddot{x}\frac{\partial}{\partial\ddot{x}} + \ddot{y}\frac{\partial}{\partial\ddot{y}} + \dddot{x}\frac{\partial}{\partial\dddot{x}} + \dddot{y}\frac{\partial}{\partial\dddot{y}}, \end{aligned} \tag{40}$$

$$X^{(3)}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = X^{(3)}\begin{pmatrix} \dddot{x} + \omega\ddot{x} \\ \dddot{y} + \omega\ddot{y} \end{pmatrix} = \begin{pmatrix} \dddot{x} + \omega\ddot{x} \\ \dddot{y} + \omega\ddot{y} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \dddot{x} + \omega\ddot{x} \\ \dddot{y} + \omega\ddot{y} \end{pmatrix}. \tag{41}$$

The conformal factor is

$$l_\alpha^l = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{42}$$

From Eq.(29), we can also obtain the conformal factor

$$\begin{aligned} l_\alpha^l &= C_{\alpha k}\left(\frac{\partial\xi_k}{\partial q_m} - \dot{q}_k\frac{\partial\xi_0}{\partial q_m}\right)C^{ml} \\ &\quad -3\delta_\alpha^l\left(\frac{\partial\xi_0}{\partial t} + \frac{\partial\xi_0}{\partial q_m}\dot{q}_m\right) \\ &\quad +X^{(1)}(C_{\alpha k})C^{kl} \\ &= \frac{\partial\xi_\alpha}{\partial q_k}. \end{aligned} \tag{43}$$

Obviously, the above solution Eq.(43) is in accordance with Eq.(42). The determining equation of conformal invariance is

$$X^{(3)}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix}, \tag{44}$$

then the conformal invariance of the system would also have Lie symmetry.

Substituting Eq.(39) into the structure equation (32), we can obtain

$$G = \frac{1}{2}m_0\left(1 + \frac{1}{\omega}\right)(\ddot{x}^2+\ddot{y}^2) + \frac{1}{2}k(\dot{x}^2+\dot{y}^2). \tag{45}$$

According to Proposition 2, we can obtain the following conserved quantity:

$$I = \frac{1}{2}\frac{m_0}{\omega}(\ddot{x}^2+\ddot{y}^2) + m_0(\dot{x}\ddot{x}+\dot{y}\ddot{y}) + \frac{1}{2}k(\dot{x}^2+\dot{y}^2) = \text{const.} \tag{46}$$

The conserved quantity Eq.(46) is a new type of conserved quantity with an acceleration energy dimension. It is similar to the conserved quantity with a unit mass acceleration energy dimension $K = 1/2(\ddot{x} + A\dot{x})^2 + 1/2B(\dot{x}+Ax)^2$, which was proposed by Linz for a special Newtonian jerky dynamical system.[42] With the development of acceleration energy and jerk, we can further seek new types of conserved quantities in classical mechanics.

7. Conclusion

The conformal invariance and conserved quantity of third-order Lagrange equations for non-conserved

mechanical systems are studied in this paper. The conformal factor of conformal invariance under single-parameter infinitesimal transformations is obtained by using the Lie symmetry, which is the necessary and sufficient condition that conformal invariance would have Lie symmetry. Conformal invariance can also lead to conserved quantities under certain conditions. The study of conformal invariance and conserved quantities of third-order Lagrange equations for mechanical systems makes the theory of third-order Lagrange equations and symmetrical theories in analytical mechanics more perfect and general.

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