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Approximation of the soliton solution for the generalized Vakhnenko equation*

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A class of generalized Vakhnemko equation is considered. First, we solve the nonlinear differential equation by the homotopic mapping method. Then, an approximate soliton solution for the original generalized Vakhnemko equation is obtained. By this method an arbitrary order approximation can be easily obtained and, similarly, approximate soliton solutions of other nonlinear equations can be acquired.

Keywords: homotopic mapping, soliton, Vakhnemko equation

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1. Introduction

The investigation of the solutions to nonlinear evolution equations plays an important role in studying soliton theory. Many different types of solitons in numerous media have been studied, which are often based on a few universal nonlinear equations, such as the nonlinear Schrödinger equation, the KdV equation, the Boussinesq equation, the sine-Gordon equation, the Burgers equation and so on. Many natural sciences such as chemistry, biology, mathematics, communication and, particularly, almost all branches of physics like fluid dynamics, plasma physics, field theory, optics, condensed matter physics, etc are all related to soliton theory.[1−8] Recently, more new methods have appeared, such as the hyperbolic tangent function method, the homogeneous equilibrium method, the Jacobi elliptic function method and the auxiliary equation method.[9] These methods have been applied to the study of shock waves, scattering light waves, quantum mechanics, atmospheric physics, networks of neurons and so on.[1,2,10,11] The approximate method for the nonlinear theory of solitons is a new subject of study. The key point of these methods is to transfer nonlinear problems to linear ones by using asymptotic expansion. The homotopic mapping method[12,13] is one of the asymptotic methods. Many other approximate methods have also been developed and improved, including the method of averaging, the boundary layer method, the method of matched asymptotic expansion and the method of multiple scales (see the studies by Ni and Wei,[14] Bartier,[15] Llibre, da Silva and Teixeira[16] and Guarguaglini and Natalini[17] and references therein). Using the method of differential inequalities and others, Mo *et al* also considered reaction diffusion problems,[18] activator inhibitor systems,[19] ecological environments,[20] shock waves,[21] solitons,[22−24] laser pulses,[25] ocean science,[26,27] atmospheric physics,[28−32] etc. In the present paper, we consider a generalized Vakhnenko equation and obtain the approximate solution of a soliton. By using a simple and valid technique and homotopic mapping method[12,13] we obtain the approximate solution of a soliton.

2. Generalized Vakhnenko equation and homotopic mapping

Consider the following generalized Vakhnenko equation:[11]

$$u_{tx} + u_x^2 + u\,u_{xx} + u = f(u), \tag{1}$$

where f is a disturbed term, which is a sufficiently smooth function in a corresponding domain.

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In order to obtain the solution of Eq.(1), set

$$u = \sum_{i=0}^{\infty} u_i(t,x) r^i, \tag{2}$$

where r is the parameter. We introduce the following homotopic mapping $H(u,r)$:[12,13]

$$\begin{aligned} &H(u,r) \\ &= L[u] - L[u_0] + r(L[u_0] + u_x^2 + u u_{xx} - f(u)), \end{aligned} \tag{3}$$

where u_0 is an initial approximate solution to Eq.(1), which will be determined below, and the linear operator L is expressed as

$$L[u] = u_{tx} + u.$$

Obviously, from mapping (3), $H(u,1) = 0$ is the same as Eq.(1). Thus the solution of Eq.(1) is the same as the solution of $H(u,r) = 0$ as $r \to 1$.

3. Approximate solution

We first consider the generalized Vakhnenko equation for $f = 0$, i.e.

$$u_{tx} + u_x^2 + u\, u_{xx} + u = 0. \tag{4}$$

It is a standard Vakhnenko equation, which is a nonlinear equation with loop solton solutions describing the propagation of high-frequency waves in a relaxing medium. We can obtain a soliton solution as follows:[11]

$$u(t,x) = 6k^2 - 6k^2 \left(\frac{C_1 \exp(2k\varepsilon_0) + C_2}{C_1 \exp(2k\varepsilon_0) - C_2} \right)^2,$$

where $x - 4k^2 t = -4k^2\varepsilon_0 + x_0 + a_0 + 6k(C_1 \exp(2k\varepsilon_0) + C_2)/(C_1 \exp(2k\varepsilon_0) - C_2)$, and $k > 0$, a_0, C_1, C_2 and x_0 are arbitrary constants.

We choose the initial approximate solution u_0 of Eq.(1), which is a soliton solution for Eq.(4) , i.e.

$$u_0(t,x) = 6k^2 - 6k^2 \left(\frac{C_1 \exp(2k\varepsilon_0) + C_2}{C_1 \exp(2k\varepsilon_0) - C_2} \right)^2, \tag{5}$$

where $x - 4k^2 t = -4k^2\varepsilon_0 + x_0 + a_0 + 6k(C_1 \exp(2k\varepsilon_0) + C_2)/(C_1 \exp(2k\varepsilon_0) - C_2)$, and $k > 0$, a_0, C_1, C_2 and x_0 are arbitrary constants.

Substituting expression (5) into $H(u,r) = 0$, expanding nonlinear terms into the power series in powers of r, we compare the coefficients of the same power of r on both sides of the equation. From the coefficients of the first power of r, i.e. r^1, we have

$$L(u_1) - f(u_0) = 0. \tag{6}$$

Using Fourier's transform method, it is easy to see that the solution u_1 of Eq.(6) with zero initial value is

$$\begin{aligned} &\mathrm{i}\lambda \frac{\mathrm{d}F[u_1]}{\mathrm{d}t} + F[u_1] - F[f(u_0)] = 0, \\ &F[u_1]|_{t=0} = 0, \end{aligned} \tag{7}$$

where $F[\cdot]$ denotes the operator of the Fourier transform.

The solution of linear equation (7) is

$$F[u_1] = -\frac{\mathrm{i}}{\lambda} \int_0^t [F(f(u_0(\tau,x)))] \exp(\mathrm{i}\lambda^{-1}(t-\tau)) \mathrm{d}\tau.$$

Thus we have

$$u_1(t,x) = -\frac{\mathrm{i}}{2\pi} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\lambda} \left[f(u_0(\tau,\xi)) \exp(\mathrm{i}(\lambda^{-1}(t-\tau) - \lambda\xi + \lambda x)) \right] \mathrm{d}\xi\, \mathrm{d}\lambda \mathrm{d}\tau. \tag{8}$$

Substituting expressions (5) and (8) into equation $H(u,r) = 0$, we compare the coefficients of the same power of r on both sides of the equation. From the coefficients of the second power of r, i.e. r^2, we have

$$L(u_2) + 2\frac{\partial u_0}{\partial x}\frac{\partial u_1}{\partial x} + u_1 \frac{\partial^2 u_0}{\partial x^2} + u_0 \frac{\partial^2 u_1}{\partial x^2} - u_1 \frac{\mathrm{d}f(u_0)}{\mathrm{d}u_0} = 0.$$

Similarly, we can obtain the solution $u_2(t,x)$ for the above linear equation as follows:

$$\begin{aligned} u_2(t,x) = -\frac{\mathrm{i}}{2\pi} \int_0^t \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \frac{1}{\lambda} \Big[\Big(&-2\frac{\partial u_0}{\partial \xi}(\tau,\xi)\frac{\partial u_1}{\partial \xi}(\tau,\xi) - u_1(\tau,\xi)\frac{\partial^2 u_0}{\partial \xi^2}(\tau,\xi) - u_0(\tau,\xi)\frac{\partial^2 u_1}{\partial \xi^2}(\tau,\xi) \\ &+ u_1(\tau,\xi)\frac{\mathrm{d}f_0(u_0)}{\mathrm{d}u_0}(\tau,\xi) \Big) \exp \mathrm{i}(\lambda^{-1}(t-\tau) - \lambda\xi + \lambda x) \Big] \mathrm{d}\xi \mathrm{d}\lambda \mathrm{d}\tau, \end{aligned} \tag{9}$$

where u_0 and u_1 are expressed in expressions (5) and (8), respectively.

Thus from expressions (5), (8) and (9) and mapping (3), we have the first and second order approximate soliton solutions $U_1(t,x)$ and $U_2(t,x)$ of the generalized Vakhnenko equation (1) as follows:

$$U_1(t,x) = 6k^2 - 6k^2\left(\frac{C_1\exp(2k\varepsilon_0)+C_2}{C_1\exp(2k\varepsilon_0)-C_2}\right)^2 - \frac{\mathrm{i}}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\lambda}[f(u_0(\tau,\xi)) \times\exp(\mathrm{i}(\lambda^{-1}(t-\tau)-\lambda\xi+\lambda x))]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau, \tag{10}$$

$$U_2(t,x) = 6k^2 - 6k^2\left(\frac{C_1\exp(2k\varepsilon_0)+C_2}{C_1\exp(2k\varepsilon_0)-C_2}\right)^2 -\frac{\mathrm{i}}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\lambda}\left[\left(-2\frac{\partial u_0}{\partial\xi}(\tau,\xi)\frac{\partial u_1}{\partial\xi}(\tau,\xi)-u_1(\tau,\xi)\frac{\partial^2 u_0}{\partial\xi^2}(\tau,\xi)\right)-u_0(\tau,\xi)\frac{\partial^2 u_1}{\partial\xi^2}(\tau,\xi) +f(u_0(\tau,\xi))+u_1(\tau,\xi)\frac{\mathrm{d}f_0(u_0)}{\mathrm{d}u_0}(\tau,\xi)\right)\exp(\mathrm{i}(\lambda^{-1}(t-\tau)-\lambda\xi+\lambda x))\Big]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau, \tag{11}$$

where $x - 4k^2t = -4k^2\varepsilon_0 + x_0 + a_0 + 6k(C_1\exp(2k\varepsilon_0)+C_2)/(C_1\exp(2k\varepsilon_0)-C_2)$, and $k>0$, a_0, C_1, C_2 and x_0 are arbitrary constants.

Similarly, we can also obtain the arbitrary nth order approximate soliton solution $U_n(t,x)$ $(n\geq 3)$ of the generalized Vakhnenko equation (1).

4. Examples

As special examples, we study some disturbed Vakhnenko equations.

(1) Let the disturbed term $f(u)=\varepsilon u^3$ of Eq.(1), where $0<\varepsilon\ll 1$. Then we consider the following disturbed Vakhnenko equation:

$$Nu \equiv u_{tx} + u_x^2 + u\,u_{xx} + u - \varepsilon\,u^3 = 0, \quad 0<\varepsilon\ll 1. \tag{12}$$

Using the above homotopic mapping method, setting $u=\sum_{i=0}^{\infty}u_i(t,x)r^i$. From Eq.(5), we first choose $u_0(t,x)$ to be

$$u_0(t,x) = 6k^2 - 6k^2\left(\frac{C_1\exp(2k\varepsilon_0)+C_2}{C_1\exp(2k\varepsilon_0)-C_2}\right)^2, \tag{13}$$

where $x - 4k^2t = -4k^2\varepsilon_0 + x_0 + a_0 + 6k(C_1\exp(2k\varepsilon_0)+C_2)/(C_1\exp(2k\varepsilon_0)-C_2)$, and $k>0$, a_0, C_1, C_2 and x_0 are arbitrary constants. Thus from expression (8), we have

$$u_1(t,x) = -\frac{\mathrm{i}\varepsilon}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\lambda}\left[u_0^3(\tau,\xi)\exp(\mathrm{i}(\lambda^{-1}(t-\tau)-\lambda\xi+\lambda x))\right]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau. \tag{14}$$

Then from expression (11), the second order approximate soliton solution $U_2(t,x)$ of the small disturbed Vakhnenko equation (12) is obtained to be

$$\begin{aligned} &U_2(t,x)\\ &= 6k^2 - 6k^2\left(\frac{C_1\exp(2k\varepsilon_0)+C_2}{C_1\exp(2k\varepsilon_0)-C_2}\right)^2 - \frac{\mathrm{i}\varepsilon}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\lambda}\left[\left(-2\frac{\partial u_0}{\partial\xi}(\tau,\xi)\frac{\partial u_1}{\partial\xi}(\tau,\xi)-u_1(\tau,\xi)\frac{\partial^2 u_0}{\partial\xi^2}(\tau,\xi)\right)\right.\\ &\left.-u_0(\tau,\xi)\frac{\partial^2 u_1}{\partial\xi^2}(\tau,\xi)+u_0^3(\tau,\xi)+3u_0^2(\tau,\xi)u_1(\tau,\xi)\right)\exp(\mathrm{i}(\lambda^{-1}(t-\tau)-\lambda\xi+\lambda x))\Big]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau, \end{aligned} \tag{15}$$

where $x-4k^2t = -4k^2\varepsilon_0 + x_0 + a_0 + 6k(C_1\exp(2k\varepsilon_0)+C_2)/(C_1\exp(2k\varepsilon_0)-C_2)$, and $k>0$, a_0, C_1, C_2 and x_0 are arbitrary constants.

Now we consider the accuracy of the homotopic mapping solution $U_{2\,\mathrm{hom}} \equiv U_2(t,x)$.

Firstly, we know that Eq.(12) is a slightly disturbed Vakhnenko equation. Thus we can obtain the disturbed solution $U_{\mathrm{per}}(t,x)$ by using the perturbation method.

Let $U_{\mathrm{per}}(t,x) = \sum\limits_{i-0}^{\infty} u_{i\,\mathrm{per}}\varepsilon^i$ and substitute it into Eq.(12), then we will have

$$\begin{aligned}&(u_{0\mathrm{per}})_{tx} + u_{0\mathrm{per}} + (u_{0\mathrm{per}})_x^2\\&+u_{0\mathrm{per}}\,(u_{0\mathrm{per}})_{xx} = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}&(u_{1\mathrm{per}})_{tx} + u_{1\mathrm{per}} + (2u_{0\mathrm{per}}u_{1\mathrm{per}})_x\\&+u_{0\mathrm{per}}\,(u_{1\mathrm{per}})_{xx} + u_{1\mathrm{per}}\,(u_{0\mathrm{per}})_{xx} = (u_{0\mathrm{per}})^3.\end{aligned} \tag{17}$$

We can obtain solutions $u_{0\mathrm{per}}(t,x)$ and $u_{1\mathrm{per}}(t,x)$ to Eqs.(16) and (17) with $u_{0\mathrm{per}}(0,x) = u_0(0,t)$ and $u_{1\mathrm{per}}(0,x) = 0$, respectively, for fixed constants k, a_0, C_1, C_2, and x_0. Then we have the first approximate disturbed solution

$$U_{1\mathrm{per}}(t,x) = u_{0\mathrm{per}} + u_{1\mathrm{per}}\varepsilon. \tag{18}$$

Let the exact solution $u_{\mathrm{ext}}(t,x)$ of the disturbed Vakhnenko equation (12) have the same initial condition as $U_{1\mathrm{per}}(t,x)$. Thus it is easy to see that

$$|u_{\mathrm{ext}}(t,x) - U_{1\mathrm{per}}(t,x)| = O(\varepsilon), \quad 0 < \varepsilon \ll 1, (t,x) \in [0,T_0]\times\bar{\Omega}, \tag{19}$$

$$\begin{aligned}|u_{\mathrm{ext}}(0,x) - U_{1\mathrm{per}}(0,x)| = O(\varepsilon), \quad |U_{1\mathrm{per}}(0,x) - U_{2\mathrm{hom}}(0,x)| = O(\varepsilon),\\ 0 < \varepsilon \ll 1, \quad x \in \bar{\Omega},\end{aligned} \tag{20}$$

where T_0 is a large enough positive constant and $\bar{\Omega}$ is an arbitrary bounded closed interval in R.

On the other hand, in Eqs.(12)–(18), there exists a positive constant M, such that

$$\begin{aligned}&|N(U_{1\mathrm{per}}(t,x) - U_{2\,\mathrm{hom}}(t,x))|\\ \le\ &|(u_{0\mathrm{per}})_{tx} + u_{0\mathrm{per}} + (u_{0\mathrm{per}})_x^2 + u_{0\mathrm{per}}\,(u_{0\mathrm{per}})_{xx}| + |(u_{1\mathrm{per}})_{tx} + u_{1\mathrm{per}} + (2u_{0\mathrm{per}}u_{1\mathrm{per}})_x\\ &+u_{0\mathrm{per}}\,(u_{1\mathrm{per}})_{xx} + u_{1\mathrm{per}}\,(u_{0\mathrm{per}})_{xx} - (u_{0\mathrm{per}})^3| + \big|(u_0)_{tx} + u_0 + (u_0)_x^2 + u_0(u_0)_{xx}\big|\\ &+\big|(u_1)_{tx} + u_1 - u_0^3\big| + |(u_2)_{tx} + u_2 + 2(u_0)_x(u_1)_x + u_1(u_0)_{xx} + u_0(u_1)_{xx} - 3u_0^2u_1| + M\varepsilon\\ =\ &M\varepsilon, \quad 0 < \varepsilon \ll 1, \ \ (t,x) \in [0,T_0]\times\bar{\Omega}.\end{aligned} \tag{21}$$

From Eqs.(20) and (21) and the fixed theorem,[33] we then have

$$|U_{1\mathrm{per}}(t,x) - U_{2\,\mathrm{hom}}(t,x)| = O(\varepsilon), \quad 0 < \varepsilon \ll 1, \ \ (t,x) \in [0,T_0]\times\bar{\Omega}. \tag{22}$$

From Eq.(19) and (22), we then have

$$|u_{\mathrm{ext}}(t,x) - U_{2\,\mathrm{hom}}(t,x)| = O(\varepsilon), \quad 0 < \varepsilon \ll 1, \ \ (t,x) \in [0,T_0]\times\bar{\Omega}.$$

Therefore, we know that the homotopic mapping solution $U_{2\,\mathrm{hom}} \equiv U_2(t,x)$ decided from expression (15) of a soliton wave for the disturbed Vakhnenko equation (12) has a good accuracy.

Similarly, from homotopic mapping (3), by using the same method, we can obtain higher order approximate soliton solutions $U_n(t,x)$, $n = 3,4,\ldots$, of disturbed Vakhnenko equation (12).

(2) Let the disturbed term $f(u) = \varepsilon\exp u$ in Eq.(1), where $0 < \varepsilon \ll 1$, then we will consider the following slightly disturbed Vakhnenko equation

$$Nu \equiv u_{tx} + u_x^2 + u\,u_{xx} + u - \varepsilon\exp u = 0, \quad 0 < \varepsilon \ll 1. \tag{23}$$

By using the above homotopic mapping method, assuming $u = \sum\limits_{i=0}^{\infty} u_i(t,x)r^i$, and selecting $u_0(t,x)$ as expression (5), from expression (8) we have

$$u_1(t,x) = -\frac{\mathrm{i}\varepsilon}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\lambda}[\exp u_0(\tau,\xi)\exp(\mathrm{i}(\lambda^{-1}(t-\tau) - \lambda\xi + \lambda x))]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau. \tag{24}$$

Then from expression (11), the second order approximate soliton solution $U_2(t,x)$ of the slightly disturbed Vakhnenko equation (12) is

$$U_2(t,x) = 6k^2 - 6k^2\left(\frac{C_1\exp(2k\varepsilon_0)+C_2}{C_1\exp(2k\varepsilon_0)-C_2}\right)^2 - \frac{\mathrm{i}\varepsilon}{2\pi}\int_0^t\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\frac{1}{\lambda}\Big[\Big(-2\frac{\partial u_0}{\partial\xi}(\tau,\xi)\frac{\partial u_1}{\partial\xi}(\tau,\xi) - u_1(\tau,\xi)\frac{\partial^2 u_0}{\partial\xi^2}(\tau,\xi)\Big) - u_0(\tau,\xi)\frac{\partial^2 u_1}{\partial\xi^2}(\tau,\xi) + \exp u_0(\tau,\xi) + u_1(\tau,\xi)\exp u_0(\tau,\xi))\exp(\mathrm{i}(\lambda^{-1}(t-\tau)-\lambda\xi+\lambda x))\Big]\mathrm{d}\xi\mathrm{d}\lambda\mathrm{d}\tau, \quad (25)$$

where $x - 4k^2t = -4k^2\varepsilon_0 + x_0 + a_0 + 6k(C_1\exp(2k\varepsilon_0)+C_2)/(C_1\exp(2k\varepsilon_0)-C_2)$, and $k > 0$, a_0, C_1, C_2 and x_0 are arbitrary constants.

Using the same method and fixed theorem as in example (1), we know also that the homotopic mapping solution $U_{2\,\mathrm{hom}} \equiv U_2(t,x)$ determined by expression (25) of a soliton wave for the disturbed Vakhnenko equation (23) has a good accuracy.

Similarly, from homotopic mapping (3), by using the same method we can also obtain higher order approximate soliton solutions $U_n(t,x)$, $n = 3,4,\ldots$, of the disturbed Vakhnenko equation (25).

5. Discussion

Soliton solutions for the disturbed Vakhnenko equation describe a kind of complicated natural phenomenon. Hence basic models need to be reduced. They are solved by using the approximate method. The homotopic mapping method is a simple and valid method. The homotopic mapping method is an approximate analytic method, which differs from the general numerical method. The expansion of solutions via the homotopic mapping method can be continuously performed analytically. Thus, with expression (10) and Eq.(11) we can further study the behaviours of the solitary wave qualitatively or quantitatively.

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