Vol 18 No 10, October 2009
1674-1056/2009/18(10)/4598–05

Chinese Physics B

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Order parameters and synchronization of FitzHugh–Nagumo small-world networks*

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(Received 17 January 2009; revised manuscript received 7 February 2009)

This paper numerically investigates the order parameter and synchronisation in the small world connected FitzHugh–Nagumo excitable systems. The simulations show that the order parameter continuously decreases with increasing D, the quality of the synchronisation worsens for large noise intensity. As the coupling intensity goes up, the quality of the synchronisation worsens, and it finds that the larger rewiring probability becomes the larger order parameter. It obtains the complete phase diagram for a wide range of values of noise intensity D and control parameter g.

Keywords: order parameter, small world network, noise
PACC: 8730C, 0250

The phenomenon of synchronisation is observed in many fields, such as in chemical reaction,[1] neural networks,[2] and biological population.[3] The importance of synchronisation for functioning of biological systems has been extensively discussed.[3−5] It has been emphasised that these effects should play a significant role in operation of the brain[6−8] since the active synchronisation of different networks is necessary to the communication in neural systems. Therefore, synchronisation of coupled neural networks has inspired great interest and much work has been done, such as the mutual synchronisation of globally coupling neural networks[9,10] and diluted neural networks.[11,12]

Mammalian nervous systems exhibit a diversity of synchronised behaviors such as a transition from a periodic orbit to a chaotic attractor, attraction to a periodic orbit, and noise induced synchronisation.[13−16] It has been supposed that theoretical studies of such synchronised behaviors in neuronal assemblies play an important role in our understanding of information processing in the nervous systems. A prominent example in which the nonlinear dynamic behavior has been successfully studied is the synchronous firing of interconnected neurons. Such synchronous firing can be found in the sensory processing of cat visual cortex.[17,18] Synchronised behavior in the nervous systems can be frequently illustrated as a nonlinear dynamical model of large and small numbers of coupled oscillators.[19−21] Thus far there have been many models for the synaptic interactions between neurons.

For a real neural network, however, couplings are neither local nor global with the degree of randomness locating between the two extremes of regular and random couplings. In recent years, much attention has been paid to small world (SW) networks with the finite degree of heterogeneity in couplings, which is characterised by the high clustering and the small average distance between nodes.[22−24] The SW property is realised in various kinds of biological, social, and technological systems. It has been shown that by introducing the coupling heterogeneity into SW networks, the synchronisation is increased because the average distance in SW networks is shorter than that in regular networks.[23−26]

In this paper, we study synchronisation parameters in FitzHugh–Nagumo networks with small world (SW) connected and noise induced. Our goal is to delve into some aspects of the synchronisation, such

*Project supported by the National Natural Science Foundation of China (Grant No 10847140) and the Doctorial Start-up Fund of Lanzhou University of Technology (Grant No 409).
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as order parameter and phase transition of SW connected excitable systems under the presence of noise. It is known that in those systems noise can induce phenomena such as stochastic resonance[27] or coherence resonance.[28,29] The latter is a mechanism by which an unforced excitable system shows a maximum degree of regularity in the period between emitted pulses in the presence of the right amount of noise. We focus here on the stationary synchronisation properties of the common firings. We find that there is a non-equilibrium phase transition between synchronised and desynchronised states. Using the proper order parameter to characterise this transition, we obtain numerically the phase diagram.

We consider the FitzHugh–Nagumo model, which provides the simplest representation of firing dynamics and has been widely used as a model for spiking neurons as well as for cardiac cell. The model is defined in terms of activation x and inhibition y variable as follows:[28,29]

$$\eta\dot{x} = x - \frac{1}{3}x^3 - y, \tag{1}$$

$$\dot{y} = x + \alpha + D\xi(t), \tag{2}$$

where, a Gaussian white noise $\xi(t)$ of zero mean and correlations $\langle\xi(t)\xi(t')\rangle = \delta(t-t')$ has been added to the slow variable y. D will be called the noise intensity. The difference in the time scales of x and y is measured by η, a small number. We work exclusively in the so-called excitable regime, characterised by $|\alpha| > 1$.

There is a single stable fixed point (x_0, y_0), which, in the absence of any external perturbation $D = 0$, is reached independently of the initial condition. When random perturbations are present, the trajectories eventually exit basin of attraction of the stable fixed point and return to it after making an excursion in phase space, i.e. a pulse.

Based on the ideas proposed by Watts and Strogatz,[22] an SW network is constructed in the following way: starting with a ring of N vertices, each connected to its $2K$ nearest neighbors by indirect edges, and then each local link is visited once with the rewiring probability p, it is removed and reconnected to a randomly chosen node. Duplicate and self-connected edges are forbidden. After the whole sweep of the entire network, an SW graph is constructed with a linked degree $k = 2K/N$.

Our SW neural network model consists of N analog FitzHugh–Nagumo neurons. This coupled system is defined as follows:[30]

$$\eta\dot{x}_i = x_i - \frac{1}{3}x_i^3 - y_i + \frac{g}{N}\sum^{N}(x_j - x_i), \tag{3}$$

$$\dot{y}_i = x_i + \alpha + D\xi_i(t), \tag{4}$$

where, $i = 1, \ldots, N, g$ is the coupling strength, $\xi(t)$ is the Gaussian white noise with $\langle\xi_i(t)\xi_j(t)\rangle = \delta_{ij}\delta(t - t')$.

The first step consists in defining phase-like variables ϕ_i for our model. They should satisfy that their variation between 0 and 2π represents the pulse movement starting from the fixed point, traveling through all the cycle, and ending again at the fixed point. Several different approaches have been taken in order to evaluate the phases ϕ_i. The most naive definition is based upon the fact that the limit cycles in which the variables (x_i, y_i) evolve, are approximately centered around the origin. Then, the easiest choice is

$$\phi_i = \arctan\left(\frac{y_i}{x_i}\right). \tag{5}$$

However, this choice is only valid for particular cases of the parameters of the FitaHugh–Nagumo model. A definition of more general validity uses the so-called Hilbert transform.[31] To measure the degree of synchronisation, an order parameter is defined as follows:

$$\rho(t)\mathrm{e}^{\mathrm{i}\,\psi(t)} = \frac{1}{N}\sum^{N}\mathrm{e}^{\mathrm{i}\,\phi_i(t)}. \tag{6}$$

Equations (3) and (4) are solved with the fourth-order accurate Runge–Kutta method at a normalised 1.0×10^6 times of calculation with the time step of 1.0×10^{-3}. For every simulation, the initial transients of the model (500 steps) were discarded.

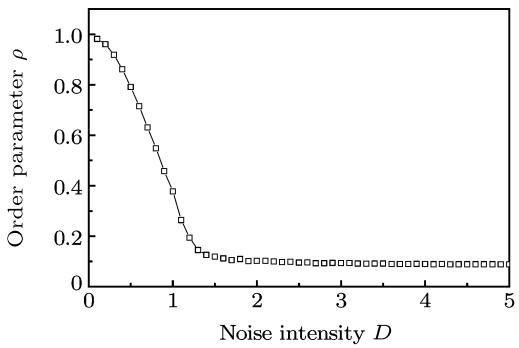


Fig.1. Order parameter ρ as a function of noise intensity for the SW connected FitzHugh–Nagumo excitable systems with $N = 100$, coupling intensity $g = 0.1$, rewiring probability $p = 0.032$, controlled parameter $\alpha = 1.1$, $\eta = 0.01$, linked degree $k = 0.1$.

We plot order parameter ρ as a function of the noise intensity D with $N = 100$, coupling intensity $g = 0.1$, rewiring probability $p = 0.032$, controlled parameter $\alpha = 1.1$, $\eta = 0.01$, linked degree $k = 0.1$ (see Fig.1). It turns out that the order parameter continuously decreases with increasing D, thus showing that the quality of the synchronisation worsens for large noise intensity.

In Fig.2, we present the order parameter ρ as a function of coupling intensity g for the SW connected FitzHugh–Nagumo excitable systems at different noise intensities: (a) $D = 2$, (b) $D = 1$, and (c) $D = 0.5$ with $N = 100$, noise intensity $D = 2$, rewiring probability $p = 0.032$, controlled parameter $\alpha = 1.1$, $\eta = 0.01$, linked degree $k = 0.1$. One can see that there is an optimal coupling intensity $g_m = 0.2$ for our SW connected FitzHugh–Nagumo excitable systems when the noise intensity is $D = 2$ and other parameters are fixed. As the coupling intensity goes up, the quality of the synchronisation worsens.

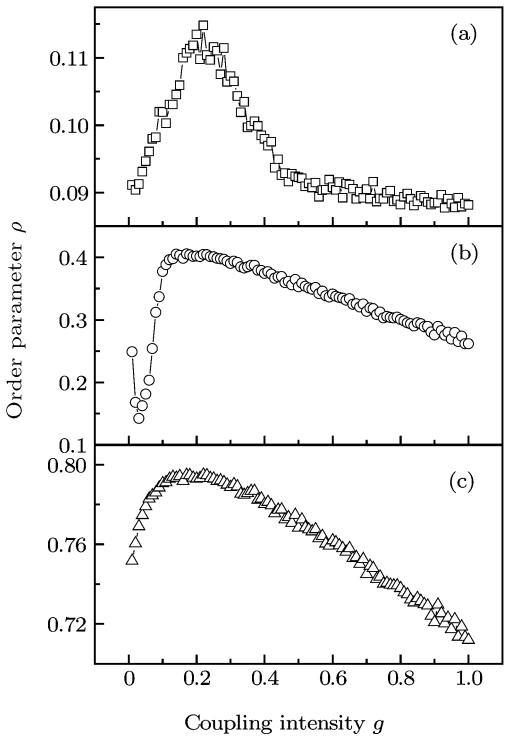


Fig.2. Order parameter ρ as a function of coupling intensity g for the SW connected FitzHugh–Nagumo excitable systems at different noise intensities: (a) $D = 2$, (b) $D = 1$ and (c) $D = 0.5$ with $N = 100$, noise intensity $D = 2$, rewiring probability $p = 0.032$, controlled parameter $\alpha = 1.1$, $\eta = 0.01$, linked degree $k = 0.1$.

Fixed other control parameters, we study the relationship between order parameter ρ and rewiring probability p at different noise intensities: (a) $D = 2$, (b) $D = 1$, and (c) $D = 0.1$ as shown in Fig.3, and find that when the noise intensity is $D = 2$, the larger rewiring probability becomes the larger order parameter, namely, in this SW network of FitzHugh–Nagumo excitable systems, the quality of the synchronisation for stochastic networks is better than regular ones.

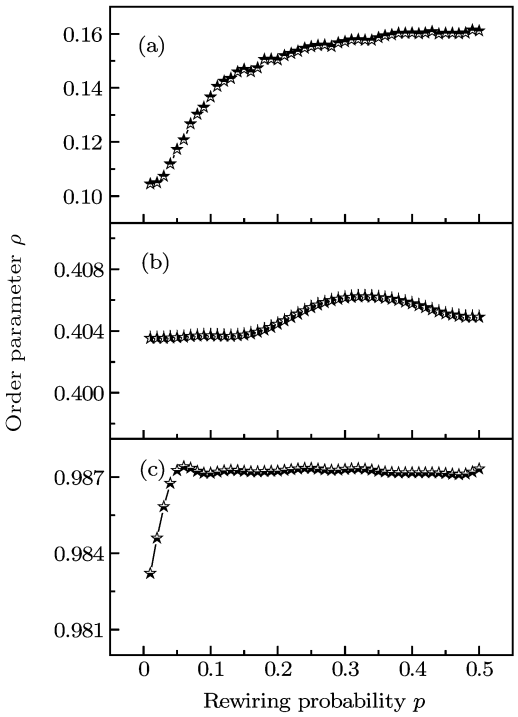


Fig.3. Order parameter ρ as a function of rewiring probability p for the SW connected FitzHugh–Nagumo excitable systems at different noise intensities: (a) $D = 2$, (b) $D = 1$, (c) $D = 0.1$ with $N = 100$, $D = 2$, $g = 0.2$, $\alpha = 1.1$, $\eta = 0.01$, $k = 0.1$.

Simulation shows that the order parameter goes up to the maximal value then declines when the linked degree increases (Fig.4(a)), the relationships of order parameter ρ as a function of linked degree k at different noise intensities: (a) $D = 2$, (b) $D = 1$ and (c) $D = 0.1$, are shown in Fig.4, where $N = 100$, coupling intensity $g = 0.2$, rewiring probability $p = 0.032$, controlled parameter $\alpha = 1.1$, $\eta = 0.01$, noise intensity $D = 2$.

The complete phase diagram for a wide range of values of noise intensity D and control parameter g is plotted in Fig.5. It is apparent a phase transition to desynchronization.

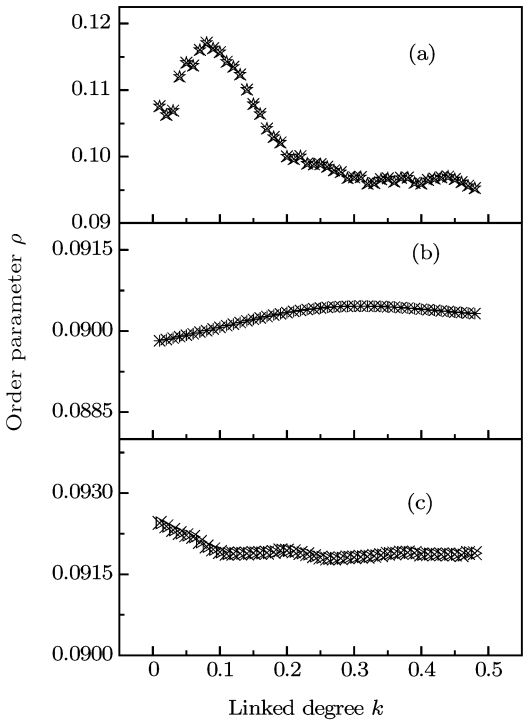


Fig.4. Order parameter ρ as a function of linked degree k for the small world connected FitzHugh–Nagumo excitable systems at different noise intensities: (a) $D = 2$, (b) $D = 1$, (c) $D = 0.1$, with $N = 100$, $D = 2$, $g = 0.2$, $\alpha = 1.1$, $\eta = 0.01$, $p = 0.032$.

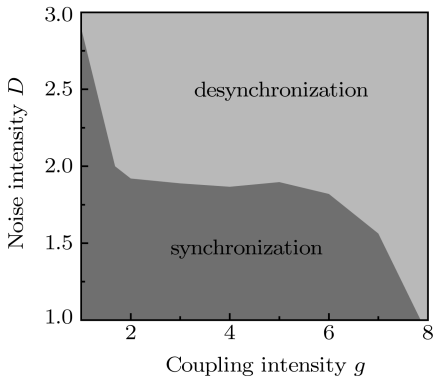


Fig.5. The order parameter ρ is plotted against the control parameter g and D, it is apparent a phase transition to de-synchronisation.

In summary, the order parameters are numerically investigated in the SW connected FitzHugh–Nagumo excitable systems. The simulations show that the order parameter continuously decreases with increasing D, the quality of the synchronisation worsens for large noise intensity. As the coupling intensity goes up, the quality of the synchronisation worsens, we find that the larger rewiring probability becomes the larger order parameter. We obtained the complete phase diagram for a wide range of values of noise intensity D and control parameter g.

The synchronisation/desynchronisation transition requires a proper definition of the order parameter for its characterisation, since the usual measures used in coupled oscillators do not properly identify the transition point. We have found that order parameter clearly displays such a transition. This order parameter is obtained from phase like variables defined through the use of the Hilbert transform and we have given details of a numerically efficient method to compute the phase variables. Further work will aim at characterising this non-equilibrium transition and its class of universality.

It is known that collective oscillations of a group of neurons are common in the brain and are thought to play a critical role in various physiological schemes. Meanwhile, the brain is the most complex and fascinating processor, and the neuron activity in the brain is subjected to many factors. Our results indicate that controlling the network topological structure and the level of correlation of the noise can modulate the collective spiking regularity of the neural network. These results discussed above will provide further insights into the detailed dynamics of complex phenomena taking place in some real neuronal circuits or neuron systems whose connection structures are similar to those discussed in this paper. The role of long-range connections has been explored in detail in topological analyses of neural connectivity.[29,30] However, to our knowledge, there are few investigations about the combined influences of network topological structures, noise spatiotemporal correlation, and noise strength on the dynamical behaviors in a system. We hope that the interplay of these parameters can result in some new phenomena in a noisy environment.

References

[1] N Khrustova, G Veser and A Mikhailov 1995 *Phys. Rev. Lett.* **75** 3564

[2] Abarbanel H D, Rabinovich M I, Selverston A, Bazhenov M V, Huerta R, Sushchik M M and Rubchinskii R R 1996 *Phys. Usp.* **39** 337

[3] Winfree A T 1980 *The Geometry of Biological Time* (Berlin: Springer) p22

[4] Kaneko K 1986 *Physica* D **23** 436

[5] Strogatz S H, Mirollo R E and Matthews P C 1992 *Phys. Rev. Lett.* **68** 2730
[6] Montejo N, Lorenzo M N, Villar V P and Muñuzuri V P 2005 *Phys. Rev.* E **72** 011902
[7] Li Y L, Chen Z Y, Ma J and Chen Y H 2008 *Acta Neuropsychiatrica* **20** 25
[8] Zhan Y, Zhang S H, Zhao T J, An H L, Zhang Z D, Han Y R, Liu H and Zhang Y H 2008 *Chin. Phys.* B **17** 2297
[9] Amengual A, Hernandez-Garcia E, Montagne R and Miguel M S 1997 *Phys. Rev. Lett.* **78** 4379
[10] Gao M and Cui B T 2009 *Chin. Phys.* B **18** 76
[11] Kitano K and Aoyagi T 1998 *Phys. Rev.* E **57** 5914
[12] Tang Y, Zhong H H and Fang J A 2008 *Chin. Phys.* B **17** 4080
[13] Basar E 1998 *Brain Function and Oscillations I: Brain Oscillations* (Berlin: Springer-Verlag) p103
[14] Haken H 2002 *Brain Dynamics-Synchronization and Activity Patterns in Pulse-Coupled Neural Nets with Delays and Noise* (Berlin: Springer-Verlag) p55
[15] Lindner B, Ojalvo J G, Neiman A and Geier L S 2004 *Phys. Rep.* **392** 321
[16] Wang Q Y, Lu Q S, Chen G R and Guo D H 2006 *Phys. Lett.* A **356** 17
[17] Eckhorn R, Frien A, Bauer R, Woelbern T and Kehr H 1993 *Neuro Rep.* **4** 243
[18] Kreiter A K and Singer W 1996 *J. Neurosci.* **16** 2381
[19] Karantonis A, Miyakita Y and Nakabayashi S 2002 *Phys. Rev.* E **65** 046213
[20] Miyakita Y, Karantonis A and Nakabayashi S 2002 *Chem. Phys. Lett.* **362** 461
[21] Brunel N, Hakim V and Richardson M J E 2003 *Phys. Rev.* E **67** 051916
[22] Watts D J and Strogatz S H 1998 *Nature* (London) **393** 440
[23] Hatchett J P L, Skantzos N S and Nikoletopoulos T 2005 *Phys. Rev.* E **72** 066105
[24] Petermann T and Rios P D L 2006 *Phys. Rev.* E **73** 026114
[25] Percha B, Dzakpasu R and Żchowski M 2005 *Phys. Rev.* E **72** 031909
[26] Kwon O and Moon H 2002 *Phys. Lett.* A **298** 319
[27] Wiesenfeld K, Pierson D, Pantazelou E, Dames C and Moss F 1994 *Phys. Rev. Lett.* **72** 2125
[28] Toral R, Mirasso C R and Gunton J D 2003 *Europhys. Lett.* **61** 162
[29] Lindner B and Schimansky G L 1999 *Phys. Rev.* E **60** 7270
[30] Li Q S and Gao Y 2008 *Phys. Rev.* E **77** 036117
[31] Pikovsky A S, Rosemblum M and Kurths J 2001 *A Universal Concept in Nonlinear Sciences* (Cambridge: Cambridge University Press) p36