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Variable coefficient nonlinear systems derived from an atmospheric dynamical system*

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Variable coefficient nonlinear systems, the Korteweg de Vries (KdV), the modified KdV (mKdV) and the nonlinear Schrödinger (NLS) type equations, are derived from the nonlinear inviscid barotropic nondivergent vorticity equation in a beta-plane by means of the multi-scale expansion method in two different ways, with and without the so-called y-average trick. The non-auto-Bäcklund transformations are found to transform the derived variable coefficient equations to the corresponding standard KdV, mKdV and NLS equations. Thus, many possible exact solutions can be obtained by taking advantage of the known solutions of these standard equations. Further, many approximate solutions of the original model are ready to be yielded which might be applied to explain some real atmospheric phenomena, such as atmospheric blocking episodes.

Keywords: nonlinear inviscid barotropic nondivergent vorticity equation, variable coefficient equations, non-auto-Bäcklund transformation

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1. Introduction

Nonlinear atmospheric dynamics is an important and active research topic in atmospheric science.[1−6] In the study of atmospheric and oceanic dynamical systems, the nonlinear inviscid barotropic nondivergent vorticity equation in a beta-plane[2,7]

$$(\partial_t + u\partial_x + v\partial_y)\,(v_x - u_y - \lambda_0^2\psi) + \beta_0\psi_x = 0 \quad (1)$$

is one of the simplest but meaningful models, where $u = -\psi_y$ and $v = \psi_x$. In Eq.(1), ψ is the stream function, u and v are components of velocity, the Rossby parameter β_0 is given by $\beta_0 = (\omega_0/R_0)\cos\phi_0$, in which R_0 is the earth's radius, ω_0 is the angular frequency of the earth's rotation and ϕ_0 is the latitude, $\lambda_0 = f_0/\sqrt{gH}$, f_0 is the Coriolis parameter, g is the gravitational acceleration, H is the average height of the atmosphere.

From Eq.(1), some types of constant coefficient nonlinear systems, such as the Korteweg de Vries (KdV), modified KdV (mKdV), and nonlinear Schrödinger (NLS) equations, have been derived by using some types of different approximations like the long wave approximation and by introducing multiple time and space scales.[2,3] The obtained equations are integrable, and have been substantially investigated. In fact, they have also been derived in a similar way in many other fields, such as plasma physics,[8] condensed matter,[9] oceanic internal waves,[10], the interaction between the Gulf Stream and eddies,[11] equatorial soliton waves,[12] atmospheric intraseasonal oscillation[4] and cyclogenesis,[5] etc. The known results of these deduced equations lead to many useful analytical solutions of the original system (1), and have been well discussed for some actual phenomena; for instance, the solitary Rossby waves found in this way have been widely used to explain the vortex pair blocking observed in the atmosphere (see Refs.[2], [13] and references therein). However, atmospheric phenomena are so complex that solutions derived from the

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constant coefficient equations can only explain some of the simplest and special cases. In many situations, variable coefficient nonlinear equations have been responsible for many kinds of nonlinear phenomena, for instance, the variable coefficient (VC) KdV (VCKdV) equation with conditions given at the inflow site is derived as governing the spatial dynamics in a simplified one-dimensional model for pulse wave propagation through fluid-filled tubes with elastic walls.[14] Therefore, in this paper, we aim to derive some variable coefficient equations so as to provide some routes to describe more true atmospheric phenomena like blockings.

One important observation is that in the usual treatment of deriving either KdV or NLS type equations, one always separates the mean current, assumed to linearly depend on y (namely, latitude) only, from the whole stream and takes zero boundary values. In reality, the mean westerly changes with time though it almost remains the same shear type, which has not yet been considered analytically so far. Consequently, we are motivated to introduce time into the mean flow and boundary conditions to reinvestigate the system (1) following similar procedures of deriving constant coefficient equations but with some fundamental modifications.

Another essential observation is that people are used to utilizing a y-average trick to get rid of the variable y in the final derived equations. Recently, some types of coupled KdV equations have been derived from a two-layer fluid system without the y-average trick.[15,16] Since there are no exact reasons for using the y-average trick in the derivation procedures, in the next section, we therefore write down the whole process of deriving the variable coefficient KdV (VCKdV), mKdV (VCmKdV) and NLS (VCNLS) equations with and without the y-average trick respectively to compare the results obtained by the two different techniques so as to attest that the trick can be removed.

The paper is organized as follows. In Section 2, for comparison, we derive the VCKdV, VCmKdV and VCNLS equations via the multi-scale expansion method with and without the so-called y-average method. In Section 3, we use a simple direct method to find the non-auto-Bäcklund transformations (NABT) which link the variable coefficient equations and the corresponding constant coefficient integrable ones. Thus, many exact solutions of the VC equations can be yielded from the known solutions of the corresponding constant coefficient equations through the NABTs. A theoretical dipole type blocking episode is graphically displayed based on the solution of the VCKdV equation. The last section contains a short summary and discussion.

2. The derivations of the variable coefficient nonlinear systems

In order to obtain some variable coefficient nonlinear systems, we have to assume that the real atmospheric system possesses an only x-independent basic flow. Hence, the stream function can be written in the form of

$$\psi = \psi_0(y,t) + \psi', \tag{2}$$

where ψ_0 is just the mean current with respect to not only the latitude y but also the time t. Due to the assumption (2), the original dynamical system (1) becomes

$$\begin{aligned}&(\partial_t - \psi_{0y}\partial_x)\,\Delta_{\rm h}\psi' + J(\psi',\ \Delta_{\rm h}\psi') - \lambda_0^2\psi'_t\\&+ (\beta_0 + \psi_{0yyy})\psi'_x - (\lambda_0^2 - \partial_y^2)\psi_{0t} = 0,\end{aligned} \tag{3}$$

where $\Delta_{\rm h} = \partial_x^2 + \partial_y^2$ denotes the two dimensional Laplacian operators, $J(a,b) = a_x b_y - a_y b_x$ is the Jacobian operator. For simplicity of notation, we will drop out primes " $'$ " in Eq.(3) and the remaining parts of this paper. In the following three subsections, the details of the derivation of the VCKdV, VCmKdV and VCNLS equations are presented with and without the y-average trick, respectively.

2.1. Derivation of the VCKdV equation

In this case, we introduce the multiple space and time scales as

$$\begin{aligned}&\psi = \psi(\xi, y, \tau) \equiv \psi(\epsilon^{1/2}(x - c_0 t), y, \epsilon^{3/2}t),\\&\psi_0 = \psi_0(y, \epsilon^{3/2}t) \equiv \psi_0(y, \tau),\end{aligned} \tag{4}$$

with ϵ being a small parameter and c_0 an arbitrary constant, as usual for the derivation of constant coefficient KdV type equations. Taking the substitution of Eq.(4) into Eq.(3) yields

$$\psi_{\xi\xi\tau}\epsilon^2+\left[\psi_\xi\psi_{\xi\xi y}-(\psi_{0\tau}+\psi_\tau)\lambda_0^2-(\psi_{0y}+\psi_y+c_0)\psi_{\xi\xi\xi}+(\psi+\psi_0)_{yy\tau}\right]\epsilon$$
$$-(\psi_{0y}+c_0+\psi_y)\psi_{\xi yy}+(\psi_{yyy}+\psi_{0yyy}+\beta_0+{\lambda_0}^2c_0)\psi_\xi=0. \tag{5}$$

The asymptotic expansions are imported for both the stream function ψ and the base stream ψ_0 as well in the following form

$$\psi=\epsilon\psi_1+\epsilon^2\psi_2+O(\epsilon^3),\quad \psi_0=\Psi_0+\epsilon\Psi_1+\epsilon^2\Psi_2+O(\epsilon^3), \tag{6}$$

where $\psi_i\equiv\psi_i(\xi,y,\tau)$, $\Psi_j\equiv\Psi_j(y,\tau)$, $(i=1,2,\ j=1,2,3)$ are functions of the indicated variables. We substitute Eq.(6) into Eq.(5), and then require all the coefficients of different powers of ϵ to be zero.

Vanishing the coefficients of $O(\epsilon)$ and $O(\epsilon^2)$ gives

$$(\Psi_{0y}+c_0)\psi_{1\xi yy}-(\lambda_0^2c_0+\beta_0+\Psi_{0yyy})\psi_{1\xi}=\Psi_{0yy\tau}-\lambda_0^2\Psi_{0\tau}, \tag{7}$$

and

$$(\Psi_{0y}+c_0)\psi_{2\xi yy}-(\lambda_0^2c_0+\beta_0+\Psi_{0yyy})\psi_{2\xi}=-(\Psi_{0y}+c_0)\psi_{1\xi\xi\xi}-\lambda_0^2(\Psi_1+\psi_1)_\tau$$
$$-\psi_{1yy\xi}(\Psi_1+\psi_1)_y+(\psi_1+\Psi_1)_{yy\tau}+\psi_{1\xi}(\psi_1+\Psi_1)_{yyy}, \tag{8}$$

respectively. Obviously, Eq.(7) is linear so it can be solved by the usual variable separation approach. Suppose ψ_1 has a variable separation solution of the form

$$\psi_1=G_0(y,\tau)A(\xi,\tau)\equiv G_0A. \tag{9}$$

The substitution of the above ansatz (9) into (7) produces

$$\Psi_{0yy}=\lambda_0^2\Psi_0+F_1, \tag{10}$$
$$(\Psi_{0y}+c_0)G_{0yy}-G_0(\lambda_0^2c_0+\beta_0+\Psi_{1y}+\lambda_0^2\Psi_{0y})=0, \tag{11}$$

where $F_1\equiv F_1(y)$ is an arbitrary integration function. It is noted that A remains undetermined, and G_0 is determined by Eq.(11) with Eq.(10).

2.1.1. VCKdV equation deduced with the y-average trick

In this case, just substituting Eq.(9) with Eqs.(10), (11) and $\psi_2=0$ into Eq.(8), then following the usually used y-average trick, say, integrating the coefficients of the resulting equation with respect to y from 0 to y_0, a VCKdV equation emerges immediately

$$A_\tau+e_1A_{\xi\xi\xi}+e_2AA_\xi+e_3A_\xi+e_4A+e_5=0 \tag{12}$$

with $(e_i\equiv e_i(\tau),\ i=1,2,3,4,5)$

$$e_1=-\int_0^{y_0}\frac{(\Psi_{0y}+c_0)^2}{F_{1y}+\beta_0}\mathrm{d}y, \tag{13}$$
$$e_2=\int_0^{y_0}\frac{G_0[(\Psi_{0y}+c_0)F_{1yy}-(\lambda_0^2\Psi_0+F_1)(F_{1y}+\beta_0)]}{(\Psi_{0y}+c_0)(F_{1y}+\beta_0)}\mathrm{d}y, \tag{14}$$
$$e_3=\int_0^{y_0}\frac{(\Psi_{0y}+c_0)\Psi_{1yyy}-(F_{1y}+\beta_0+\lambda_0^2c_0+\lambda_0^2\Psi_{0y})\Psi_{1y}}{F_{1y}+\beta_0}\mathrm{d}y, \tag{15}$$
$$e_4=\int_0^{y_0}\frac{(c_0+\Psi_{0y})G_{0\tau}-\Psi_{0y\tau}G_0}{G_0(\Psi_{0y}+c_0)}\mathrm{d}y, \tag{16}$$
$$e_5=\int_0^{y_0}\frac{(\Psi_{0y}+c_0)(\Psi_{1yy}-\lambda_0^2\Psi_1)_\tau}{F_{1y}+\beta_0}\mathrm{d}y, \tag{17}$$

where Ψ_0 and G_0 are determined by Eqs.(10) and (11), F_1 and Ψ_1 are two arbitrary functions.

2.1.2. VCKdV equation deduced without the y-average trick

In order to discard the y-average trick utilized in the traditional treatment, we have to introduce the next higher order ψ_2 as

$$\begin{aligned}\psi_2 &= B_1(y,\tau)A^2 + B_2(y,\tau)A + B_3(y,\tau)A_{\xi\xi} + B_4(y,\tau) + B_5(y,\tau)\int A\mathrm{d}\xi + B_0(y,\tau)\xi\\ &\equiv B_1A^2 + B_2A + B_3A_{\xi\xi} + B_4 + B_5\int A\mathrm{d}\xi + B_0\xi. \end{aligned}\tag{18}$$

It can be easily verified that the VCKdV equation can be obtained in the form of Eq.(12) if we take B_i $(i = 0,1,2,3,5)$ to satisfy

$$\begin{aligned}&(c_0+\Psi_{0y})^2B_{0yy} - (c_0+\Psi_{0y})[(\lambda_0^2c_0+\lambda_0^2\Psi_{0y}+\beta_0+F_{1y})B_0 - \lambda_0^2\Psi_{1\tau}+\Psi_{1yy\tau}]\\ &\quad + G_0e_5(\beta_0+F_{1y}) = 0, \end{aligned}\tag{19}$$

$$\begin{aligned}&2(c_0+\Psi_{0y})^3B_{1yy} - 2(c_0+\Psi_{0y})^2(\lambda_0^2\Psi_{0y}+F_{1y}+\beta_0+\lambda_0^2c_0)B_1\\ &\quad + G_0(c_0+\Psi_{0y})(e_2F_{1y}+e_2\beta_0-F_{1yy}G_0) + G_0^2(\lambda_0^2\Psi_0+F_1)(F_{1y}+\beta_0) = 0, \end{aligned}\tag{20}$$

$$\begin{aligned}&(c_0^2+\Psi_{0y})^2B_{2yy} + (G_0\Psi_{1y}-c_0B_2-\Psi_{0y}B_2)(\lambda_0^2\Psi_{0y}+F_{1y}+\beta_0+\lambda_0^2c_0)\\ &\quad + G_0e_3(F_{1y}+\beta_0) - G_0(c_0+\Psi_{0y})U_{1yyy} = 0, \end{aligned}\tag{21}$$

$$\begin{aligned}&(c_0+\Psi_{0y})^2B_{3yy} - (c_0+\Psi_{0y})(\lambda_0^2U_{0y}+F_{1y}+\beta_0+\lambda_0^2c_0)B_3 + (F_{1y}+\beta_0)G_0e_1\\ &\quad + G_0(c_0+\Psi_{0y})^2 = 0, \end{aligned}\tag{22}$$

$$\begin{aligned}&(c_0+\Psi_{0y})^3B_{5yy} - (c_0+\Psi_{0y})^2(\lambda_0^2c_0+\lambda_0^2\Psi_{0y}+\beta_0+F_{1y})B_5\\ &\quad + G_0(c_0+\Psi_{0y})(\beta_0+F_{1y})e_4 - (G_{0\tau}c_0 - G_0\Psi_{0y\tau}+G_{0\tau}\Psi_{0y})(\beta_0+F_{1y}) = 0, \end{aligned}\tag{23}$$

and B_4 arbitrary, where $e_i \equiv e_i(\tau)$ $(i = 1,2,3,4,5)$ are arbitrary functions with respect to the indicated variable.

2.2. Derivation of the VCmKdV equation

Similar to the VCKdV case, to derive the mKdV type equations, the long wave approximation ansatz reads

$$\psi = \psi(\xi,\eta,\tau) \equiv \psi(\epsilon(x-c_0t), y, \epsilon^3t), \qquad \psi_0 = \psi_0(y,\epsilon^3t) \equiv \psi_0(y,\tau),\tag{24}$$

where ϵ is an infinitesimal parameter, and c_0 is an arbitrary constant.

Substituting Eq.(24) into Eq.(3) yields

$$\begin{aligned}&\psi_{\xi\xi\tau}\epsilon^4 + [(\psi+\psi_0)_{yy\tau} - \psi_{\xi\xi\xi}(\psi_y+\psi_{0y}+c_0) - \lambda_0^2(\psi+\psi_0)_\tau + \psi_\xi\psi_{\xi\xi y}]\epsilon^2\\ &\quad + (\psi_{yyy}+\psi_{0yyy}+c_0\lambda_0^2+\beta_0)\psi_\xi - (\psi_y+\psi_{0y}+c_0)\psi_{yy\xi} = 0. \end{aligned}\tag{25}$$

To calculate further, ψ and ψ_0 are expanded as

$$\psi = \epsilon\psi_1 + \epsilon^2\psi_2 + \epsilon^3\psi_3 + O(\epsilon^4), \quad \psi_0 = \Psi_0 + \epsilon\Psi_1 + \epsilon^2\Psi_2 + \epsilon^3\Psi_3 + O(\epsilon^4),\tag{26}$$

where ψ_i $(i = 1,2,3)$ are functions of $\{\xi, y, \tau\}$, Ψ_i $(i = 0,1,2,3)$ are functions of $\{y,\tau\}$. Substituting the expansions (26) into Eq.(25), and vanishing the coefficients of $O(\epsilon)$, $O(\epsilon^2)$ and $O(\epsilon^3)$, we have

$$(\Psi_{0y}+c_0)\psi_{1\xi yy} - (\Psi_{0yyy}+\lambda_0^2c_0+\beta_0)\psi_{1\xi} = 0,\tag{27}$$

$$\begin{aligned}&(\Psi_{0y}+c_0)\psi_{2\xi yy} - (\Psi_{0yyy}+\lambda_0^2c_0+\beta_0)\psi_{2\xi} = (\psi_1+\Psi_1)_{yyy}\psi_{1\xi} - (\psi_1+\Psi_1)_y\psi_{1yy\xi}\\ &\quad - \lambda_0^2\Psi_{0\tau} + \Psi_{0yy\tau}, \end{aligned}\tag{28}$$

$$\begin{aligned}&(\Psi_{0y}+c_0)\psi_{3\xi yy} - (\Psi_{0yyy}+\lambda_0^2c_0+\beta_0)\psi_{3\xi} = (\Psi_2+\psi_2)_{yyy}\psi_{1\xi} + (\Psi_1+\psi_1)_{yyy}\psi_{2\xi} - \lambda_0^2\Psi_{1\tau}\\ &\quad - (c_0+\Psi_{0y})\psi_{1\xi\xi\xi} - (\psi_2+\Phi_2)_y\psi_{1\xi yy} - (\psi_1+\Psi_1)_y\psi_{2\xi yy} + (\psi_1+\Psi_1)_{yy\tau} - \lambda_0^2\psi_{1\tau}. \end{aligned}\tag{29}$$

As mentioned in the last subsection, we can suppose that Eq.(27) possesses the following variable separation solution

$$\psi_1 = G_0(y,\tau)A(\xi,\tau) \equiv G_0A.\tag{30}$$

Substituting Eq.(30) into Eq.(27), a relation between G_0 and Ψ_0 is generated as

$$(\Psi_{0y}+c_0)G_{0yy}-(\Psi_{0yyy}+\lambda_0^2c_0+\beta_0)G_0=0. \tag{31}$$

Next, supposing

$$\psi_2=G_2(y,\tau)A^2+G_1(y,\tau)A\equiv G_2A^2+G_1A, \tag{32}$$

and substituting it into Eq.(28) with Eqs.(30) and (32), we have

$$\Psi_{0yy}-\lambda_0^2\Psi_0+F_1=0, \tag{33}$$

$$\begin{aligned}&(\Psi_{0y}+c_0)^2G_{1yy}-(c_0G_1+G_1\Psi_{0y}-\Psi_{1y}G_0)(\lambda_0^2c_0+\beta_0+F_{1y}+\lambda_0^2\Psi_{0y})\\&\quad-(\Psi_{0y}+c_0)\Psi_{1yyy}G_0=0,\end{aligned} \tag{34}$$

$$\begin{aligned}&2(\Psi_{0y}+c_0)^3G_{2yy}-[(\Psi_{0y}+c_0)F_{1yy}-(\lambda_0^2\Psi_0+F_1)(\beta_0+F_{1y})]G_0^2\\&\quad-2G_2(\Psi_{0y}+c_0)^2(\lambda_0^2c_0+\beta_0+F_{1y}+\lambda_0^2\Psi_{0y})=0.\end{aligned} \tag{35}$$

Similarly, $F_1\equiv F_1(y)$ is an arbitrary integration function, and A remains free in this order.

2.2.1. VCmKdV equation deduced with the y-average trick

In this case, we just substitute the obtained results presented above with $\psi_3=0$ into Eq.(29), and then integrate the result with respect to y from 0 to y_0. In this way, a generalized VCmKdV equation is derived immediately

$$A_\tau+e_1A_{\xi\xi\xi}+(e_2A^2+e_3A+e_5)A_\xi+e_4A+e_6=0 \tag{36}$$

with $(e_i\equiv e_i(\tau),\ i=1,2,3,4,5,6)$

$$e_1=\int_0^{y_0}\frac{G_0(\Psi_{0y}+c_0)}{\lambda_0^2G_0-G_{0yy}}\mathrm{d}y, \tag{37}$$

$$e_2=\int_0^{y_0}\frac{G_{0yy}G_{2y}-G_0G_{2yyy}-2G_{0yyy}G_2+2G_{0y}G_{2yy}}{\lambda_0^2G_0-G_{0yy}}\mathrm{d}y, \tag{38}$$

$$e_3=\int_0^{y_0}\frac{2\Psi_{1y}G_{2yy}-2\Psi_{1yyy}G_2+(G_{0y}G_{1y})_y-G_{0yyy}G_1-G_0G_{1yyy}}{\lambda_0^2G_0-G_{0yy}}\mathrm{d}y, \tag{39}$$

$$e_4=\int_0^{y_0}\Big(\ln(\lambda_0^2G_0-G_{0yy})\Big)_\tau\mathrm{d}y, \tag{40}$$

$$e_5=\int_0^{y_0}\frac{G_{0yy}\Psi_{2y}-G_0\Psi_{2yyy}+\Psi_{1y}G_{1yy}-\Psi_{1yyy}G_1}{\lambda_0^2G_0-G_{0yy}}\mathrm{d}y, \tag{41}$$

$$e_6=\int_0^{y_0}\frac{\lambda_0^2\Psi_{1\tau}-\Psi_{1yy\tau}}{\lambda_0^2G_0-G_{0yy}}\mathrm{d}y, \tag{42}$$

where $G_0,\ G_1,\ G_2$ satisfy Eqs.(31)–(35) with Eq.(33), Ψ_1 is an arbitrary function of $\{y,\tau\}$.

2.2.2. VCmKdV equation deduced without the y-average trick

Similarly, in order to discard the usage of the y-average trick, ψ_3 cannot be assumed zero, instead; it can be fixed as

$$\psi_3=B_0+B_1A_{\xi\xi}+B_2A^3+B_3A^2+B_4A+B_5\xi+B_6\int A\mathrm{d}\xi, \tag{43}$$

where $B_i\equiv B_i(y,\tau)$ $(i=0,1,\ldots,6)$ are functions of the indicated variables. Substituting Eqs.(30)–(35) and (43) into Eq.(29) can lead to VCmKdV equation in the form of Eq.(36) when B_i $(i=0,1,\ldots,6)$ satisfies

$$(\Psi_{0y}+c_0)B_{1yy}-(e_1\lambda_0^2-c_0-\Psi_{0y})G_0-(\lambda_0^2c_0+\beta_0+\Psi_{0yyy})B_1+e_1G_{0yy}=0, \tag{44}$$

$$\begin{aligned}&3(\Psi_{0y}+c_0)B_{2yy}-(G_{2yyy}+e_2\lambda_0^2)G_0-3(\Psi_{0yyy}+\beta_0+\lambda_0^2c_0)B_2+2G_{0y}G_{2yy}\\&\quad+(G_{2y}+e_2)G_{0yy}-2G_{0yyy}G_2=0,\end{aligned} \tag{45}$$

$$2(\Psi_{0y}+c_0)B_{3yy}+(G_{1y}+e_3)G_{0yy}-2(\Psi_{0yyy}+\lambda_0^2c_0+\beta_0)B_3-(G_{1yyy}+e_3\lambda_0^2)G_0$$
$$-2\Psi_{1yyy}G_2+G_{0y}G_{1yy}+2\Psi_{1y}G_{2yy}-G_{0yyy}G_1=0, \tag{46}$$
$$(\Psi_{0y}+c_0)B_{4yy}-(\lambda_0^2c_0+\beta_0+\Psi_{0yyy})B_4-(\Psi_{2yyy}+e_5\lambda_0^2)G_0+(e_5+\Psi_{2y})G_{0yy}$$
$$+\Psi_{1y}G_{1yy}-\Psi_{1yyy}G_1=0, \tag{47}$$
$$(\Psi_{0y}+c_0)B_{5yy}-(\lambda_0^2c_0+\beta_0+\Psi_{0yyy})B_5-(\lambda_0^2G_0-G_{0yy})e_6-\Psi_{1yy\tau}+\lambda_0^2\Psi_{1\tau}=0, \tag{48}$$
$$(\Psi_{0y}+c_0)B_{6yy}-(\lambda_0^2c_0+\beta_0+\Psi_{0yyy})B_6-(\lambda_0^2G_0-G_{0yy})e_4+\lambda_0^2G_{0\tau}-G_{0yy\tau}=0, \tag{49}$$

where $e_i\equiv e_i(\tau)$ $(i=0,1,\dots,6)$ are arbitrary functions of the indicated variable.

2.3. Derivation of the VCNLS equation

In order to derive the nonlinear Schrödinger type equations, we should use the multiple scales

$$\psi=\psi(x,y,t,\xi,\tau)\equiv\psi(x,y,t,\epsilon(x-c_0t),\epsilon^2t),\quad \psi_0=\psi_0(y,\tau)\equiv\psi_0(y,\epsilon^2t), \tag{50}$$

where ϵ is an infinitesimal parameter, and c_0 is an arbitrary constant. Substituting Eq.(50) into Eq.(3) yields

$$\psi_{\xi\xi\tau}\epsilon^4+\big[\psi_\xi\psi_{y\xi\xi}+2\psi_{x\xi\tau}-(\psi_{0y}+\psi_y+c_0)\psi_{\xi\xi\xi}\big]\epsilon^3+\big[2\psi_\xi\psi_{xy\xi}+\psi_x\psi_{y\xi\xi}$$
$$-\lambda_0^2(\psi_0+\psi)_\tau+\psi_{xx\tau}-(3\psi_{0y}+3\psi_y+2c_0)\psi_{x\xi\xi}+\psi_{\xi\xi t}+(\psi+\psi_0)_{yy\tau}\big]\epsilon^2$$
$$+\big[(\beta_0+\lambda_0^2c_0+\psi_{xxy}+\psi_{yyy}+\psi_{0yyy})\psi_\xi+2\psi_x\psi_{xy\xi}-(\psi_{0y}+\psi_y+c_0)\psi_{yy\xi}$$
$$+2\psi_{xt\xi}-(3\psi_{0y}+3\psi_y+c_0)\psi_{xx\xi}\big]\epsilon+(\partial_t-\psi_{0y}\partial_x)\,\Delta_{\rm h}\psi+J(\psi,\ \Delta_{\rm h}\psi)$$
$$-\lambda_0^2\psi_t+(\beta_0+\psi_{0yyy})\psi_x=0. \tag{51}$$

We now introduce the asymptotic expansions

$$\psi=\epsilon\psi_1+\epsilon^2\psi_2+\epsilon^3\psi_3+O(\epsilon^4),\quad \psi_0=\Psi_0+\epsilon\Psi_1+\epsilon^2\Psi_2+\epsilon^3\Psi_3+O(\epsilon^4), \tag{52}$$

with $\psi_i\equiv\psi_i(x,y,t,\xi,\tau)$ $(i=1,2,3)$ and $\Psi_j\equiv\Psi_j(y,\tau)$ $(j=0,1,2,3)$ being functions of the indicated variables. Similarly, we can write down the coefficients of $O(\epsilon)$, $O(\epsilon^2)$ and $O(\epsilon^3)$ which are obtained by substituting Eq.(52) into Eq.(51). They are

$$(\beta_0+\Psi_{0yyy})\psi_{1x}-\lambda_0^2\psi_{1t}+\Delta_{\rm h}\psi_{1t}-\Psi_{0y}\Delta_{\rm h}\psi_{1x}=0, \tag{53}$$
$$(\beta_0+\Psi_{0yyy})\psi_{2x}-\lambda_0^2\psi_{2t}+\Delta_{\rm h}\psi_{2t}-\Psi_{0y}\Delta_{\rm h}\psi_{2x}=2\Psi_{0y}\psi_{1xx\xi}-2\psi_{1xt\xi}$$
$$+(c_0+\Psi_{0y})\Delta_{\rm h}\psi_{1\xi}-(\Psi_{1yyy}+\Delta_{\rm h}\psi_{1y})\psi_{1x}+(\Psi_{1y}+\psi_{1y})\Delta_{\rm h}\psi_{1x}$$
$$+\lambda_0^2\Psi_{0\tau}-\Psi_{0yy\tau}-(\Psi_{0yyy}+\lambda_0^2c_0+\beta_0)\psi_{1\xi}, \tag{54}$$
$$(\beta_0+\Psi_{0yyy})\psi_{3x}-\lambda_0^2\psi_{3t}+\Delta_{\rm h}\psi_{3t}-\Psi_{0y}\Delta_{\rm h}\psi_{3x}=(2c_0+3\Psi_{0y})\psi_{1x\xi\xi}-\Delta_{\rm h}\psi_{1\tau}$$
$$+(\Psi_{1y}+\psi_{1y})(\Delta_{\rm h}\psi_{2x}+2\psi_{1xx\xi}+\Delta_{\rm h}\psi_{1\xi})-(\Delta_{\rm h}\psi_{1y}+\Psi_{1yyy})\psi_{2x}-\psi_{1x}\Delta_{\rm h}\psi_{2y}$$
$$+(\Psi_{0y}+c_0)\Delta_{\rm h}\psi_{2\xi}-(\beta_0+\lambda_0^2c_0+\Psi_{0yyy})\psi_{2\xi}-(\Psi_{2yyy}+2\psi_{1xy\xi})\psi_{1x}$$
$$+2\Psi_{0y}\psi_{2xx\xi}-\psi_{1\xi}\Delta_{\rm h}\psi_{1y}+(\psi_{2y}+\Psi_{2y})\Delta_{\rm h}\psi_{1x}-\psi_{1t\xi\xi}+\lambda_0^2\psi_{1\tau}$$
$$+\lambda_0^2\Psi_{1\tau}-\Psi_{1yy\tau}-2\psi_{2xt\xi}-\Psi_{1yyy}\psi_{1\xi}. \tag{55}$$

Obviously, the solution of Eq.(53) can be assumed as

$$\psi_1=G_0(y,\tau)A(\xi,\tau)\mathrm{e}^{\mathrm{i}(kx-\omega t)}+\text{c.c.}\equiv G_0A\mathrm{e}^{\mathrm{i}(kx-\omega t)}+\text{c.c.}, \tag{56}$$

where 'c.c.' means the complex conjugation of the previous one. The substitution of Eq.(56) into Eq.(53) gives

$$\left(\frac{k\Psi_{0yyy}+k^3\Psi_{0y}+k\beta_0+\omega k^2+\omega\lambda_0^2}{k\Psi_{0y}+\omega}-\frac{G_{0yy}}{G_0}\right)A\mathrm{e}^{\mathrm{i}(kx-\omega t)}+\text{c.c.}=0. \tag{57}$$

It can be demonstrated that the first part, i.e., Ψ_0-dependent, is related to the secular terms. Thus, the vanishing of this part requires Ψ_0 to be fixed by

$$k\Psi_{0yyy}+k(k^2+F_0)\Psi_{0y}+k\beta_0+\omega k^2+\omega\lambda_0^2+\omega F_0=0. \tag{58}$$

Because of Eq.(58), G_0 can be determined as

$$G_0 = F_1 \sin(\sqrt{F_0}y + F_2), \tag{59}$$

where F_0, F_1 and F_2 are arbitrary functions with respect to τ.

Inserting Eq.(56) with Eqs.(58), (59) and

$$\begin{aligned}\psi_2 &= G_0G_1(\tau)A_\xi \mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.} + G_2(y,\tau)|A|^2 \\ &\equiv G_0G_1A_\xi \mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.} + G_2|A|^2,\end{aligned} \tag{60}$$

into Eq.(54), we have

$$\begin{aligned}&\Big\{\mathrm{i}\,G_0k(\Psi_{1yyy} + \Psi_{1y}F_0 + \Psi_{1y}k^2)A \\ &+ \frac{G_0}{k}\Big(c_0kF_0 - \omega F_0 + \omega k^2 - \omega\lambda_0^2 + \lambda_0^2c_0k \\ &+ c_0k^3 + 2\Psi_{0y}k^3\Big)A_\xi\Big\}\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \text{c.c.} \\ &+ (\Psi_{0yy} - \lambda_0^2\Psi_0)_\tau = 0.\end{aligned} \tag{61}$$

Obviously, the above equation indicates that

$$(\Psi_{0yy} - \lambda_0^2\Psi_0)_\tau = 0, \tag{62}$$

because it is related to the secular term. Solving Eqs.(58) and (62) simultaneously, we may have a special solution

$$\Psi_0 = C_1y + C_0, \tag{63}$$

with

$$F_0 = -\frac{\lambda_0^2\omega + \beta_0k + \omega k^2 + C_1k^3}{\omega + C_1k}. \tag{64}$$

By integrating the coefficient of $\mathrm{e}^{\mathrm{i}(kx-\omega t)}$ in Eq.(61), and vanishing the result with considerations of Eqs.(63) and (64), we find that Ψ_1 must be fixed by

$$\Psi_{1yyy} + \frac{\lambda_0^2\omega + \beta_0k}{\omega + C_1k}\Psi_y = 0. \tag{65}$$

Due to Eqs.(63)–(65), Eq.(61) possesses nontrivial wave solutions only for

$$c_0 = \frac{4C_1k^2\omega + 2C_1^2k^3 + \lambda_0^2\omega C_1 + 2\omega^2k + \omega\beta_0}{k(\beta_0 - C_1\lambda_0^2)}, \tag{66}$$

while A remains free and shall be determined in the next order.

2.3.1. VCNLS equation deduced with the y-average trick

Now substituting the above results and

$$\psi_3 = 0 \tag{67}$$

into the third order equation (55), we get

$$\gamma_1\mathrm{e}^{\mathrm{i}(kx-\omega t)} + \gamma_1^*\mathrm{e}^{-\mathrm{i}(kx-\omega t)} + \gamma_0 = 0, \tag{68}$$

where

$$\begin{aligned}\gamma_1 = {}& G_0k(\beta_0 - C_1\lambda_0^2)A_\tau + \Big(\mathrm{i}(\omega + C_1k)(G_1k - \mathrm{i})\Psi_{1yyy} - \mathrm{i}\,G_1k(\beta_0k + \lambda_0^2\omega)\Psi_{1y} - (\beta_0k + \lambda_0^2\omega \\ &- 2C_1k^3 - 2\omega k^2)\Psi_{1y}\Big)G_0A_\xi + \mathrm{i}\,kG_0\Big((C_1k + \omega)G_{2yyy} - (\beta_0k + \lambda_0^2\omega)G_{2y}\Big)|A|^2A \\ &+ k\Big(\mathrm{i}\,G_0(\omega + C_1k)\Psi_{2yyy} + (C_1\lambda_0^2 - \beta_0)G_{0\tau} + G_0(\beta_0k + \lambda_0^2\omega)\Psi_{2y}\Big)A - \mathrm{i}\,\frac{\omega + C_1k}{C_1\lambda_0^2 - \beta_0} \\ &\times G_0\Big(3kC_1^2\lambda_0^2 - 3kC_1\beta_0 + 3\lambda_0^2\omega C_1 - 3\omega\beta_0 - 8k^2C_1\omega - 4k^3C_1^2 - 4k\omega^2\Big)A_{\xi\xi},\end{aligned} \tag{69}$$

$$\begin{aligned}\gamma_0 = {}& \Big[4k^2G_{0y}G_0 - \frac{(\omega + kC_1)(C_1\lambda_0^2 - 2k^2C_1 - 2k\omega - \beta_0)}{k(C_1\lambda_0^2 - \beta_0)}G_{2yy} \\ &+ \frac{\lambda_0^4\omega C_1 - 4k^2\lambda_0^2C_1\omega - 2k^3\lambda_0^2C_1^2 - \lambda_0^2\omega\beta_0 - 2k\lambda_0^2\omega^2 + k\beta_0C_1\lambda_0^2 - k\beta_0^2}{k(C_1\lambda_0^2 - \beta_0)}G_2\Big]\left(|A|^2\right)_\xi \\ &+ (\Psi_{1yy} - \lambda_0^2\Psi_1)_\tau.\end{aligned} \tag{70}$$

Vanishing the coefficients of different exponentials requires

$$\gamma_0 = 0, \tag{71}$$

$$\gamma_1 = 0. \tag{72}$$

From Eqs.(70) and (71) we know that the Ψ_1-dependent part of Eq.(70) is related to the secular terms. So in order to eliminate the secular terms, Ψ_1 has to be constrained further

$$(\Psi_{1yy} - \lambda_0^2\Psi_1)_\tau = 0. \tag{73}$$

Solving Eqs.(65) and (73) simultaneously generates the general solution

$$\Psi_1 = C_2 + C_3 \sin\left(\sqrt{-\frac{\lambda_0^2\omega+\beta_0 k}{\omega+C_1 k}}y + C_4\right), \tag{74}$$

where C_2, C_3 and C_4 are arbitrary constants. Due to Eq.(71) with Eqs.(70) and (73), G_2 is associated with G_0 by

$$\begin{aligned}&(\omega+kC_1)(C_1\lambda_0^2-2k^2C_1-2k\omega-\beta_0)G_{2yy}-(\lambda_0^4\omega C_1-4k^2\lambda_0^2C_1\omega-2k^3\lambda_0^2C_1^2\\&-\lambda_0^2\omega\beta_0-2k\lambda_0^2\omega^2+k\beta_0C_1\lambda_0^2-k\beta_0^2)G_2-4k^3(C_1\lambda_0^2-\beta_0)G_0G_{0y}=0.\end{aligned} \tag{75}$$

Finally, integrating Eq.(72) with Eq.(69) with respect to y from 0 to y_0 results in the VCNLS equation

$$\mathrm{i}\,A_\tau + e_0A_{\xi\xi} + \mathrm{i}\,e_1A_\xi + e_2|A|^2A + (e_3+\mathrm{i}\,e_4)A = 0 \tag{76}$$

with e_i $(i=0,1,2,3,4)$ given by

$$e_0 = \frac{y_0(\omega+kC_1)^2(4k^2C_1+4k\omega-3C_1\lambda_0^2+3\beta_0)}{k(C_1\lambda_0^2-\beta_0)^2}, \tag{77}$$

$$e_1 = \frac{2kC_3(\omega+kC_1)}{C_1\lambda_0^2-\beta_0}\left[\sin(C_4)-\sin\left(\sqrt{-\frac{\lambda_0^2\omega+k\beta_0}{\omega+kC_1}}y_0+C_4\right)\right], \tag{78}$$

$$\begin{aligned}e_2\equiv e_2(\tau) =\;& \frac{2k^3F_1^2}{C_1\lambda_0^2-2k^2C_1-2k\omega-\beta_0}\sqrt{\frac{k\beta_0+\lambda_0^2\omega+k^2\omega+k^3C_1}{\omega+kC_1}}\\&\times\left[\sin(2F_2)-\sin\left(2y_0\sqrt{-\frac{k\beta_0+\lambda_0^2\omega+k^2\omega+k^3C_1}{\omega+kC_1}}+2F_2\right)\right]\\&-\frac{2k^2(\omega+kC_1)}{C_1\lambda_0^2-2k^2C_1-2k\omega-\beta_0}\int_0^{y_0}G_{2y}\mathrm{d}y,\end{aligned} \tag{79}$$

$$e_3\equiv e_3(\tau) = \int_0^{y_0}\frac{\omega+kC_1}{C_1\lambda_0^2-\beta_0}\Psi_{2yyy}-\frac{\lambda_0^2\omega+k\beta_0}{C_1\lambda_0^2-\beta_0}\Psi_{2y}\mathrm{d}y, \tag{80}$$

$$e_4\equiv e_4(\tau) = \int_0^{y_0}(\ln G_0)_\tau\mathrm{d}y, \tag{81}$$

where Ψ_2 is an undetermined (arbitrary) function, G_0 is given by Eq.(59), and G_2 is related to G_0 by Eq.(75).

2.3.2. VCNLS equation deduced without the y-average trick

In this case, instead of Eq.(67), we can suppose

$$\begin{aligned}\psi_3 &= [G_3(y,\tau)A_{\xi\xi}+\mathrm{i}\,G_4(y,\tau)A_\xi+G_5(y,\tau)|A|^2A+(G_6(y,\tau)+\mathrm{i}\,G_7(y,\tau))A]\mathrm{e}^{\mathrm{i}(kx-\omega t)}+\text{c.c.},\\&\equiv [G_3A_{\xi\xi}+\mathrm{i}\,G_4A_\xi+G_5|A|^2A+(G_6+\mathrm{i}\,G_7)A]\mathrm{e}^{\mathrm{i}(kx-\omega t)}+\text{c.c.}.\end{aligned} \tag{82}$$

Now substituting Eqs.(56), (60) and (82) with Eqs.(59), (63), (64) and (66) into Eq.(55) arrives at a resulting equation similar to Eq.(68) with γ_0 also given by Eq.(7) while γ_1 given by

$$\begin{aligned}\gamma_1 =\;& \mathrm{i}\Big[(k^3C_1+k^2\omega+k\beta_0+\lambda_0^2\omega)(\omega+kC_1)G_5-k(\lambda_0^2\omega+k\beta_0)G_0G_{2y}-(\omega+kC_1)^2G_{5yy}\\&+k(\omega+kC_1)G_0G_{2yyy}\Big]|A|^2A+\Big[\mathrm{i}\,k(\omega+kC_1)G_0\Psi_{2yyy}-\mathrm{i}\,k(\lambda_0^2\omega+k\beta_0)G_0\Psi_{2y}\\&-\mathrm{i}(\omega+kC_1)^2G_{6yy}+\mathrm{i}(k^3C_1+k^2\omega+k\beta_0+\lambda_0^2\omega)(\omega+kC_1)(G_6+\mathrm{i}\,G_7)\\&-k(C_1\lambda_0^2-\beta_0)G_{0\tau}+(\omega+kC_1)^2G_{7yy}\Big]A+\mathrm{i}\Big[(k^3C_1+k^2\omega+k\beta_0+\lambda_0^2\omega)(\omega+kC_1)G_3\end{aligned}$$

$$-(\omega+kC_1)^2G_{3yy}-\frac{(\omega+kC_1)^2}{C_1\lambda_0^2-\beta_0}(3C_1\lambda_0^2-4k^2C_1-4k\omega-3\beta_0)G_0\Big]A_{\xi\xi}$$
$$+\Big[\mathrm{i}(\omega+kC_1)(kG_1-\mathrm{i})G_0\Psi_{1yyy}-\mathrm{i}(\lambda_0^2G_1k\omega+k^2G_1\beta_0-\mathrm{i}\,\lambda_0^2\omega+2\,\mathrm{i}\,k^3C_1+2\,\mathrm{i}\,k^2\omega$$
$$-\mathrm{i}\,k\beta_0)G_0\Psi_{1y}+(\omega+kC_1)^2G_{4yy}-(\omega+kC_1)(k^3C_1+k^2\omega+k\beta_0+\lambda_0^2\omega)G_4\Big]A_{\xi}$$
$$-k(C_1\lambda_0^2-\beta_0)G_0A_{\tau}. \tag{83}$$

Since γ_0 is still read as Eq.(70), then Ψ_1 is also solved out as Eq.(74) and G_2 is also related to G_0 by Eq.(75). Hereupon, from Eq.(72) with Eq.(83), we can obtain a type of VCNLS equation with the same form as Eq.(76) while G_i $(i=3,4,5,6,7)$ are determined by

$$e_0kG_0(C_1\lambda_0^2-\beta_0)^2+(\omega+kC_1)(C_1\lambda_0^2-\beta_0)\Big[(k^3C_1+k^2\omega+k\beta_0+\lambda_0^2\omega)G_3$$
$$-(\omega+kC_1)G_{3yy}\Big]+G_0(3C_1\lambda_0^2-4k^2C_1-4k\omega-3\beta_0)(\omega+kC_1)^2=0, \tag{84}$$
$$e_1kG_0(C_1\lambda_0^2-\beta_0)-(\omega+kC_1)\Big[(k^3C_1+k^2\omega+k\beta_0+\lambda_0^2\omega)G_4-(\omega+kC_1)G_{4yy}\Big]$$
$$+2k^2G_0C_3(\omega+kC_1)\sqrt{-\frac{\lambda_0^2\omega+k\beta_0}{\omega+kC_1}}\cos\left(\sqrt{-\frac{\lambda_0^2\omega+k\beta_0}{\omega+kC_1}}y+C_4\right)=0, \tag{85}$$
$$e_2kG_0(C_1\lambda_0^2-\beta_0)-\Big[(\omega+kC_1)(kG_0G_{2yyy}+(k^3C_1+k^2\omega+k\beta_0+\lambda_0^2\omega)G_5)$$
$$-k(\lambda_0^2\omega+k\beta_0)G_0G_{2y}-(\omega+kC_1)^2G_{5yy}\Big]=0, \tag{86}$$
$$e_3kG_0(C_1\lambda_0^2-\beta_0)-\Big[(\omega+kC_1)(kG_0\Psi_{2yyy}+(k^3C_1+k^2\omega+k\beta_0+\lambda_0^2\omega)G_6)$$
$$-k(\lambda_0^2\omega+k\beta_0)G_0\Psi_{2y}-(\omega+kC_1)^2G_{6yy}\Big]=0, \tag{87}$$
$$kG_0(e_4-(\ln G_0)_\tau)(C_1\lambda_0^2-\beta_0)-(\omega+kC_1)\Big[(k^3C_1+k^2\omega+k\beta_0+\lambda_0^2\omega)G_7$$
$$-(\omega+kC_1)G_{7yy}\Big]=0 \tag{88}$$

with $e_i\equiv e_i(\tau)$ $(i=0,1,2,3,4)$, Ψ_2 being arbitrary functions, G_0 given by Eq.(59), and G_2 determined by Eq.(75).

3. Exact solutions of the variable coefficient nonlinear systems

In this section, we are going to obtain some exact solutions of the variable coefficient nonlinear equations derived in the last section. As we know, there now exist many different ways to get exact solutions of nonlinear systems; for instance, the symmetry approach is a powerful method, and can be applied to the models, either integrable or not. Actually, the general symmetry approach of Refs.[17] and [18] is equivalent to finding some non-auto-Bäcklund transformations (NABT) among different differential equations. Recently, it has been discovered that full finite Lie point symmetry transformation groups of constant coefficient integrable systems can be directly obtained with the help of gauge and space-time transformations of Lax pairs (or linear triads) without using the standard Lie algebra and Lie group theory.[19] For constant coefficient nonintegrable systems, a direct method is also established to find Lie point subalgebras.[20] Here, we will utilize the latter direct method to find NABT between variable coefficient nonlinear systems and constant coefficient ones. Then taking advantages of the NABT and the known results of the constant coefficient nonlinear equations, the solutions of the variable coefficient nonlinear systems can be generated. Firstly let us generally outline the direct method to show how to find the NABT, and then apply it to the equations VCKdV (12), VCmKdV (36) and VCNLS (76) to yield their solutions which finally lead to several approximate solutions of the original model (1).

3.1. Outline of the direct method to find NABT between two equations

In Ref.[21], Clarkson and Kruskal introduced a direct method to obtain similarity reductions of partial differential equations (PDEs) and utilized it to solve the Boussinesq equation. It is a very simple method because it does not make use of group theory. The main idea is to look for a reduction of a given (1+1)-dimensional PDE in the form

$$u(x,t) = W(x,t,U(z)), \quad z \equiv z(x,t), \tag{89}$$

which is the most general form for a similarity reduction.[22] For an nth order given PDE

$$F(x,t,u,u_x,u_{x^it^j},...) = 0, \quad (i,\ j = 1,2,...,n), \tag{90}$$

substituting Eq.(89) into Eq.(90) and then requiring the result to be an ordinary differential equation for $U(z)$ imposes conditions upon W, z and their derivatives that enable one to solve out W and z.

It is interesting that for many real physical systems, it is enough to seek the symmetry reductions in a simple form

$$u(x,t) = \alpha(x,t) + \beta(x,t)U(z), \quad z \equiv z(x,t), \tag{91}$$

instead of the general Eq.(89). The above method can be readily extended to higher dimensional PDEs.[23]

Actually, the Clarkson and Kruskal direct symmetry reduction method implies that it is possible to find the NABT between two systems by a simple direct method without using any group theory. To realize this idea is quite straightforward just by substituting

$$\begin{aligned} &u(x_1,x_2,\dots,x_n) \\ &= W\big(x_1,x_2,\dots,x_n,U(X_1,X_2,\dots,X_n)\big), \end{aligned}$$

$$(X_i = X_i(x_1,x_2,\dots,x_n), \quad i = 1,2,\dots,n) \tag{92}$$

into a given PDE Eq.(90) and requiring U to satisfy a particular PDE (which is rightly the aim equation) under the transformation $\{u,x_1,\dots,x_n\} \to \{U,X_1,\dots,X_n\}$. It is not surprising but interesting that it is enough to suppose Eq.(92) be the form of

$$\begin{aligned} &u(x_1,x_2,\dots,x_n) \\ &= \alpha(x_1,x_2,\dots,x_n) + \beta(x_1,x_2,\dots,x_n) \\ &\quad \times U(X_1,X_2,\dots,X_n), \end{aligned} \tag{93}$$

for various nonlinear systems, especially for those for which the CK's direct reduction method works. In the following, we will apply the form of Eq.(93) with $n = 2$ to the VCKdV, VCmKdV and VCNLS equations and ask for U to satisfy the corresponding standard constant coefficient KdV, mKdV and NLS equations respectively; hereafter, solving the conditions on α, β, X_1 and X_2 leads to the NABT and then the solutions of the VC equations.

3.2. Exact solutions of the VCKdV equation (12)

As described in the last subsection, we suppose

$$\begin{aligned} A &= \alpha(\xi,\tau) + \beta(\xi,\tau)P\big(X(\xi,\tau),T(\xi,\ \tau)\big) \\ &\equiv \alpha + \beta P(X,T), \end{aligned} \tag{94}$$

and then substitute it into Eq.(12) with the requirement that P satisfies

$$P_T + 6PP_X + P_{XXX} = 0. \tag{95}$$

Here we avoid detailed calculations and just directly write down the results (as in the other two cases). After computation, we obtain

$$\alpha = -\frac{e_3}{e_2} - \sqrt{\frac{6}{\beta_0}}\frac{e_1\xi_0}{e_2} - \frac{e_1}{e_2}\int \frac{e_1e_2e_5 - e_1e_{3\tau} + e_{1\tau}e_3}{e_1^2}\mathrm{d}\tau, \quad \beta = \frac{\beta_0 e_1}{e_2}, \tag{96}$$

$$X = \sqrt{\frac{\beta_0}{6}}\xi + \int e_1\xi_0 + \sqrt{\frac{\beta_0}{6}}e_1\int \frac{e_1e_2e_5 - e_1e_{3\tau} + e_{1\tau}e_3}{e_1^2}\mathrm{d}\tau\mathrm{d}\tau, \tag{97}$$

$$T = \frac{\sqrt{6}}{36}\beta_0^{\frac{3}{2}}\int e_1\mathrm{d}\tau \tag{98}$$

with the condition

$$e_1e_2e_4 - e_1e_{2\tau} + e_{1\tau}e_2 = 0, \tag{99}$$

where β_0 and ξ_0 are arbitrary constants.

Therefore, the solutions of Eq.(95) can produce solutions of the VCKdV Eq.(12) with Eqs.(97)–(99). For example, a simple well-known soliton solution of Eq.(95) which reads

$$P = 2K^2\mathrm{sech}^2(KX - 4K^3T) \tag{100}$$

with K being an arbitrary constant gives a corresponding solution of Eq.(12) as

$$\begin{aligned} A = &-\frac{e_3}{e_2} - \sqrt{\frac{6}{\beta_0}}\frac{e_1\xi_0}{e_2} - \frac{e_1}{e_2}\int \frac{e_1e_2e_5 - e_1e_{3\tau} + e_{1\tau}e_3}{e_1^2}\mathrm{d}\tau \\ &+ \frac{2K^2\beta_0e_1}{e_2}\mathrm{sech}^2\left[K\sqrt{\frac{\beta_0}{6}}\xi + K\int e_1\xi_0 + \sqrt{\frac{\beta_0}{6}}e_1\int \frac{e_1e_2e_5 - e_1e_{3\tau} + e_{1\tau}e_3}{e_1^2}\mathrm{d}\tau \right. \\ &\left. -\frac{\sqrt{6}}{9}K^2\beta_0^{3/2}e_1\mathrm{d}\tau\right], \end{aligned} \tag{101}$$

with the condition (99).

3.3. Exact solutions of the VCmKdV equation (36)

For the VCmKdV Eq.(36), it can be transformed to the standard constant coefficient mKdV equation

$$P_T + 6P^2P_X + P_{XXX} = 0 \tag{102}$$

with the substitution of Eq.(94) into Eq.(36). After detailed calculations, we get

$$\begin{aligned} &\alpha = -\frac{e_3}{2e_2}, \quad \beta = T_0\sqrt{\frac{6e_1}{e_2}}, \\ &T = T_0^3\int e_1\mathrm{d}\tau, \\ &X = T_0\xi + T_0\int\left(\frac{e_3^2}{4e_2} - e_5\right)\mathrm{d}\tau, \end{aligned} \tag{103}$$

with T_0 being an arbitrary constant and the conditions

$$\begin{aligned} &e_1e_{2\tau} - e_{1\tau}e_2 - 2e_1e_2e_4 = 0, \\ &2e_1e_2e_{3\tau} - (e_1e_2)_\tau e_3 - 4e_1e_2^2e_6 = 0. \end{aligned} \tag{104}$$

It is known that Eq.(102) has the following single soliton solution

$$P = K\mathrm{sech}(KX - K^3T), \tag{105}$$

where K is an arbitrary constant, which then generates the solution of Eq.(36) in the form of

$$\begin{aligned} A = -\frac{e_3}{2e_2} + KT_0\sqrt{\frac{6e_1}{e_2}}\mathrm{sech}\Bigg[&KT_0\xi + KT_0\int\left(\frac{e_3^2}{4e_2}\right. \\ &\left. - e_5 - K^2T_0^2e_1\right)\mathrm{d}\tau\Bigg], \end{aligned} \tag{106}$$

with conditions (104).

3.4. Exact solutions of the VCNLS equation (76)

In this case, the similar form of the assumption (94) is made only with the difference that α, β, P are complex while X, T are real. After calculations, it is revealed that the substitution of the assumption into Eq.(76) with e_i $(i = 0, 1, 2, 3, 4)$ all being functions of τ can transform the VCNLS Eq.(76) to the standard constant coefficient NLS equation

$$\mathrm{i}\,P_T + P_{XX} + |P|^2P = 0 \tag{107}$$

with

$$\alpha = 0, \quad X = \frac{Be_2\xi}{e_0e^{\int 2e_4\mathrm{d}\tau}} + B\int\frac{e_2\sqrt{e_1^2 + 4e_0e_3}}{e_0e^{\int 2e_4\mathrm{d}\tau}}\mathrm{d}\tau, \quad T = \int\frac{B^2e_2^2}{e_0e^{\int 4e_4\mathrm{d}\tau}}\mathrm{d}\tau, \tag{108}$$

$$\beta = B\sqrt{\frac{e_2}{e_0}}\exp\left(\mathrm{i}\,\frac{e_{0\tau}e_2 - e_0e_{2\tau} + 2e_0e_2e_4}{4e_0^2e_2}\xi^2 - \mathrm{i}\,\frac{e_1 + \sqrt{e_1^2 + 4e_0e_3}}{2e_0}\xi - 2\int e_4\mathrm{d}\tau\right), \tag{109}$$

where B is an arbitrary constant, and the conditions

$$\begin{aligned} &e_2(e_{0\tau}e_1 - e_0e_{1\tau})\sqrt{e_1^2+4e_0e_3} + 2(2e_{2\tau}e_3 - (4e_3e_4+e_{3\tau})e_2)e_0^2 \\ &\quad + ((e_{2\tau}-2e_2e_4)e_1^2 - e_1e_{1\tau}e_2 - 2e_{0\tau}e_2e_3)e_0 = 0, \qquad (110) \\ &(2(e_{4\tau}+2e_4^2)e_2^2 - (4e_{2\tau}e_4+e_{2\tau\tau})e_2 + 2e_{2\tau}^2)e_0^2 \\ &\quad + ((2e_{0\tau}e_4+e_{0\tau\tau})e_2^2 - e_{0\tau}e_2e_{2\tau})e_0 - e_{0\tau}^2e_2^2 = 0. \qquad (111) \end{aligned}$$

Familiarly, Eq.(107) possesses such a solution

$$P = -\sqrt{2}K_1\mathrm{sech}\big(K_1X + 2KK_1T\big)\mathrm{e}^{-\,\mathrm{i}\,\left(KX-(K_1^2-K^2)T\right)} \qquad (112)$$

with K and K_1 being arbitrary constants, which then produces a solution of Eq.(76)

$$\begin{aligned} A = &-BK_1\sqrt{\frac{2e_2}{e_0}}\mathrm{sech}\left[\frac{BK_1e_2}{e_0e^{\int 2e_4\mathrm{d}\tau}}\xi + BK_1\int\left(\frac{e_2\sqrt{e_1^2+4e_0e_3}}{e_0e^{\int 2e_4\mathrm{d}\tau}} + \frac{2KBe_2^2}{e_0e^{\int 4e_4\mathrm{d}\tau}}\right)\mathrm{d}\tau\right] \\ &\times\exp\left\{\mathrm{i}\,\frac{e_{0\tau}e_2 - e_0e_{2\tau} + 2e_0e_2e_4}{4e_0^2e_2}\xi^2 - \mathrm{i}\left(\frac{BKe_2}{e_0e^{\int 2e_4\mathrm{d}\tau}} + \frac{e_1+\sqrt{e_1^2+4e_0e_3}}{2e_0}\right)\xi\right. \\ &\left.-\int\left(\frac{\mathrm{i}\,BKe_2\sqrt{e_1^2+4e_0e_3}}{e_0e^{\int 2e_4\mathrm{d}\tau}} - \frac{\mathrm{i}(K_1^2-K^2)B^2e_2^2}{e_0e^{\int 4e_4\mathrm{d}\tau}} + 2e_4\right)\mathrm{d}\tau\right\}, \end{aligned} \qquad (113)$$

with conditions (110)–(111).

3.5. An illustration: a theoretical dipole blocking episode

The solutions presented above can all be used to describe some real atmospheric phenomena such as various blocking systems. In Ref.[24], a special solution generated from the KdV equation derived with the y average method has successfully captured the salient features of a dipole type blocking event, and was then further discussed through a real case. However, it is rather difficult to adjust the functions and parameters to produce an explicit desirable expression. Fortunately, the non-traditional method makes the job much easier for the existence of more arbitrary functions. In detail, all the variable coefficients e_i $(i = 1, 2, \ldots, 5)$ of the VCKdV equation (12) derived without the y-average method are entirely arbitrary and should be determined during the derivation with the y-average trick. As an illustration, we give a theoretical dipole blocking episode based on the VCKdV soliton solution (101). In this case, the original stream function up to the first order reads

$$\begin{aligned} \psi \approx\ & a_0y + a_1 + 2\epsilon\beta_0K^2F(\epsilon^{3/2}t) \\ &\times\sin\left(\sqrt{-\frac{\lambda_0^2c_0+\beta_0}{a_0+c_0}}y\right) \\ &\times\mathrm{sech}^2\left[\frac{K\sqrt{6\beta_0}}{18}\left(3\sqrt{\epsilon}(x-c_0t) - 2K^2\beta_0\epsilon^{3/2}t\right)\right], \end{aligned} \qquad (114)$$

where $F(\epsilon^{3/2}t)$ is an arbitrary function of the indicated variable, if we assume $e_3 = e_5 = \xi_0 = \Psi_1(y,\tau) = 0$, $\Psi_0(y,\tau) = a_0y + a_1$ with constants a_0 and a_1. Figure 1 displays a dipole type blocking event with a lifetime of seven days, where the function F is taken as

$$F = \mathrm{sech}\left[20\epsilon^{3/2}(t-4)\right], \qquad (115)$$

and all the constants are

$$\begin{aligned} &a_0 = -1, \quad a_1 = 3, \quad \epsilon = 0.1, \\ &\beta_0 = 1, \quad K = 8, \quad \lambda_0 = 0.25. \end{aligned} \qquad (116)$$

It is remarkable that due to the arbitrary functions and parameters in the analytical expression of the stream, the type and the lifetime of an atmospheric blocking can be easily fixed to match a real blocking case. More importantly, the background stream remains arbitrary in functions y and t, therefore a more real background westerly can be taken, for instance, a wester wind with linear and nonlinear shears. Further, we hope that blocking events can be somehow predicted based on some weather and climate information, especially on the basic westerly.

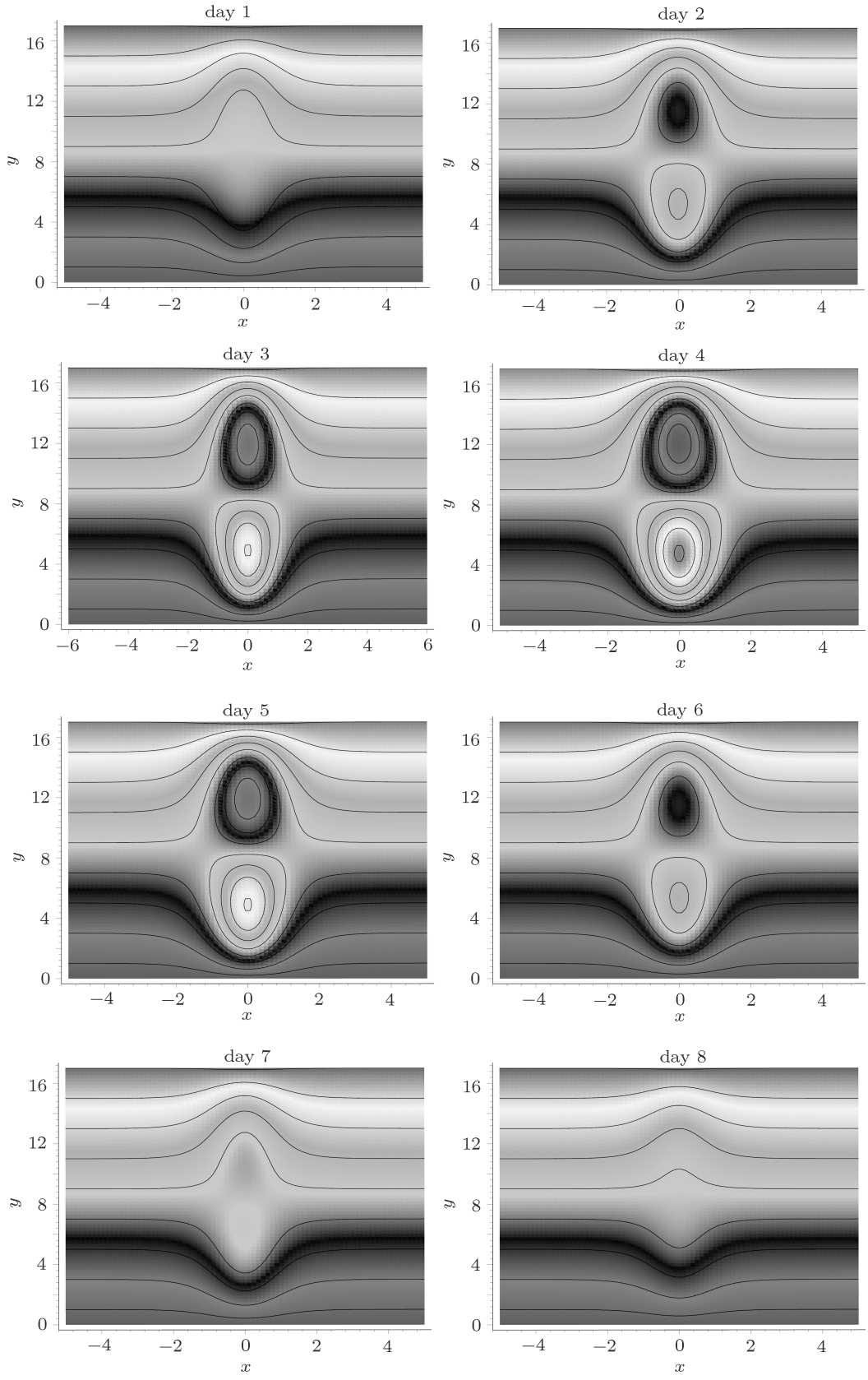


Fig.1. A dipole type blocking life cycle from the theoretical solution (114) with F given by Eq.(115) and the parameters given by Eq.(116). The contour interval (CI)=2.

4. Summary and discussions

Three types of variable coefficient nonlinear systems are derived from the nonlinear inviscid barotropic nondivergent vorticity equation in a beta-plane with and without the y-average trick. It is thus testified that the y-average trick can be removed in the derivation process though it is commonly used in the literature without exact reasons. The variable coefficients are all functions of time which essentially originate from the background basic flow because the uniform background basic flow is considered as a function of $\{y, t\}$ instead of a linear function of y only as in the usual treatment. In addition, when expanding the flow dream without the basic flow, we simultaneously expand the basic flow in an appropriate manner.

Through a simple direct method, we successfully find NABTs between the variable coefficient systems and the corresponding standard constant coefficient ones. Thus, in principle, many types of exact solutions of the variable coefficient models can be obtained by taking advantage of the NABTs and the known solutions of the constant coefficient ones. However, only one simple type of exact solutions, i.e., single soliton solutions, are written down for the derived variable coefficient nonlinear systems which therefore give three possible approximate solutions of the original nonlinear dynamical system in some particular order.

It can be seen that many functions of time appear in these solutions, some of which are arbitrary. Moreover, because the time dependent functions are related to the background westerly and/or unfixed boundary conditions, they thus can be determined according to different situations of the background westerly and boundary conditions in some real atmospheric phenomena, such as atmospheric blockings, and therefore they may be interpreted by our analytical solutions. More real applications of our solutions will be further discussed in our continuous research work.

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