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Conformal invariance and Noether symmetry, Lie symmetry of holonomic mechanical systems in event space*

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This paper is devoted to studying the conformal invariance and Noether symmetry and Lie symmetry of a holonomic mechanical system in event space. The definition of the conformal invariance and the corresponding conformal factors of the holonomic system in event space are given. By investigating the relation between the conformal invariance and the Noether symmetry and the Lie symmetry, expressions of conformal factors of the system under these circumstances are obtained, and the Noether conserved quantity and the Hojman conserved quantity directly derived from the conformal invariance are given. Two examples are given to illustrate the application of the results.

Keywords: holonomic system, conformal invariance, symmetry, event space
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1. Introduction

It is of importance not only in geometry but also in mechanics to study the dynamics of mechanical systems in event space. Since the generalized coordinates and the time are on an equal footing in event space, one can choose parameters flexibly to establish the equations in a simpler form, and then by using parametric equations in event space not only can the dynamical equations in the usual configuration space be obtained, but the energy integral can also be obtained directly. In 1960, Synge[1] studied the dynamics of holonomic conservative systems in event space. In 1983, Rumyantsev[2] established parametric equations with multipliers in event space for nonholonomic conservative systems. Mei[3,4] studied the variational principle, the parametric equations and the method of their integration for nonholonomic nonconservative systems.

Research on symmetries and conserved quantities of dynamical systems is a modern developmental aspect of analytical mechanics.[5−23] In recent years, some important results on the study of symmetries and conserved quantities of mechanical systems in the event space have been obtained.[24−28] Mei[22,23] studied the Noether symmetry, the Lie symmetry and a form invariance of holonomic or nonholonomic systems in event space. Li *et al*[24] studied the Noether theory of systems with unilateral constraints in event space. Xu *et al*[25] gave the symmetries and the Hojman conserved quantities of holonomic systems in event space. Zhang[26] presented a type of Hojman high-order adiabatic invariants of holonomic systems in event space. Zhang[27] gave the Hojman conserved quantities of Birkhoffian systems in event space. Jia *et al*[28] studied the Mei symmetry and Mei conserved quantity of nonholonomic systems of non-Chetaev's type in event space. In 1997, Galiullin *et al*[29] studied a conformal invariance of a Birkhoffian system. In fact, a conformal invariance means that each equation of a system after undergoing infinitesimal transformations is identical to the one multiplied with a conformal factor. The conformal invariance has proved to be a useful tool in quantum field theory, statistical mechanics, geometric probability and others. In the present paper, the conformal invariance and the Noether symmetry and the Lie symmetry of holonomic mechanical systems in the event space are further studied, the relations between the conformal invariance and Noether symmetry and Lie symmetry are revealed, and the expressions of corresponding conformal factors are given. Noether conserved quantities and Hojman conserved quantities directly derived from the conformal invariance are discussed.

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2. Equations of motion of holonomic mechanical systems in event space

We consider a mechanical system whose configuration is determined by n generalized coordinates $q_s\,(s=1,\dots,n)$. Now we construct an $(n+1)$-dimensional event space. The coordinates of space points are $q_s\,(s=1,\dots,n)$ and t, which may be regarded as totally independent variables. Introducing the notation

$$x_1=t,\quad x_{s+1}=q_s\,,\quad (s=1,\dots,n)\,,\tag{1}$$

then, all the variables $x_\alpha\,(\alpha=1,\dots,n+1)$ may be given as a function of the parameter τ each. Let $x_\alpha=x_\alpha\,(\tau)$ be some curves of class C^2 such that

$$\frac{\mathrm{d}x_\alpha}{\mathrm{d}\tau}=x'_\alpha,\quad (\alpha=1,\dots,n+1)\tag{2}$$

are not all zero at the same time, and we have

$$\dot{x}_\alpha=\frac{\mathrm{d}x_\alpha}{\mathrm{d}t}=\frac{x'_\alpha}{x'_1},\quad (\alpha=1,\dots,n+1)\,.\tag{3}$$

Suppose the Lagrangian of the system in the configuration space is $L=L\,(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, the non-potential generalized forces are $Q''_s=Q''_s\,(t,\boldsymbol{q},\dot{\boldsymbol{q}})$, then the Lagrangian in the event space is

$$\Lambda\,(x_\alpha,x'_\alpha)=x'_1L\left(x_\alpha,\frac{x'_2}{x'_1},\dots,\frac{x'_{n+1}}{x'_1}\right),\tag{4}$$

and non-potential forces are

$$\begin{aligned}&P_1=-Q''_sx'_{s+1},\\&P_{s+1}=x'_1Q''_s\left(x_\alpha,\frac{x'_2}{x'_1},\dots,\frac{x'_{n+1}}{x'_1}\right).\end{aligned}\tag{5}$$

The differential equations of motion for the holonomic mechanical system in event space can be written as

$$E_\alpha\,(\Lambda)=P_\alpha,\quad (\alpha=1,\dots,n+1)\,,\tag{6}$$

where

$$E_\alpha=\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial}{\partial x'_\alpha}-\frac{\partial}{\partial x_\alpha}.$$

Obviously, the equations in Eq.(6) are not independent. The first equation of Eq.(6) is the result of the last n equations. The last n equations of Eq.(6) can be written as

$$F_{s+1}\equiv h_{s+1,k+1}x''_{k+1}+R_{s+1}=0,\quad (s=1,\dots,n)\,,\tag{7}$$

where

$$h_{s+1,k+1}=\frac{\partial^2\Lambda}{\partial x'_{s+1}\partial x'_{k+1}},$$

$$\begin{aligned}R_{s+1}=&\frac{\partial^2\Lambda}{\partial x'_{s+1}\partial x'_1}x''_1+\frac{\partial^2\Lambda}{\partial x'_{s+1}\partial x_\beta}x'_\beta\\&-\frac{\partial\Lambda}{\partial x_{s+1}}-P_{s+1}.\end{aligned}\tag{8}$$

Supposing that the system is nonsingular, i.e. $\det\,(h_{s+1,k+1})\neq 0$, we may obtain x''_{s+1} from Eq.(7) as

$$x''_{s+1}=h_{s+1}\,(x_\alpha,x'_\alpha,x''_1)\,,\quad (s=1,\dots,n)\,.\tag{9}$$

3. Conformal invariance and Noether symmetry of holonomic mechanical systems in event space

We introduce the infinitesimal transformations of a group with respect to the coordinates x_α and parameter τ in event space, such as

$$\begin{aligned}&\tau^*=\tau,\quad x^*_\alpha\,(\tau^*)=x_\alpha\,(\tau)+\Delta x_\alpha,\\&(\alpha=1,\dots,n+1)\,.\end{aligned}\tag{10}$$

Their expanded forms are

$$\begin{aligned}&\tau^*=\tau,\quad x^*_\alpha\,(\tau^*)=x_\alpha\,(\tau)+\varepsilon\xi_\alpha\left(x_\beta,x'_\beta\right),\\&(\alpha=1,\dots,n+1)\,,\end{aligned}\tag{11}$$

where ε is an infinitesimal parameter, and ξ_α are the infinitesimal generators. Below are introduced the infinitesimal generator vector

$$X^{(0)}=\xi_\alpha\frac{\partial}{\partial x_\alpha},\tag{12}$$

its first extended vector

$$X^{(1)}=X^{(0)}+\xi'_\alpha\frac{\partial}{\partial x'_\alpha},\tag{13}$$

and its second extended vector

$$X^{(2)}=X^{(1)}+\xi''_\alpha\frac{\partial}{\partial x''_\alpha}.\tag{14}$$

Definition 1 For the holonomic mechanical system in event space, if there is a nonsingular matrix $\left(\boldsymbol{C}^\beta_\alpha\right)$ such that the following equations come into existence:

$$\begin{aligned}&X^{(2)}\left[E_\alpha\,(\Lambda)-P_\alpha\right]=\boldsymbol{C}^\beta_\alpha\left[E_\beta\,(\Lambda)-P_\beta\right],\\&(\alpha,\beta=1,\dots,n+1)\,,\end{aligned}\tag{15}$$

then this kind of invariance is called the conformal invariance of Eq.(6) under the infinitesimal transformations (11), and the matrix $\left(\boldsymbol{C}_{\alpha}^{\beta}\right)$ is the corresponding conformal factors.

Equation (15) can be called the determining equation of conformal invariance of the holonomic mechanical system (6) in event space.

For holonomic mechanical systems in event space, the Noether theory points out[22,23] that if there exists a gauge function $G_{\mathrm{N}}=G_{\mathrm{N}}\left(x_{\beta},x'_{\beta}\right)$ such that the generators ξ_{α} of the infinitesimal transformations (11) satisfy the following Noether identity:

$$X^{(1)}\left(\Lambda\right)+P_{\alpha}\xi_{\alpha}+G'_{\mathrm{N}}=0, \tag{16}$$

then the corresponding invariance is the Noether symmetry of the system.

Another method to identify the Noether symmetry is to find a solution for the Killing equations. The Killing equations have the form[22]

$$\frac{\partial\Lambda}{\partial x_{\alpha}}\xi_{\alpha}+\frac{\partial\Lambda}{\partial x'_{\alpha}}\frac{\partial\xi_{\alpha}}{\partial x_{\beta}}x'_{\beta}+P_{\alpha}\xi_{\alpha}+\frac{\partial G_{\mathrm{N}}}{\partial x_{\alpha}}x'_{\alpha}=0, \tag{17}$$

$$\frac{\partial\Lambda}{\partial x'_{\alpha}}\frac{\partial\xi_{\alpha}}{\partial x'_{\beta}}+\frac{\partial G_{\mathrm{N}}}{\partial x'_{\beta}}=0,\quad(\beta=1,\dots,n+1). \tag{18}$$

In regard to the relation between the conformal invariance and the Noether symmetry of the holonomic mechanical system in event space, we have the following proposition.

Proposition 1 For the infinitesimal transformations (11), which correspond to a Noether symmetry of the holonomic mechanical system (6) in event space, and the non-potential generalized forces P_{α} satisfying the conditions

$$\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial P_{\beta}}{\partial x'_{\alpha}}-\frac{\partial P_{\beta}}{\partial x_{\alpha}}+\frac{\partial P_{\alpha}}{\partial x_{\beta}}\right)\xi_{\beta}+\left(\frac{\partial P_{\alpha}}{\partial x'_{\beta}}+\frac{\partial P_{\beta}}{\partial x'_{\alpha}}\right)\xi'_{\beta}=0,\quad(\alpha=1,\dots,n+1), \tag{19}$$

if and only if the conformal factors $\left(\boldsymbol{C}_{\alpha}^{\beta}\right)$ are

$$C_{\alpha}^{\beta}=E_{\alpha}\left(\xi_{\beta}\right),\quad(\alpha,\beta=1,\dots,n+1), \tag{20}$$

i.e. the determining equation (15) of conformal invariance can be written as

$$X^{(2)}\left[E_{\alpha}\left(\Lambda\right)-P_{\alpha}\right]=E_{\alpha}\left(\xi_{\beta}\right)\left[E_{\beta}\left(\Lambda\right)-P_{\beta}\right],\quad(\alpha=1,\dots,n+1), \tag{21}$$

then the infinitesimal transformations (11) are also related to the conformal invariance of the system.

Proof According to a relation given by Mei,[23] we have

$$X^{(2)}\left[E_{\alpha}\left(\Lambda\right)\right]=E_{\alpha}\left[X^{(1)}\left(\Lambda\right)\right]-\frac{\mathrm{d}^{2}}{\mathrm{d}\tau^{2}}\left(\frac{\partial\Lambda}{\partial x'_{\beta}}\frac{\partial\xi_{\beta}}{\partial x'_{\alpha}}\right)+E_{\alpha}\left(\xi_{\beta}\right)E_{\beta}\left(\Lambda\right)+\frac{\partial\xi_{\beta}}{\partial x'_{\alpha}}\frac{\mathrm{d}}{\mathrm{d}\tau}E_{\beta}\left(\Lambda\right). \tag{22}$$

Hence we obtain

$$\begin{aligned}X^{(2)}\left[E_{\alpha}\left(\Lambda\right)-P_{\alpha}\right]=&E_{\alpha}\left[X^{(1)}\left(\Lambda\right)+P_{\beta}\xi_{\beta}+G'_{\mathrm{N}}\right]-E_{\alpha}\left(P_{\beta}\xi_{\beta}\right)-E_{\alpha}\left(G'_{\mathrm{N}}\right)-X^{(1)}\left(P_{\alpha}\right)\\&-\frac{\mathrm{d}^{2}}{\mathrm{d}\tau^{2}}\left(\frac{\partial\Lambda}{\partial x'_{\beta}}\frac{\partial\xi_{\beta}}{\partial x'_{\alpha}}\right)+E_{\alpha}\left(\xi_{\beta}\right)E_{\beta}\left(\Lambda\right)+\frac{\partial\xi_{\beta}}{\partial x'_{\alpha}}\frac{\mathrm{d}}{\mathrm{d}\tau}E_{\beta}\left(\Lambda\right).\end{aligned} \tag{23}$$

Since

$$E_{\alpha}\left(P_{\beta}\xi_{\beta}\right)=E_{\alpha}\left(P_{\beta}\right)\xi_{\beta}+P_{\beta}E_{\alpha}\left(\xi_{\beta}\right)+\frac{\partial P_{\beta}}{\partial x'_{\alpha}}\xi'_{\beta}+\frac{\mathrm{d}P_{\beta}}{\mathrm{d}\tau}\frac{\partial\xi_{\beta}}{\partial x'_{\alpha}} \tag{24}$$

and

$$E_{\alpha}\left(G'_{\mathrm{N}}\right)=\frac{\mathrm{d}^{2}}{\mathrm{d}\tau^{2}}\left(\frac{\partial G_{\mathrm{N}}}{\partial x'_{\alpha}}\right), \tag{25}$$

substituting Eqs.(24) and (25) into Eq.(23), we have

$$\begin{aligned} X^{(2)}\left[E_\alpha\left(\Lambda\right)-P_\alpha\right] &= E_\alpha\left[X^{(1)}\left(\Lambda\right)+P_\beta\xi_\beta+G'_{\mathrm{N}}\right]-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{\partial\Lambda}{\partial x'_\beta}\frac{\partial\xi_\beta}{\partial x'_\alpha}+\frac{\partial G_{\mathrm{N}}}{\partial x'_\alpha}\right)\\ &\quad+E_\alpha\left(\xi_\beta\right)\left[E_\beta\left(\Lambda\right)-P_\beta\right]-\xi_\beta E_\alpha\left(P_\beta\right)-\frac{\partial P_\beta}{\partial x'_\alpha}\xi'_\beta-\frac{\mathrm{d}P_\beta}{\mathrm{d}\tau}\frac{\partial\xi_\beta}{\partial x'_\alpha}-X^{(1)}\left(P_\alpha\right)+\frac{\partial\xi_\beta}{\partial x'_\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}E_\beta\left(\Lambda\right)\\ &= E_\alpha\left[X^{(1)}\left(\Lambda\right)+P_\beta\xi_\beta+G'_{\mathrm{N}}\right]-\frac{\mathrm{d}^2}{\mathrm{d}\tau^2}\left(\frac{\partial\Lambda}{\partial x'_\beta}\frac{\partial\xi_\beta}{\partial x'_\alpha}+\frac{\partial G_{\mathrm{N}}}{\partial x'_\alpha}\right)+E_\alpha\left(\xi_\beta\right)\left[E_\beta\left(\Lambda\right)-P_\beta\right]\\ &\quad-\xi'_\beta\left(\frac{\partial P_\alpha}{\partial x'_\beta}+\frac{\partial P_\beta}{\partial x'_\alpha}\right)-\xi_\beta\left(\frac{\mathrm{d}}{\mathrm{d}\tau}\frac{\partial P_\beta}{\partial x'_\alpha}-\frac{\partial P_\beta}{\partial x_\alpha}+\frac{\partial P_\alpha}{\partial x_\beta}\right)+\frac{\partial\xi_\beta}{\partial x'_\alpha}\frac{\mathrm{d}}{\mathrm{d}\tau}\left[E_\beta\left(\Lambda\right)-P_\beta\right]. \end{aligned} \tag{26}$$

Equation (26) is an important relation of the holonomic mechanical system in event space under the infinitesimal transformations (11). For the system (6), if the transformations (11) are related to the Noether symmetry, i.e. the transformations (11) satisfy the Noether identity (16), the Killing equations (18), and the conditions (19), then from Eq.(26) we may obtain

$$X^{(2)}\left[E_\alpha\left(\Lambda\right)-P_\alpha\right]=E_\alpha\left(\xi_\beta\right)\left[E_\beta\left(\Lambda\right)-P_\beta\right],\quad\left(\alpha=1,\ldots,n+1\right). \tag{21}$$

According to definition 1, the infinitesimal transformations (11) correspond to the conformal invariance of the system.

By means of proposition 1 and the Noether theorem,[22] we can easily obtain the following proposition.

Proposition 2 For the holonomic mechanical system (6) in event space, if the generators ξ_α of the infinitesimal transformations (11) satisfy the Noether identity (16), the Killing equations (18) and the conditions (19), then the conformal invariance of the system can directly lead to a Noether conserved quantity, such as

$$I_{\mathrm{N}}=\frac{\partial\Lambda}{\partial x'_\alpha}\xi_\alpha+G_{\mathrm{N}}=\mathrm{const.} \tag{27}$$

Proposition 2 reveals the relation between the conformal invariance and the Noether conserved quantity of the holonomic mechanical system in event space.

Example 1 Consider a holonomic mechanical system whose Lagrangian and non-potential generalized force are

$$L=\frac{1}{2}t^2\dot{q}^2,\quad Q=-t^2q^5. \tag{28}$$

Now we come to study the conformal invariance and the Noether conserved quantity of the system in event space.

Let

$$x_1=t,\quad x_2=q, \tag{29}$$

then we have

$$\begin{aligned} &\Lambda=\frac{1}{2}x_1^2x'_1\left(\frac{x'_2}{x'_1}\right)^2,\\ &P_1=x_1^2x_2^5x'_2,\quad P_2=-x_1^2x_2^5x'_1. \end{aligned} \tag{30}$$

The Noether identity (16) gives

$$\begin{aligned} &x_1x'_1\left(\frac{x'_2}{x'_1}\right)^2\xi_1-\frac{1}{2}x_1^2\left(\frac{x'_2}{x'_1}\right)^2\xi'_1+x_1^2\left(\frac{x'_2}{x'_1}\right)\xi'_2\\ &+x_1^2x_2^5x'_2\xi_1-x_1^2x_2^5x'_1\xi_2+G'_{\mathrm{N}}=0. \end{aligned} \tag{31}$$

Take the infinitesimal generators

$$\xi_1=-2x_1,\quad\xi_2=x_2 \tag{32}$$

and substitute expression (32) into Eq.(31), then we will obtain

$$G_{\mathrm{N}}=\frac{1}{3}x_1^3x_2^6. \tag{33}$$

Hence the generators (32) correspond to the Noether symmetry of the system. Substituting expressions (32) and (30) into Eq.(19), we can easily verify that the conditions (19) are existent, and therefore, the generators (32) also correspond to the conformal invariance of the system. The conformal factors (20) give

$$\boldsymbol{C}_\alpha^\beta=\begin{pmatrix}2&0\\0&-1\end{pmatrix}, \tag{34}$$

and the determining equations (21) have the following

form:

$$X^{(2)}\begin{pmatrix} E_1(\Lambda)-P_1 \\ E_2(\Lambda)-P_2 \end{pmatrix} = \begin{pmatrix} 2 & 0 \\ 0 & -1 \end{pmatrix}\begin{pmatrix} E_1(\Lambda)-P_1 \\ E_2(\Lambda)-P_2 \end{pmatrix}, \tag{35}$$

where

$$X^{(2)} = -2x_1\frac{\partial}{\partial x_1} + x_2\frac{\partial}{\partial x_2}. \tag{36}$$

Substituting the generators (32) and the gauge function (33) into Eq.(27), we have

$$I_{\mathrm{N}} = \frac{1}{3}x_1^3x_2^6 + x_1^2x_2\left(\frac{x_2'}{x_1'}\right) + x_1^3\left(\frac{x_2'}{x_1'}\right)^2 = \text{const.} \tag{37}$$

Equation (37) is a Noether conserved quantity directly derived from the conformal invariance (32).

4. Conformal invariance and Lie symmetry of holonomic mechanical systems in event space

Take the infinitesimal transformations in the following form:

$$\tau^* = \tau, \quad x_1^* = x_1, \quad x_{s+1}^* = x_{s+1} + \varepsilon\xi_{s+1}(x_\beta), \quad (s=1,\dots,n). \tag{38}$$

Definition 2 For the holonomic mechanical system in event space, if there is a nonsingular matrix $\left(\boldsymbol{L}_s^j\right)$ such that the following equations come into existence:

$$X^{(2)}(F_{s+1}) = L_s^jF_{j+1}, \quad (s,j=1,\dots,n), \tag{39}$$

then this kind of invariance is called the conformal invariance of Eq.(7) under the infinitesimal transformations (38), and the matrix $\left(\boldsymbol{L}_s^j\right)$ is the corresponding conformal factors.

Equation (39) can be called the determining equations of conformal invariance of the holonomic mechanical system (7) in event space.

A Lie symmetry is an invariance of differential equations under the infinitesimal transformations of groups. The invariance of Eq.(7) of the holonomic mechanical system in event space under infinitesimal transformations (38) leads to the satisfaction of the following determining equations

$$X^{(2)}(F_{s+1})|_{F_{s+1=0}} = 0, \quad (s=1,\dots,n). \tag{40}$$

According to Eqs.(7) and (14), we have

$$\begin{aligned} X^{(2)}(F_{s+1}) &= X^{(1)}(h_{s+1,k+1})\,x_{k+1}'' + h_{s+1,k+1}\xi_{k+1}'' + X^{(1)}(R_{s+1}) \\ &= X^{(1)}(h_{s+1,k+1})\,x_{k+1}'' + h_{s+1,k+1}\left(\frac{\partial\xi_{k+1}}{\partial x_1}x_1'' + \frac{\partial\xi_{k+1}}{\partial x_{l+1}}x_{l+1}'' + \frac{\partial^2\xi_{k+1}}{\partial x_\alpha\partial x_\beta}x_\alpha'x_\beta'\right) \\ &\quad + X^{(1)}(R_{s+1}). \end{aligned} \tag{41}$$

By utilizing Eqs.(7) and (9), we have

$$\begin{aligned} X^{(2)}(F_{s+1})|_{F_{s+1=0}} &= X^{(1)}(h_{s+1,k+1})\,h_{k+1} + h_{s+1,k+1}\left(\frac{\partial\xi_{k+1}}{\partial x_1}x_1'' + \frac{\partial\xi_{k+1}}{\partial x_{l+1}}h_{l+1} + \frac{\partial^2\xi_{k+1}}{\partial x_\alpha\partial x_\beta}x_\alpha'x_\beta'\right) \\ &\quad + X^{(1)}(R_{s+1}). \end{aligned} \tag{42}$$

Hence

$$X^{(2)}(F_{s+1}) - X^{(2)}(F_{s+1})|_{F_{s+1=0}} = X^{(1)}(h_{s+1,k+1})\left(x_{k+1}'' - h_{k+1}\right) + h_{s+1,k+1}\frac{\partial\xi_{k+1}}{\partial x_{l+1}}\left(x_{l+1}'' - h_{l+1}\right). \tag{43}$$

Since

$$x_{k+1}'' - h_{k+1} = x_{k+1}'' + \tilde{h}_{k+1,j+1}R_{j+1} = \tilde{h}_{k+1,j+1}\left(h_{j+1,s+1}x_{s+1}'' + R_{j+1}\right) = \tilde{h}_{k+1,j+1}F_{j+1}, \tag{44}$$

where $\tilde{h}_{k+1,j+1}h_{j+1,s+1} = \delta_{k+1,s+1}$, substituting Eq.(44) into Eq.(43) yields

$$X^{(2)}(F_{s+1}) - X^{(2)}(F_{s+1})|_{F_{s+1=0}} = \left[X^{(1)}(h_{s+1,k+1}) + h_{s+1,l+1}\frac{\partial\xi_{l+1}}{\partial x_{k+1}}\right]\tilde{h}_{k+1,j+1}F_{j+1}. \tag{45}$$

Let

$$\beta_s^j = \left[X^{(1)}\left(h_{s+1,k+1}\right) + h_{s+1,l+1}\frac{\partial \xi_{l+1}}{\partial x_{k+1}}\right]\tilde{h}_{k+1,j+1}, \tag{46}$$

then Eq.(45) can be written as

$$X^{(2)}\left(F_{s+1}\right) - X^{(2)}\left(F_{s+1}\right)|_{F_{s+1=0}} = \beta_s^j F_{j+1}. \tag{47}$$

From Eq.(47), and utilizing definition 2 and the determining equations (40), we can easily obtain the following proposition.

Proposition 3 For the infinitesimal transformation (38), which corresponds to Lie symmetry of the holonomic mechanical system (7) in event space, if and only if the conformal factors $\left(\boldsymbol{L}_s^j\right)$ are

$$L_s^j = \beta_s^j = \left[X^{(1)}\left(h_{s+1,k+1}\right) + h_{s+1,l+1}\frac{\partial \xi_{l+1}}{\partial x_{k+1}}\right]\tilde{h}_{k+1,j+1}, \quad (s,j=1,\ldots,n), \tag{48}$$

i.e. the determining equation (39) of conformal invariance can be written as

$$X^{(2)}\left(F_{s+1}\right) = \left[X^{(1)}\left(h_{s+1,k+1}\right) + h_{s+1,l+1}\frac{\partial \xi_{l+1}}{\partial x_{k+1}}\right]\tilde{h}_{k+1,j+1}F_{j+1}, \quad (s=1,\ldots,n), \tag{49}$$

then the infinitesimal transformations (38) are also related to the conformal invariance of the system.

Proposition 3 reveals the relation between the conformal invariance and the Lie symmetry of the holonomic mechanical system in event space. From proposition 3 and the Hojman theorem,[27] we can easily obtain the following proposition.

Proposition 4 For the holonomic mechanical system (7) in event space, if the infinitesimal transformations (38) are related to the conformal invariance of the system, and there exists a function $\lambda = \lambda\left(x_\alpha, x'_\alpha\right)$ satisfying the following condition

$$\frac{\partial h_{s+1}}{\partial x'_{s+1}} + \frac{\bar{\mathrm{d}}}{\mathrm{d}\tau}\ln\lambda = 0, \tag{50}$$

where

$$\frac{\bar{\mathrm{d}}}{\mathrm{d}\tau} = x'_\alpha\frac{\partial}{\partial x_\alpha} + x''_1\frac{\partial}{\partial x'_1} + h_{s+1}\frac{\partial}{\partial x'_{s+1}}, \tag{51}$$

then the conformal invariance of the system directly leads to the Hojman conserved quantity, such as

$$I_{\mathrm{H}} = \frac{1}{\lambda}\frac{\partial}{\partial x_{s+1}}\left(\lambda\xi_{s+1}\right) + \frac{1}{\lambda}\frac{\partial}{\partial x'_{s+1}}\left(\lambda\xi'_{s+1}\right) = \text{const.} \tag{52}$$

Proposition 4 gives the relation between the conformal invariance and the Hojman conserved quantity of the holonomic mechanical system in event space.

Example 2 The Lagrangian of a mechanical system in event space is

$$\Lambda = \frac{1}{2}x'_1\left(\frac{x'_2}{x'_1}\right)^2\exp\left(-\gamma x_1\right), \quad (\gamma = \text{const.}). \tag{53}$$

Now we study the conformal invariance and the Hojman conserved quantity of the system.

Equation (7) gives

$$\begin{aligned}&\frac{x''_2}{x'_1}\exp\left(-\gamma x_1\right) - \frac{x'_2}{\left(x'_1\right)^2}x''_1\exp\left(-\gamma x_1\right)\\ &-\gamma x'_2\exp\left(-\gamma x_1\right) = 0.\end{aligned} \tag{54}$$

The conformal factors (48) give

$$L_1^1 = \frac{\partial \xi_2}{\partial x_2}. \tag{55}$$

The determining equation (49) gives

$$\begin{aligned}&\frac{1}{x'_1}\exp\left(-\gamma x_1\right)\xi''_2 - \frac{x''_1}{\left(x'_1\right)^2}\exp\left(-\gamma x_1\right)\xi'_2 - \gamma\exp\left(-\gamma x_1\right)\xi'_2\\ &= \frac{\partial \xi_2}{\partial x_2}\left[\frac{x''_2}{x'_1}\exp\left(-\gamma x_1\right) - \frac{x'_2}{\left(x'_1\right)^2}x''_1\exp\left(-\gamma x_1\right) - \gamma x'_2\exp\left(-\gamma x_1\right)\right].\end{aligned} \tag{56}$$

Equation (56) has a solution

$$\xi_2 = x_2. \tag{57}$$

The generator (57) is related to the conformal invariance and the Lie symmetry of the system. Substituting expression (57) into Eq.(54), we obtain the conformal factor

$$L_1^1 = \beta_1^1 = 1. \tag{58}$$

Equation (50) gives

$$\gamma x_1' + \frac{x_1''}{x_1'} + \frac{\bar{\mathrm{d}}}{\mathrm{d}\tau} \ln \lambda = 0. \tag{59}$$

It has a solution

$$\lambda = \frac{1}{x_2'} \exp\left(\gamma x_2 - \frac{x_2'}{x_1'}\right). \tag{60}$$

Substituting expressions (57) and (60) into Eq.(52) yields

$$I_{\mathrm{H}} = \gamma x_2 - \frac{x_2'}{x_1'} = \text{const.} \tag{61}$$

The conserved quantity (61) is a Hojman conserved quantity corresponding to the conformal invariance (57).

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