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Conformal invariance and generalized Hojman conserved quantities of mechanico-electrical systems

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This paper studies conformal invariance and generalized Hojman conserved quantities of mechanico-electrical systems. The definition and the determining equation of conformal invariance for mechanico-electrical systems are provided. The conformal factor expression is deduced from conformal invariance and Lie symmetry under the infinitesimal single-parameter transformation group. The generalized Hojman conserved quantities from the conformal invariance of the system are given. An example is given to illustrate the application of the result.

Keywords: mechanico-electrical systems, conformal invariance, conformal factor, generalized Hojman conserved quantities

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1. Introduction

Symmetries and conservation laws are profound statements about how the world works, so this theory has some theoretical significance. There are mainly three kinds of symmetries: the Noether symmetry,[1] the Lie symmetry[2] and the Mei symmetry.[3] Respectively, there are mainly three kinds of conserved quantities (i.e., the Noether conserved quantity, the Hojman conserved quantity and a new type of conserved quantity) according to these main kinds of symmetries.[4−15]

Conformal invariance was introduced into physics by Bateman[16] and Conningham[17] shortly after Einstein's first paper on special relativity. Then it was applied widely to the gauge theory[18] and the quantum field theory.[19] Recently, conformal invariance has been developed rapidly in dynamical mechanical systems; Cai *et al* studied the conformal invariance and conserved quantities of mechanical systems.[20−25] In 1873, Maxwell studied the problems of the mechanico-electrical system by using Lagrange's method; he obtained the Lagrange–Maxwell equation, which is very famous today. Recently, Fu *et al* got the discrete variation principle and first integral of mechanico-electrical systems.[26−35] In this paper, we study the conformal invariance and generalized Hojman conserved quantities of mechanico-electrical systems. The determining equation and conformal factor deduced from conformal invariance and Lie symmetry of conformal invariance for mechanico-electrical systems are provided.

2. The differential equations of the systems

A mechanico-electrical system couples a mechanical process to an electromagnetic process. The mechanical part consists of N particles, and is described by n generalized coordinates $q_s(s = 1, \ldots, n)$. The electromagnetic part corresponds to m electrical circuits consisting of linear conditions and a capacitor. For a circuit k, we denote the current by i_k, the electronic potential by u_k, the capacitor charge by $e_k(k = 1, \ldots, m)$ (with $\dot{e}_k = i_k$), the resistance by R_k, and the capacitance by C_k. The Lagrange function of the mechanico-electrical system is

$$L = T(q_s, \dot{q}_s) - V(q_s) + W_{\rm m}(q_s, \dot{e}_k) - W_{\rm e}(q_s, e_k), \quad (1)$$

where T and V are, respectively, the kinetic and the potential energy. The electric field energy and magnetic field energy are, respectively, defined by

$$W_{\rm m} = \frac{1}{2}\sum_{k=1}^{m}\sum_{r=1}^{m} L_{kr} i_k i_r, \quad W_{\rm e} = \frac{1}{2}\sum_{k=1}^{m}\frac{e_k^2}{C_k}, \quad (2)$$

where L_{kr} (with $k \neq r$) is the inductance of circuit k due to circuit r, and L_{kk} is the self-inductance of

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circuit k, and the summation is taken over repeated suffixes.

Let q_s $(s=1,\dots,n,n+1,\dots,n+m)$ denote the generalized coordinates. We denote the coordinate components by $q_s(s=1,\dots,n)$ and the electricity components by $q_s(s=n+1,\dots,n+m)$. The motion of the system is given by the following Lagrange–Maxwell equations:

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}+\frac{\partial \psi}{\partial \dot{q}_s}=Q_s,\quad (s=1,\dots,n,n+1,\dots,n+m), \tag{3}$$

where $Q_s(s=1,\dots,n)$ are the generalized non-potential forces. The $Q_s(s=n+1,\dots,n+m)$ are the electric potential energy, the dissipative function ψ of the system is given by

$$\psi=\psi_{\mathrm{e}}(i_k)+\psi_{\mathrm{m}}(q_s,\dot{q}_s), \tag{4}$$

where the electric dissipation function is

$$\psi_{\mathrm{e}}=\frac{1}{2}\sum_{k=1}^{m}R_k i_k^2, \tag{5}$$

and ψ_{m} is the function for the viscous frictional damping forces, thus $-\partial\psi/\partial\dot{q}_s$ $(s=1,\dots,n)$ correspond to the dissipative forces.

From Eq.(3), we have

$$\frac{\mathrm{d}}{\mathrm{d}t}\frac{\partial L}{\partial \dot{q}_s}-\frac{\partial L}{\partial q_s}=Q_s\ -\frac{\partial \psi}{\partial \dot{q}_s},\quad (s=1,\dots,n,n+1,\dots,n+m), \tag{6}$$

where $Q_s\ -\partial\psi/\partial\dot{q}_s$ $(s=1,\dots,n,n+1,\dots,n+m)$ are generalized non-conservation forces. From Eq.(6) we obtain

$$A_{sk}\ddot{q}_k+B_s-Q_s+\frac{\partial \psi}{\partial \dot{q}_s}=0, \tag{7}$$

where

$$A_{sk}=\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k},\quad B_s=\frac{\partial^2 L}{\partial \dot{q}_s\partial q_k}\dot{q}_k+\frac{\partial^2 L}{\partial \dot{q}_s\partial t}-\frac{\partial L}{\partial q_s},\quad (s,k=1,\dots,n,n+1,\dots,n+m). \tag{8}$$

Suppose that Eq.(6) are nonsingular, namely

$$\det\left(\frac{\partial^2 L}{\partial \dot{q}_s\partial \dot{q}_k}\right)\neq 0. \tag{9}$$

From Eq.(7), we obtain

$$\ddot{q}_s=\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\ (s=1,\dots,n,n+1,\dots,n+m). \tag{10}$$

3. The conformal factor that the conformal systems possess Lie symmetry

Let

$$F_s=A_{sk}\ddot{q}_s+B_s-Q_s+\frac{\partial \psi}{\partial \dot{q}_s}. \tag{11}$$

If we want to get the conformal invariance of the mechanico-electrical systems, we can search for the dependent or independent variable transformation which the conformal invariance of Eq.(6) or Eq.(7) corresponds to. Considering the symmetries inherent in Eq.(7), we introduce a single-parameter infinitesimal transformation with respect to time and coordinates

$$t^*=t+\varepsilon\xi_0(t,\boldsymbol{q},\dot{\boldsymbol{q}}),\quad q_s^*=q_s(t)+\varepsilon\xi_s(t,\boldsymbol{q},\dot{\boldsymbol{q}}), \tag{12}$$

where ε is a small parameter, ξ_0 and ξ_s are the infinitesimal generators.

The vector field itself has the form

$$X^{(0)}=\xi_0\frac{\partial}{\partial t}+\xi_s\frac{\partial}{\partial q_s},$$

the first and second prolongations of the vector field are respectively

$$X^{(1)}=X^{(0)}+(\dot{\xi}_s-\dot{q}_s\dot{\xi}_0)\frac{\partial}{\partial \dot{q}_s}$$

and

$$X^{(2)}=X^{(1)}+(\ddot{\xi}_s-2\ddot{q}_s\dot{\xi}_0-\dot{q}_s\ddot{\xi}_0)\frac{\partial}{\partial \ddot{q}_s}.$$

Definition If there is a nonsingular matrix Γ_s^l satisfying the condition

$$X^{(2)}(F_s)=\Gamma_s^l(F_l), \tag{13}$$

then this kind of invariance is called the conformal invariance of Eq.(6) under single-parameter infinitesimal transformation Eq.(12). Equation (13) is the determining equation of conformal invariance of the mechanico-electrical system, and Γ_s^l is called the conformal factor. If the generators ξ_0 and ξ_s of the infinitesimal transformation Eq.(12) satisfy Eq.(13), then the above transformations are called conformal transformations.

Equation (10) must be both the conformal invariance and the Lie symmetry under the infinitesimal transformation if we want to get the expressions of the conformal factor of conformal invariance. We calculate the difference of

$$X^{(2)}(F_s)-X^{(2)}(F_s)\,|_{F_s=0}\ . \tag{14}$$

Since

$$\dot{\xi}_k = \frac{\partial \xi_k}{\partial t} + \frac{\partial \xi_k}{\partial q_r}\dot{q}_r + \frac{\partial \xi_k}{\partial \dot{q}_r}\ddot{q}_r, \tag{15}$$

$$\begin{aligned}\ddot{\xi}_k = {} & \frac{\partial^2 \xi_k}{\partial t^2} + 2\frac{\partial^2 \xi_k}{\partial q_r \partial t}\dot{q}_r + 2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t}\ddot{q}_r + \frac{\partial^2 \xi_k}{\partial q_r \partial q_j}\dot{q}_r\dot{q}_j + 2\frac{\partial^2 \xi_k}{\partial q_r \partial \dot{q}_j}\dot{q}_r\ddot{q}_j + \frac{\partial \xi_k}{\partial q_r}\ddot{q}_r + \frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}\ddot{q}_r\ddot{q}_j \\ & + \frac{\partial \xi_k}{\partial \dot{q}_r}\left(\frac{\partial \alpha_r}{\partial t} + \frac{\partial \alpha_r}{\partial q_j}\dot{q}_j + \frac{\partial \alpha_r}{\partial \dot{q}_j}\ddot{q}_j\right), \qquad (k = 0, 1 \ldots, n; r, j = 1, \ldots, n),\end{aligned} \tag{16}$$

then

$$\begin{aligned}X^{(2)}(F_s) &= A_{sk}(\ddot{\xi}_k - 2\ddot{q}_k\dot{\xi}_0 - \dot{q}_k\ddot{\xi}_0) + X^{(2)}(A_{sk})\ddot{q}_k + X^{(2)}(B_s - Q_s - P_s) \\ &= A_{sk}(\ddot{\xi}_k - 2\ddot{q}_k\dot{\xi}_0 - \dot{q}_k\ddot{\xi}_0) + X^{(0)}(A_{sk})\ddot{q}_k + X^{(0)}(B_s - Q_s - P_s) \\ &\quad + (\dot{\xi}_k - \dot{q}_k\dot{\xi}_0)\frac{\partial(B_s - Q_s - P_s)}{\partial \dot{q}_k} \\ &= A_{sk}\left[2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t}\ddot{q}_r + 2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial q_j}\ddot{q}_r\dot{q}_j + \frac{\partial \xi_k}{\partial q_r}\ddot{q}_r + \frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}\ddot{q}_r\alpha_j + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\ddot{q}_r\right. \\ &\quad \left. -2\ddot{q}_k\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r + \frac{\partial \xi_0}{\partial \dot{q}_r}\alpha_r\right)\right] - A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t}\ddot{q}_r + 2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial q_j}\ddot{q}_r\dot{q}_j + \frac{\partial \xi_0}{\partial q_r}\ddot{q}_r\right. \\ &\quad \left. + \frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}\ddot{q}_r\alpha_j + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\ddot{q}_r\right) + X^{(0)}(A_{sk})\ddot{q}_k \\ &\quad + \left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \frac{\partial \xi_0}{\partial \dot{q}_r}\dot{q}_k\right)\ddot{q}_r\frac{\partial(B_s - Q_s - P_s)}{\partial \dot{q}_k} + C(t, \boldsymbol{q}, \dot{\boldsymbol{q}}), \\ &\quad (s, k, r, j = 1, \ldots, n),\end{aligned} \tag{17}$$

where $C(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$ is the summation of items in which $\ddot{q}_s$ are not included. For the same reason,

$$\begin{aligned}X^{(2)}(F_s)\,|_{F_s=0} = {} & A_{sk}\left[2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t}\alpha_r + 2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial q_j}\alpha_r\dot{q}_j + \frac{\partial \xi_k}{\partial q_r}\alpha_r + \frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}\alpha_r\alpha_j + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\alpha_r\right. \\ & \left. -2\alpha_k\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r + \frac{\partial \xi_0}{\partial \dot{q}_r}\alpha_r\right)\right] - A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t}\alpha_r + 2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial q_j}\alpha_r\dot{q}_j + \frac{\partial \xi_0}{\partial q_r}\alpha_r\right. \\ & \left. + \frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}\alpha_r\alpha_j + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\alpha_r\right) + X^{(0)}(A_{sk})\alpha_k \\ & + \left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \frac{\partial \xi_0}{\partial \dot{q}_r}\dot{q}_k\right)\alpha_r\frac{\partial(B_s - Q_s - P_s)}{\partial \dot{q}_k} + C(t, \boldsymbol{q}, \dot{\boldsymbol{q}}),\end{aligned} \tag{18}$$

where $\alpha_s = -A^{sk}(B_k - Q_k - P_k)$.

Letting Eq.(18) be subtracted from Eq.(17), we can obtain

$$\begin{aligned}X^{(2)}(F_s) - X^{(2)}(F_s)\,|_{F_s=0} = {} & A_{sk}\left(2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial q_j}\dot{q}_j + \frac{\partial \xi_k}{\partial q_r} + \frac{\partial^2 \xi_k}{\partial \dot{q}_r \partial \dot{q}_j}\alpha_j + \frac{\partial \xi_k}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)(\ddot{q}_r - \alpha_r) \\ & - 2A_{sk}(\ddot{q}_k - \alpha_k)\left(\frac{\partial \xi_0}{\partial t} + \frac{\partial \xi_0}{\partial q_r}\dot{q}_r + \frac{\partial \xi_0}{\partial \dot{q}_r}\alpha_r\right) \\ & - A_{sk}\dot{q}_k\left(2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial t} + 2\frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial q_j}\dot{q}_j + \frac{\partial \xi_0}{\partial q_r} + \frac{\partial^2 \xi_0}{\partial \dot{q}_r \partial \dot{q}_j}\alpha_j + \frac{\partial \xi_0}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_r}\right)(\ddot{q}_r - \alpha_r) \\ & + X^{(0)}(A_{sk})(\ddot{q}_k - \alpha_k) + \left(\frac{\partial \xi_k}{\partial \dot{q}_r} - \frac{\partial \xi_0}{\partial \dot{q}_r}\dot{q}_k\right)(\ddot{q}_r - \alpha_r)\frac{\partial(B_k - Q_k - P_k)}{\partial \dot{q}_k}, \\ & (s, k, j, r = 1, \ldots, n).\end{aligned} \tag{19}$$

Since

$$\ddot{q}_k - \alpha_k = \ddot{q}_k + A^{kl}B_l = A^{kl}(A_{lm}\dot{q}_{\mathrm{m}} + B_l - Q_l - P_l) = A^{kl}F_l, \tag{20}$$

we have

$$\begin{aligned}X^{(2)}(F_s)-X^{(2)}(F_s)\,|_{F_s=0}\;=\;&\Bigg[A_{sk}\left(2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial t}+2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial q_j}\dot{q}_j+\frac{\partial\xi_k}{\partial q_r}+\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial\dot{q}_j}\alpha_j+\frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\right)A^{rl}\\&-A_{sk}\dot{q}_k\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial t}+2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial q_j}\dot{q}_j+\frac{\partial\xi_0}{\partial q_r}+\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial\dot{q}_j}\alpha_j+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\right)A^{rl}\\&-2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r+\frac{\partial\xi_0}{\partial\dot{q}_r}\alpha_r\right)\\&+X^{(0)}(A_{sk})A^{kl}+\left(\frac{\partial\xi_k}{\partial\dot{q}_r}-\frac{\partial\xi_0}{\partial\dot{q}_r}\dot{q}_k\right)A^{rl}\frac{\partial(B_k-Q_k-P_k)}{\partial\dot{q}_k}\Bigg]F_l\\=\;&\beta_s^lF_l,\\&(s,k,j,r,l=1,\dots,n),\end{aligned}\tag{21}$$

where $\delta_s^l=A_{sk}A^{kl}$.

If Eq.(7) is Lie symmetric under the infinitesimal transformation Eq.(12), then

$$X^{(2)}(F_s)\,|_{F_s=0}\;=0.\tag{22}$$

From Eqs.(21) and (13), we have $\Gamma_s^l=\beta_s^l$. If Eq.(7) is conformal invariant under the infinitesimal transformation Eq.(12), from Eq.(21), we get

$$(\Gamma_s^l-\beta_s^l)(F_l)=X^{(2)}(F_s)\,|_{F_s=0}\;,\quad(s,l=1,\dots,n),\tag{23}$$

then the necessary and sufficient condition that the conformal invariance would be the Lie symmetry of the system is

$$\Gamma_s^l=\beta_s^l.\tag{24}$$

Proposition 1 Under the infinitesimal transformation Eq.(12), the necessary and sufficient condition that the conformal invariance would be the Lie symmetry of Eq.(7) is the conformal factor

$$\begin{aligned}\Gamma_s^l=\;&A_{sk}\left(2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial t}+2\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial q_j}\dot{q}_j+\frac{\partial\xi_k}{\partial q_r}+\frac{\partial^2\xi_k}{\partial\dot{q}_r\partial\dot{q}_j}\alpha_j+\frac{\partial\xi_k}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\right)A^{rl}\\&-A_{sk}\dot{q}_k\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial t}+2\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial q_j}\dot{q}_j+\frac{\partial\xi_0}{\partial q_r}+\frac{\partial^2\xi_0}{\partial\dot{q}_r\partial\dot{q}_j}\alpha_j+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_r}\right)A^{rl}\\&-2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r+\frac{\partial\xi_0}{\partial\dot{q}_r}\alpha_r\right)+X^{(0)}(A_{sk})A^{kl}+(\frac{\partial\xi_k}{\partial\dot{q}_r}-\frac{\partial\xi_0}{\partial\dot{q}_r}\dot{q}_k)A^{rl}\frac{\partial(B_k-Q_k-P_k)}{\partial\dot{q}_k},\\&(s,k,j,r,l=1,\dots,n).\end{aligned}\tag{25}$$

Inference If Eq.(10) can be expressed in the standard form

$$F_s\equiv\ddot{q}_s-\alpha_s(t,\boldsymbol{q},\dot{\boldsymbol{q}})=0,\quad(s=1,\dots,n),\tag{26}$$

under the infinitesimal transformation Eq.(12), the necessary and sufficient condition that the conformal invariance would be the Lie symmetry of Eq.(26) is the conformal factor satisfying

$$\begin{aligned}\Gamma_s^l=\;&2\frac{\partial^2\xi_s}{\partial\dot{q}_l\partial t}+2\frac{\partial^2\xi_s}{\partial\dot{q}_l\partial q_j}\dot{q}_j+\frac{\partial\xi_s}{\partial q_l}+\frac{\partial^2\xi_s}{\partial\dot{q}_l\partial\dot{q}_j}\alpha_j+\frac{\partial\xi_s}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_l}\\&-\dot{q}_s\left(2\frac{\partial^2\xi_0}{\partial\dot{q}_l\partial t}+2\frac{\partial^2\xi_0}{\partial\dot{q}_l\partial q_j}\dot{q}_j+\frac{\partial\xi_0}{\partial q_l}+\frac{\partial^2\xi_0}{\partial\dot{q}_l\partial\dot{q}_j}\alpha_j+\frac{\partial\xi_0}{\partial\dot{q}_j}\frac{\partial\alpha_j}{\partial\dot{q}_l}\right)\\&-2\delta_s^l\left(\frac{\partial\xi_0}{\partial t}+\frac{\partial\xi_0}{\partial q_r}\dot{q}_r+\frac{\partial\xi_0}{\partial\dot{q}_r}\alpha_r\right)-\left(\frac{\partial\xi_k}{\partial\dot{q}_l}-\frac{\partial\xi_0}{\partial\dot{q}_l}\dot{q}_k\right)\frac{\partial\alpha_s}{\partial\dot{q}_k}.\end{aligned}\tag{27}$$

Comparing Eq.(10) with Eq.(26), we have $A_{sk} = \delta_{sk}$, $B_s - Q_s - P_s = -\alpha_s(t, \boldsymbol{q}, \dot{\boldsymbol{q}})$; using Eq.(25), we can easily obtain Eq.(27).

4. The generalized Hojman conserved quantities of conformal mechanico-electrical systems

The conformal invariance of mechanico-electrical systems can lead to a generalized Hojman conserved quantity under certain conditions.

Proposition 2 For mechanico-electrical systems, if generators ξ_0, ξ_s under the infinitesimal transformations Eq.(12) satisfy Eq.(22) and the conformal factor Eq.(25) or Eq.(27), and also satisfy the following equations

$$\frac{\partial \alpha_s}{\partial \dot{q}_s} + \frac{\bar{\mathrm{d}}}{\mathrm{d}t} \ln \mu = 0, \tag{28}$$

then the generalized Hojman conserved quantities can be induced from the systems

$$\begin{aligned} I_{\mathrm{H}} = & \frac{1}{\mu}\frac{\partial}{\partial t}(\mu \xi_0) + \frac{1}{\mu}\frac{\partial}{\partial q_s}(\mu \xi_s) \\ & + \frac{1}{\mu}\frac{\partial}{\partial \dot{q}_s}[\mu(\dot{\xi}_s - \dot{q}_s \dot{\xi}_0)] - \dot{\xi}_0 \\ = & \text{ const.}. \end{aligned} \tag{29}$$

The proof of Proposition 2 is the same as that in Ref.[36].

5. An illustrative example

Capacitor C is shown in the following Fig.1. It is an electrode plank (unit mass $m = 1$) which is

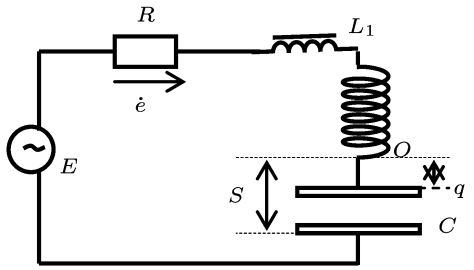


Fig.1. An electrocircuit of a capacitor.

hung on a spring with stiffness k. The plank is vibrating in the vertical direction under the action of gravity, elastic and electric field force. L_1 ($L_1 = 1$) is the inductance, R is the resistance, S is the distance, $E = \dot{e}$ is the electromotive force. Let us study the conformal invariance and the generalized Hojman conserved quantities.

For this system, we choose the position of equilibrium (point O) of the active plank of the capacitor as the origin of the coordinate, and the active plank described by generalized coordinate q. The electrodynamical part is described by the generalized electric quantity e. We denote the kinetic energy of the active plank of the capacitor by $T = \dot{q}^2/2$, by the magnetic energy of capacitor $W_{\mathrm{m}} = \dot{e}^2/2$, by the potential of spring $V = kq^2/2$, and by the electric energy of the capacitor $W_{\mathrm{e}} = Ae^2/2$; then, the Lagrange function of the system is

$$L = \frac{1}{2}\dot{q}^2 + \frac{1}{2}\dot{e}^2 - \frac{1}{2}kq^2 - \frac{1}{2}Ae^2,$$

where A is constant. The dissipation function can be written as

$$\psi = \psi_{\mathrm{e}} + \psi_{\mathrm{m}} = \frac{1}{2}R\dot{e}^2 + kq\dot{q}.$$

The voltage of the capacitor can be written as $u_r = R\dot{e}$. We denote the space coordinates and general electricity in unified generalized coordinates q_s, then the Lagrange function can be written as

$$L = \frac{1}{2}\dot{q}_1^2 + \frac{1}{2}\dot{q}_2^2 - \frac{1}{2}kq_1^2 - \frac{1}{2}Aq_2^2, \tag{30}$$

the dissipation function can be written as

$$\psi = \psi_{\mathrm{e}} + \psi_{\mathrm{m}} = \frac{1}{2}R\dot{q}_2^2 + kq_1\dot{q}_1, \tag{31}$$

and the voltage of the capacitor is written as

$$u_r = R\dot{q}_2. \tag{32}$$

Let us study the conformal invariance and the generalized Hojman conserved quantities.

The differential equations of motion are

$$\begin{aligned} F_s = & \begin{pmatrix} \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{q}_1} - \dfrac{\partial L}{\partial q_1} + \dfrac{\partial \psi}{\partial \dot{q}_1} = Q_1 \\ \dfrac{\mathrm{d}}{\mathrm{d}t}\dfrac{\partial L}{\partial \dot{q}_2} - \dfrac{\partial L}{\partial q_2} + \dfrac{\partial \psi}{\partial \dot{q}_2} = Q_2 \end{pmatrix} \\ = & \begin{pmatrix} F_1 = \ddot{q}_1 \\ F_2 = \ddot{q}_2 - Aq_2 \end{pmatrix}. \end{aligned} \tag{33}$$

Furthermore, we have

$$X^{(2)}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \left(\xi_0\frac{\partial}{\partial t} + \xi_s\frac{\partial}{\partial q_s} + \dot{\xi}_s\frac{\partial}{\partial \dot{q}_s} + \ddot{\xi}_s\frac{\partial}{\partial \ddot{q}_s}\right) \times \begin{pmatrix} F_1 = \ddot{q}_1 \\ F_2 = \ddot{q}_2 - Aq_2 \end{pmatrix}.$$

From Eq.(22), we can obtain the equations

$$\begin{aligned} &\ddot{\xi}_1 = 0, \\ &\ddot{\xi}_2 - A\xi_2 = 0. \end{aligned} \tag{34}$$

Taking the infinitesimal transformation generators as

$$\xi_0 = 0, \quad \xi_1 = q_1, \quad \xi_2 = q_2, \tag{35}$$

the equations in Eq.(34) exist, and we have

$$\begin{aligned} X^{(2)}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} &= \begin{pmatrix} \ddot{\xi}_1 \\ \ddot{\xi}_2 - A\xi_2 \end{pmatrix} = \begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 - Aq_2 \end{pmatrix} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 - Aq_2 \end{pmatrix}. \end{aligned} \tag{36}$$

Therefore, the conformal factor is

$$\Gamma = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \tag{37}$$

Utilizing Eq.(25), we can also obtain

$$\begin{aligned} \Gamma_s^l &= \frac{\partial \xi_s}{\partial q_l} + \frac{\partial \xi_s}{\partial \dot{q}_j}\frac{\partial \alpha_j}{\partial \dot{q}_l} - \frac{\partial \xi_k}{\partial \dot{q}_l}\frac{\partial \alpha_s}{\partial \dot{q}_k} \\ &= \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{38}$$

and the result is the same as Eq.(37). Therefore the determining equations of conformal invariance of Eq.(33) are

$$X^{(2)}\begin{pmatrix} F_1 \\ F_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} \ddot{q}_1 \\ \ddot{q}_2 - Aq_2 \end{pmatrix}. \tag{39}$$

Here Eq.(35) has conformal invariance and Lie symmetry simultaneously. Substituting Eq.(35) into Eq.(22) and (28), we obtain

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}t}\ln\mu = 0, \tag{40}$$

when

$$\mu = \exp \dot{q}_1^2, \tag{41}$$

and consequently we obtain the Hojman conserved quantities

$$I_{\mathrm{H}} = 2\dot{q}_1^2 + 4. \tag{42}$$

6. Conclusion

When mechanico-electrical systems have conformal invariance by the action of infinitesimal transformations, we can obtain the determining equations and conformal factor. The existence of the conformal factor is the sufficient and necessary condition of conformal invariance and Lie symmetry simultaneously, and we can also induce generalized Hojman conserved quantities under some conditions.

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