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Homotopic mapping solution of an oscillator for the El Niño / La Niña-southern oscillation*

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This paper considers a class of oscillator for the El Niño/La Niña-southern oscillation (ENSO) model. By using the homotopic mapping method, it obtains approximations of the solution for the ENSO model.

Keywords: nonlinear, homotopic mapping, ENSO model
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1. Introduction

The El Niño / La Niña-southern oscillation (ENSO) is an interannual phenomenon involving tropical Pacific ocean–atmosphere interactions. The ENSO phenomenon is a very attractive object of study in international academic circles.[1−5] The oscillatory nature of ENSO requires both positive and negative ocean–atmosphere feedbacks. The weaker trade winds drive ocean circulation changes that reinforce the SST anomaly. This positive ocean–atmosphere feedback or coupled ocean–atmosphere instability leads the equatorial Pacific to a never-ending warm state. Recently, many scholars have investigated the nonlinear problem. Approximation methods have been developed and refined, including the method of averaging, the boundary layer method, the matched asymptotic expansion method, and the multiple scales method.[6−9] Using the perturbation method, Mo *et al* considered a class of nonlinear perturbed problems.[10−24] In this paper we consider a class of ENSO model by using a simple and special method for the homotopic mapping theory.[25,26]

The conceptual oscillator model should consider the variations of both the eastern and western Pacific anomaly patterns. As will be shown below, the oscillator model can be reduced to a simple oscillator model by further simplification and assumptions. The western Pacific oscillator of Wang *et al*[4,5] emphasizes the role of the western Pacific anomaly patterns in ENSO.

2. An ENSO model

Now we consider the sea–air oscillator of an ENSO model:[5]

$$\frac{\mathrm{d}T}{\mathrm{d}t} = CT + Dh - f(T,h), \tag{1}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -ET - R_h h - g(T,h), \tag{2}$$

where T is the SST anomaly, h is the thermocline depth anomaly, perturbation terms f and g are sufficiently smooth functions which satisfy conditions $\lim\limits_{\|y\|\to 0} \dfrac{\|(f,h)\|}{\|y\|} = 0$, $\|y\| = (T^2 + h^3)^{1/2}$, C, D, E and R_h are model parameters; for their physical definitions see the Refs.[1]–[3].

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First, as $f = g = 0$, Eqs.(1) and (2) becomes

$$\frac{\mathrm{d}T}{\mathrm{d}t} = CT + Dh, \quad \frac{\mathrm{d}h}{\mathrm{d}t} = -ET - R_h h. \tag{3}$$

It is easy to see that the characteristic roots $\lambda_i,\ i = 1, 2$ of the linear Eq.(3) are

$$\lambda_1 = \frac{1}{2}\left[(C - R_h) - \sqrt{(C + R_h)^2 - 4DE}\right], \tag{4}$$

$$\lambda_2 = \frac{1}{2}\left[(C - R_h) + \sqrt{(C + R_h)^2 - 4DE}\right]. \tag{5}$$

Obviously, as $CR_h > DE$, we have $\lambda_1 < 0 < \lambda_2$. Then from singular point theory, the zero solution of the linear Eq.(3) is a saddle point on the phase plane.

Because the nonlinear terms of Eqs.(1) and (2) satisfy $\lim\limits_{\|y\|\to 0} \dfrac{\|(f, h)\|}{\|y\|} = 0$, $\|y\| = (T^2 + h^3)^{1/2}$, thus the singular solution of the nonlinear coupling Eqs.(1) and (2) has the same stability as that of the linear Eq.(3), so its singular solution is a saddle point on the phase plane and unstable too.

The solution $(\bar{T}(t), \bar{h}(t))$ of Eq.(3) with initial conditions $\bar{T}(0) = T(0),\ \bar{h}(0) = h(0)$ is

$$\begin{aligned} \bar{T}_0(t) &= T(0)\exp(Ct), \\ \bar{h}_0(t) &= \left(h(0) + \frac{ET(0)}{C + R_h}\right)\exp(-R_h t) \\ &\quad - \frac{ET(0)}{C + R_h}\exp(Ct), \end{aligned} \tag{6}$$

as $D = 0$;

$$\begin{aligned} \bar{T}_0(t) &= \frac{(\lambda_2 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}\exp(\lambda_1 t) \\ &\quad - \frac{(\lambda_1 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}\exp(\lambda_2 t), \end{aligned} \tag{7}$$

$$\begin{aligned} \bar{h}_0(t) &= \frac{(\lambda_2 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}(\lambda_1 - C)\exp(\lambda_1 t) \\ &\quad - \frac{(\lambda_1 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}(\lambda_2 - C)\exp(\lambda_2 t), \end{aligned} \tag{8}$$

as $D \neq 0$, where $\lambda_1,\ \lambda_2$ are decided by Eqs.(4) and (5) respectively.

3. Homotopic mapping

Introducing a set of homotopic mapping $H_i(T, h, p):\ X^2 \times I \to R, i = 1, 2$:

$$\begin{aligned} H_1(T, h, p) &= L_1(T, h) - L_1(\bar{T}, \bar{h}) \\ &\quad + p(L_1(\bar{T}, \bar{h}) + f(T, h)), \end{aligned} \tag{9}$$

$$\begin{aligned} H_2(T, h, p) &= L_2(T, h) - L_2(\bar{T}, \bar{h}) \\ &\quad + p(L_2(\bar{T}, \bar{h}) + g(T, h)), \end{aligned} \tag{10}$$

where $X = [0, \infty),\ I = [0, 1],\ R = (-\infty, +\infty)$, while linear operators $L_i,\ i = 1, 2$, are

$$L_1(T, h) = \frac{\mathrm{d}T}{\mathrm{d}t} - CT - Dh,$$

$$L_2(T, h) = \frac{\mathrm{d}h}{\mathrm{d}t} + ET + R_h h.$$

$(\bar{T}, \bar{h})$ are zeroth order approximations of solutions for the original Eqs.(1) and (2).

Obviously, from Eqs.(9) and (10), $H_j(T, h, 1) = \lim\limits_{p\to 1} H_j(T, h, p) = 0$, $j = 1, 2$, are the same as Eqs.(1) and (2). Then the solution $(T(t), h(t))$ of Eqs.(1) and (2) is the same as the solution of $H_j(T, h, p) = 0$, $j = 1, 2$, as $p \to 1$

Let

$$T = \sum_{k=0}^{\infty} T_k(t)p^k, \quad h = \sum_{k=0}^{\infty} h_k(t)p^k. \tag{11}$$

Considering Eqs.(6)–(11), from $H_i(T, h, p) = 0,\ i = 1, 2$, developing nonlinear terms to the power series in p, we compare the coefficients of the same power in p for both sides of the equations.

From the coefficients of zeroth order power in p, we have

$$L_i(T_0, h_0) = L_i(\bar{T}, \bar{h}), \quad i = 1, 2.$$

Obviously, then we have

$$\begin{aligned} T_0(t) &= T(0)\exp(Ct), \\ h_0(t) &= \left(h(0) + \frac{ET(0)}{C + R_h}\right)\exp(-R_h t) \\ &\quad - \frac{ET(0)}{C + R_h}\exp(Ct), \end{aligned} \tag{12}$$

as $D = 0$;

$$\begin{aligned} T_0(t) &= \frac{(\lambda_2 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}\exp(\lambda_1 t) \\ &\quad - \frac{(\lambda_1 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}\exp(\lambda_2 t), \end{aligned} \tag{13}$$

$$\begin{aligned} h_0(t) &= \frac{(\lambda_2 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}(\lambda_1 - C)\exp(\lambda_1 t) \\ &\quad - \frac{(\lambda_1 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}(\lambda_2 - C)\exp(\lambda_2 t), \end{aligned} \tag{14}$$

as $D \neq 0$.

From the coefficients of first order power in p for the equation $H_i(T, h, p) = 0,\ i = 1, 2$, we obtain

$$\begin{aligned} &\frac{\mathrm{d}T_1}{\mathrm{d}t} - CT_1 - Dh_1 = f(T_0, h_0), \\ &\frac{\mathrm{d}h_1}{\mathrm{d}t} + ET_1 + R_h h_1 = g(T_0, h_0). \end{aligned} \tag{15}$$

Thus from Eqs.(12)–(15) with zero initial value, we obtain

$$T_1(t) = \int_0^t f(T_0(\tau), h_0(\tau)) \exp(-C(t-\tau))\mathrm{d}\tau,$$
$$h_1(t) = \int_0^t g(T_0(\tau), h_0(\tau)) \exp(-R_h(t-\tau))\mathrm{d}\tau - E\int_0^t \int_0^\tau f(T_0(\tau_1), h_0(\tau_1)) \exp(-R_h t + (R_h - C)\tau - C\tau_1)\mathrm{d}\tau_1\mathrm{d}\tau,$$

as $D = 0$;

$$T_1(t) = \frac{1}{\lambda_2 - \lambda_1}\int_0^t [((\lambda_2 - C)f(T_0(\tau), h_0(\tau)) - Dg(T_0(\tau), h_0(\tau)) \exp(\lambda_1(t-\tau)) + (-(\lambda_1 - C)f(T_0(\tau), h_0(\tau)) - D(\lambda_2 - C)g(T_0(\tau), h_0(\tau)) \exp(\lambda_2(t-\tau))]\mathrm{d}\tau,$$

$$h_1(t) = \frac{1}{D(\lambda_2 - \lambda_1)}\int_0^t [(\lambda_1 - C)((\lambda_2 - C)f(T_0(\tau), h_0(\tau)) - Dg(T_0(\tau), h_0(\tau)) \exp(\lambda_1(t-\tau) + (\lambda_2 - C))(-(\lambda_1 - C)f(T_0(\tau), h_0(\tau)) - D(\lambda_2 - C)g(T_0(\tau), h_0(\tau)) \exp(\lambda_2(t-\tau))]\mathrm{d}\tau,$$

as $D \neq 0$, where λ_1, λ_2 are decided by Eqs.(4) and (5), T_0 and h_0 are denoted by Eqs.(13) and (14) repectively.

Then we obtan the first order approximations $T_{1\mathrm{app}}$, $h_{1\mathrm{app}}$ of solutions for the sea–air oscillator of the ENSO models (1) and (2):

(1) As $D = 0$

$$T_{1\mathrm{app}}(t) = T_0(0)\exp(Ct) + \int_0^t f(T_0(\tau), h_0(\tau)) \exp(-C(t-\tau))\mathrm{d}\tau, \tag{16}$$

$$h_{1\mathrm{app}} = \left(h(0) + \frac{ET(0)}{C + R_h}\right)\exp(-R_h t) - \frac{ET(0)}{C + R_h}\exp(Ct) + \int_0^t g(T_0(\tau), h_0(\tau)) \exp(-R_h(t-\tau))\mathrm{d}\tau - E\int_0^t \int_0^\tau f(T_0(\tau_1), h_0(\tau_1)) \exp(-R_h t + (R_h - C)\tau - C\tau_1)\mathrm{d}\tau_1\mathrm{d}\tau, \tag{17}$$

(2) As $D \neq 0$,

$$T_{1\mathrm{app}}(t) = \frac{(\lambda_2 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}\exp(\lambda_1 t) - \frac{(\lambda_1 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}\exp(\lambda_2 t) + \frac{1}{\lambda_2 - \lambda_1}\int_0^t [((\lambda_2 - C)f(T_0(\tau), h_0(\tau)) - Dg(T_0(\tau), h_0(\tau)) \exp(\lambda_1(t-\tau)) + (-(\lambda_1 - C)f(T_0(\tau), h_0(\tau)) - D(\lambda_2 - C)g(T_0(\tau), h_0(\tau)) \exp(\lambda_2(t-\tau))]\mathrm{d}\tau, \tag{18}$$

$$h_{1\mathrm{app}}(t) = \frac{(\lambda_2 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}(\lambda_1 - C)\exp(\lambda_1 t) - \frac{(\lambda_1 - C)T(0) - Dh(0)}{D(\lambda_2 - \lambda_1)}(\lambda_2 - C)\exp(\lambda_2 t) + \frac{1}{D(\lambda_2 - \lambda_1)}\int_0^t [(\lambda_1 - C)((\lambda_2 - C)f(T_0(\tau), h_0(\tau)) - Dg(T_0(\tau), h_0(\tau)) \exp(\lambda_1(t-\tau) + (\lambda_2 - C))(-(\lambda_1 - C)f(T_0(\tau), h_0(\tau)) - D(\lambda_2 - C)g(T_0(\tau), h_0(\tau)) \exp(\lambda_2(t-\tau))]\mathrm{d}\tau. \tag{19}$$

Using the same method, we can obtain a higher order approximate solution of Eqs.(1) and (2).

4. Comparison of precision

In order to show the precision of the above result, we now compare the following special case of infinitesimal disturbance.

Let $f = \varepsilon T^3$, $g = \varepsilon h^3$, $0 < \varepsilon \ll 1$ and $D = 0$, then the ENSO models (1) and (2) are

$$\frac{\mathrm{d}T}{\mathrm{d}t} = CT - \varepsilon T^3, \tag{20}$$

$$\frac{\mathrm{d}h}{\mathrm{d}t} = -ET - R_h h - \varepsilon h^3. \tag{21}$$

It is easy to see that the exact solution of $T(t)$ for the Eq.(20) is

$$T_{\mathrm{exa}}(t) = \left[\frac{\varepsilon}{C} + \left(\frac{1}{(T(0))^2} - \frac{\varepsilon}{C}\right)\exp(-2Ct)\right]^{-\frac{1}{2}}. \tag{22}$$

Thus for $0 < \varepsilon \ll 1$ we have

$$T_{\mathrm{exa}}(t) = T(0)\exp(Ct) - \frac{\varepsilon(T(0))^3}{2C}\exp(3Ct)[1 - \exp(-2Ct)] + O(\varepsilon^2). \tag{23}$$

From Eqs.(21) and (22), we have

$$\frac{\mathrm{d}h}{\mathrm{d}t} + R_h h = -\varepsilon h^3 - E\left[\frac{\varepsilon}{C} + \left(\frac{1}{(T(0))^2} - \frac{\varepsilon}{C}\right)\exp(-2Ct)\right]^{-1/2}. \tag{24}$$

Using the perturbation method, let

$$h = \sum_{i=0}^{\infty} h_i(t)\varepsilon^i, \tag{25}$$

substituting Eq.(25) into Eq.(24), developing nonlinear terms to the power series in ε, we compare the coefficients of the same power in ε for both sides of the equation. For $i = 0, 1$, we obtain

$$\frac{\mathrm{d}h_0}{\mathrm{d}t} + R_h h_0 = -ET(0)\exp(Ct), \tag{26}$$

$$\frac{\mathrm{d}h_1}{\mathrm{d}t} + R_h h_1 = -h_0^3 - \frac{ET^3(0)}{C}\left(1 - \exp(-2Ct)\right). \tag{27}$$

The solutions of Eqs.(26) and (27) are

$$h_0(t) = h(0)\exp(-R_h t) - \frac{ET(0)}{C + R_h}\exp(Ct), \tag{28}$$

$$h_1(t) = -\int_0^t \left[h_0^3(\tau)\exp(-R_h(t-\tau)) + \frac{ET^3(0)}{C}(1 - \exp(-2C\tau))\exp(-R_h(t-\tau))\right]\mathrm{d}\tau. \tag{29}$$

Thus from Eqs.(28), (29) and Eq.(25), we obtain

$$\begin{aligned} h_{1\mathrm{per}}(t) \;=\; & h(0)\exp(-R_h t) - \frac{ET(0)}{C + R_h}\exp(Ct) \\ & -\varepsilon\int_0^t \left[(h(0)\exp(-R_h t) - \frac{ET(0)}{C + R_h}\exp(Ct))^3\exp(-R_h(t-\tau))\right. \\ & \left. + \frac{ET^3(0)}{C}(1 - \exp(-2C\tau))\exp(-R_h(t-\tau))\right]\mathrm{d}\tau + O(\varepsilon^2), \quad 0 < \varepsilon \ll 1. \end{aligned} \tag{30}$$

On the other hand, using the above homotopic mapping Eqs.(9) and (10), from Eqs.(16) and (17), as ε is small enough we obtained the first order approximations $T_{1\mathrm{app}}(t)$, $h_{1\mathrm{app}}(t)$ of solution for Eqs.(20) and (21),

$$T_{1\mathrm{app}}(t) \;=\; T(0)\exp(Ct) - \frac{\varepsilon(T(0))^3}{2C}\exp(3Ct)[1 - \exp(-2Ct)] + O(\varepsilon^2), \quad 0 < \varepsilon \ll 1. \tag{31}$$

$$\begin{aligned} h_{1\mathrm{app}}(t) \;=\; & h(0)\exp(-R_h t) - \frac{ET(0)}{C + R_h}\exp(Ct) \\ & -\varepsilon\int_0^t \left[(h(0)\exp(-R_h t) - \frac{ET(0)}{C + R_h}\exp(Ct))^3\exp(-R_h(t-\tau))\right. \\ & \left. + \frac{ET^3(0)}{C}(1 - \exp(-2C\tau))\exp(-R_h(t-\tau))\right]\mathrm{d}\tau + O(\varepsilon^2), \quad 0 < \varepsilon \ll 1. \end{aligned} \tag{32}$$

Comparing Eqs.(23), (30) and Eqs.(31), (32), we know that they are the same. Therefore, we know that the obtained solution by using the homotopic mapping method possesses a finer approximate degree.

5. Summary

In this paper, based on a class of oscillator of the ENSO model, employing the homotopic mapping method, we obtain first order approximations of the SST anomaly $T_{1\mathrm{app}}$ and the thermocline depth anomaly $h_{1\mathrm{app}}$ in the equatorial eastern Pacific. Meanwhile, the first order approximations by the homotopic mapping method possess a higher approximate degree. The homotopic mapping method is an approximate analytic method, which differs from the general numerical method. The expansions of solution can be a continuously analytic operation. Thus, from Eqs.(16)–(19), we can study further the qualitative and quantitative behaviours of the SST anomaly and the thermocline depth anomaly of the sea–air oscillation for the ENSO model (1) and (2).

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