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Single point velocity distributions in homogeneous isotropic turbulence*

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The starting point for this paper lies in the results obtained by Tatsumi (2004) for isotropic turbulence with the self-preserving hypothesis. A careful consideration of the mathematical structure of the one-point velocity distribution function equation obtained by Tatsumi (2004) leads to an exact analysis of all possible cases and to all admissible solutions of the problem. This paper revisits this interesting problem from a new point of view, and obtains a new complete set of solutions. Based on these exact solutions, some physically significant consequences of recent advances in the theory of homogenous statistical solution of the Navier–Stokes equations are presented. The comparison with former theory was also made. The origin of non–Gaussian character could be deduced from the above exact solutions.

Keywords: isotropic turbulence, probability distribution function
PACC: 4725, 4725C

1. Introduction

The complete statistical information of turbulence is provided by the probability distribution functional of the velocity field in space and time. The evolution equation for this distribution functional was given by Hopf[1] in terms of the characteristic functional equation. So far, however, no mathematical method has been known for dealing with such a functional equation generally, and only a few particular solutions given by Hopf himself have been available. A statistically equivalent formalism to distribution functional is given by an infinite set of the joint-probability distribution of the velocities at an arbitrary number of points in space-time. The system of equations governing such joint-probability distributions was given by Lundgren[2] and Monin[3] independently. The set of equations, however, is indeterminate since each equation involves a new unknown higher-order distribution. Such indeterminacy is known as the common difficulty of nonlinear physics and the closure of the set of equations is the central problem of turbulence. Recently, the hypothesis of cross-independence of two turbulent velocities has been proposed by Tatsumi[4,5] for closing the equations for the velocity distributions given by Lundgren[2] and Monin.[3] Using this hypothesis, we derive the equations for one and two-point velocity distributions in homogeneous isotropic turbulence in closed form and obtain the velocity distributions from these equations. One-point velocity distribution is obtained as an inertial normal distribution (N1) including the energy dissipation rate as the only parameter. The inertial similarity of homogeneous isotropic turbulence associated with the normal one and two-point velocity distributions seems to give good prospects for the extension of the further approach to inhomogeneous turbulent flows and more complex turbulent phenomena.

The starting point lies in the work of Tatsumi[4,5] with the hypothesis of cross-independence of two turbulent velocities. New similarity exact solutions could be found. This would change subtly the results presented by Tatsumi.[4,5]

2. Equations for velocity distributions

Let us consider turbulence in an incompressible viscous fluid, in particular, homogeneous isotropic turbulence whose probability distributions are uniform and isotropic in space. Take the coordinates $x = (x_1, x_2, x_3)$, the time t and denote the fluid velocity at point (x, t) by $u(x, t)$ and the pressure by $p(x, t)$. We denote the velocities at the two space points x_1, x_2 and at time t by $u(x_1, t), u(x_2, t)$ respec-

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tively, the one and two-point velocity distributions are defined, respectively, as

$$f(v_1, x_1, t) = \langle \delta(u(x_1, t) - v_1) \rangle, \tag{1}$$

$$f_2(v_1, v_2; x_1, x_2; t) = \langle \delta(u(x_1, t) - v_1)\, \delta(u(x_2, t) - v_2) \rangle, \tag{2}$$

where v_i, $(i = 1, 2)$ denotes the probability variable corresponding to the velocities $u(x_i, t)$, δ the three-dimensional delta function, and $\langle\ \rangle$ the probability mean with respect to a certain initial distribution. Tatisumi[4,5] introduced the sum and the difference of two velocities u_1, u_2 as

$$u_+ = \frac{1}{2}(u_1 + u_2), \tag{3}$$

$$u_- = \frac{1}{2}(u_2 - u_1), \tag{4}$$

and call them the cross-velocities of two velocities $u_1,\ u_2$. The distributions of the cross-velocities are defined in accordance with the former.

$$g_\pm(v_\pm, x_1, t) = \langle \delta(u_\pm(x_1, t) - v_\pm) \rangle, \tag{5}$$

$$g_2(u_\pm, v_\pm; x_1, x_2; t) = \langle \delta(u_\pm(x_1, t) - v_1)\, \delta(u(x_2, t) - v_2) \rangle, \tag{6}$$

where

$$v_+ = \frac{1}{2}(v_1 + v_2), \tag{7}$$

$$v_- = \frac{1}{2}(v_2 - v_1). \tag{8}$$

Tatsumi[4] introduced a new independent relation between the cross-velocities by

$$g_2(v_+, v_-; r, t) = g_+(v_+, r, t)\, g_-(v_-, r, t), \tag{9}$$

that may be called the cross-independence in contrast to the ordinary independence. Under this consideration, one can obtain the following closed form equation for one-point velocity distribution (Ref.[5], Eq.[50]):

$$\left[\frac{\partial}{\partial t} + \alpha(t)\left|\frac{\partial}{\partial v}\right|^2\right] \cdot f(v, t) = 0, \tag{10}$$

where

$$\alpha(t) = \frac{1}{3}\bar{\varepsilon}(t), \tag{11}$$

in which $\bar{\varepsilon}$ represents the mean energy dissipation rate.

3. Similarity solution for one-point velocity distribution

Let us consider homogeneous turbulence without external energy supply and obtain the velocity distribution as the solution of the governing equation (10). We introduce the self-preserving hypothesis as

$$f(v, t) \equiv b(t)\, Y\left(\frac{v}{l(t)}\right), \tag{12}$$

$$\xi = \frac{\nu}{l(t)}. \tag{13}$$

Substituting Eq.(12) into Eq. (10), we get

$$\frac{\mathrm{d}^2 Y}{\mathrm{d}\xi^2} + \left(\frac{2}{\xi} + \frac{a_1}{2}\xi\right)\frac{\mathrm{d}Y}{\mathrm{d}\xi} + \frac{a_2}{2}Y = 0, \tag{14}$$

$$Y(0) = 1, \quad Y(\infty) = 0, \tag{15}$$

where

$$-\frac{1}{\alpha(t)\, l(t)} \cdot \frac{\mathrm{d}l}{\mathrm{d}t} = \frac{a_1}{2}, \tag{16}$$

$$\frac{l^2(t)}{\alpha(t)\, b(t)} \cdot \frac{\mathrm{d}b}{\mathrm{d}t} = \frac{a_2}{2}. \tag{17}$$

We call these the scaling equations. Now, let us confine ourselves to the self-similar solution in time. Under this consideration

$$a_i = \text{constant}, \quad i = 1, 2. \tag{18}$$

The turbulence energy equation is

$$\frac{\mathrm{d}\bar{E}}{\mathrm{d}t} = -\bar{\varepsilon}. \tag{19}$$

For decaying turbulence, we have

$$\bar{E}(t) = E_0 t^{-1}, \tag{20}$$

$$\bar{\varepsilon}(t) = \varepsilon_0 t^{-2}. \tag{21}$$

Substituting these into the scaling equations, we have

$$l^2(t) = \frac{a_1 \varepsilon_0}{6} t^{-1} + l_0^2, \tag{22}$$

$$\frac{b(t)}{b_0} = (1 + \omega t)^{-\sigma}, \tag{23}$$

where

$$\omega = \frac{6 l_0^2}{a_1 \varepsilon_0},$$

$$\sigma = \frac{a_2}{a_1}.$$

The complete solutions are obtained as follows (The details could be seen in Appendix).

When $\sigma = 3$ (denotes type SI),

$$Y_1(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right). \tag{24}$$

When $\kappa = \sigma - 3/2$ (denotes type SII),

$$Y_2(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right) \mathrm{F}\left(\frac{9}{4} - \sigma, \frac{3}{2}, \frac{a_1}{4}\xi^2\right). \quad (25)$$

When $\kappa = 3/2 - \sigma$ (denotes type SIII),

$$Y_3(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right) \mathrm{F}\left(\sigma - \frac{3}{4}, \frac{3}{2}, \frac{a_1}{4}\xi^2\right). \quad (26)$$

When $\sigma = 3/2$ (denotes SIV)

$$Y_4(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right) \mathrm{F}\left(\frac{1}{4}, \frac{1}{2}, \frac{a_1}{4}\xi^2\right). \quad (27)$$

We could deduce the corresponding asymptotic expansions of the above solutions, and give the existence conditions for different types of solutions.

(i) Existence condition for SI

In order to satisfy the boundary condition at infinity, we must have

$$a_1 > 0. \quad (28)$$

This is the only existence condition for SI, and also one of the existence conditions for the other three solutions.

(ii) Existence condition for SII

$$F_2(\xi) \propto \xi^{2(3/4-\sigma)}. \quad (29)$$

The boundary condition leads to

$$\sigma > \frac{3}{4}, \quad (30)$$

which is one of the existence conditions for SII.

(iii) Existence condition for SIII

$$F_3(\xi) \propto \xi^{2(\sigma-9/4)}. \quad (31)$$

So

$$\sigma < \frac{9}{4}. \quad (32)$$

(iv) Existence condition for SIV

$$F_4(\xi) \propto \xi^{-1/2}. \quad (33)$$

So the only existence condition for SIV is $a_1 > 0$.

We can draw the following conclusion on the existence conditions for the complete set of solutions:

i) For all kinds of solution: $a_1 > 0$.

ii) For the second kind of solution: $\sigma > 3/4$; $\sigma \neq 3/2$.

iii) For the third kind of solution: $0 < \sigma < 9/4$; $\sigma \neq 3/2$.

A simple comparison shows that the special solution found by Tatsumi[4] belongs to one special kind of our new set of solutions. Other differences between Tatsumi's solution and our new set of solutions would be discussed below.

In order to give a complete description of turbulence features, we must introduce a probability structure based on these solutions. Let p_j, $(j = 1, 2, 3, 4)$ denotes the probability of the j-th class solution, and the set P to be the state vector of this process defined by

$$P \equiv (p_1, p_2, p_3, p_4). \quad (34)$$

The state vector is a function of the parameter σ, we could easily write the following results:

1) If $0 < \sigma \leq 3/4$, the probability vector is

$$P \equiv (p_1, p_2, p_3, p_4) = (0, 0, 1, 0). \quad (35)$$

2) If $3/4 < \sigma < 3/2$, we have

$$P \equiv (p_1, p_2, p_3, p_4) = \left(0, \tfrac{1}{2}, \tfrac{1}{2}, 0\right). \quad (36)$$

3) If $\sigma = 3/2$, we have

$$P \equiv (p_1, p_2, p_3, p_4) = (0, 0, 0, 1). \quad (37)$$

4) If $3/2 < \sigma < 9/4$,

$$P \equiv (p_1, p_2, p_3, p_4) = \left(0, \tfrac{1}{2}, \tfrac{1}{2}, 0\right). \quad (38)$$

5) If $9/4 \leq \sigma < 3$, we have

$$P \equiv (p_1, p_2, p_3, p_4) = (0, 1, 0, 0). \quad (39)$$

6) If $\sigma = 3$,

$$P \equiv (p_1, p_2, p_3, p_4) = \left(\tfrac{1}{2}, \tfrac{1}{2}, 0, 0\right). \quad (40)$$

7) If $\sigma > 3$, we have

$$P \equiv (p_1, p_2, p_3, p_4) = (0, 1, 0, 0). \quad (41)$$

Based on the above deduced probability structure, the average turbulence could be given, which represents the statistical distributions of turbulence. In the following discussion, we will deal with the one-point velocity distribution function, in principle, we could give the average of these physical quantities.

$$\bar{f} = \sum_{j=1}^{4} p_j f_j. \quad (42)$$

(I) If $0 < \sigma \leq 3/4$, the probability vector is

$$\bar{f} = f_3. \quad (43)$$

(II) If $3/4 < \sigma < 3/2$, we have

$$\bar{f} = \frac{1}{2}(f_2 + f_3). \quad (44)$$

(III) If $\sigma = 3/2$, we have

$$\bar{f} = f_4. \tag{45}$$

(IV) If $3/2 < \sigma < 9/4$,

$$\bar{f} = \frac{1}{2}(f_2 + f_3). \tag{46}$$

(V) If $9/4 \leq \sigma < 3$, we have

$$\bar{f} = f_2. \tag{47}$$

(VI) If $\sigma = 3$,

$$\bar{f} = \frac{1}{2}(f_2 + f_1). \tag{48}$$

(VII) If $\sigma > 3$, we have

$$\bar{f} = f_2. \tag{49}$$

Actually, in a turbulent viscid fluid, the statistics of the velocity field at a fixed point is non-Gaussian. For example, let us set $\sigma = 1.0$, and it belongs to the second case:

If $3/4 < \sigma < 3/2$, we have

$$\bar{f} = \frac{1}{2}(f_2 + f_3). \tag{50}$$

So, we could easily write the expression of the one-point velocity distribution as (only presented the part in terms of the $Y_j(\xi)$)

$$\bar{Y} = \frac{1}{2}(Y_2 + Y_3), \tag{51a}$$

$$\begin{aligned}\bar{Y} &= \frac{1}{2}\exp\left(-\frac{a_1}{4}\xi^2\right)\left\{\mathrm{F}\left(\frac{5}{4}, \frac{3}{2}, \frac{a_1}{4}\xi^2\right)\right. \\ &\quad \left. + \mathrm{F}\left(\frac{1}{4}, \frac{3}{2}, \frac{a_1}{4}\xi^2\right)\right\} \\ &\approx \exp\left(-\frac{a_1}{4}\xi^2\right)\left\{1 + \frac{2}{3}\cdot\frac{a_1}{4}\xi^2 + \cdots\right\}.\end{aligned} \tag{51b}$$

We can see that $\bar{Y}$ is non-Guassian type distribution. The comparison is presented in Fig.1.

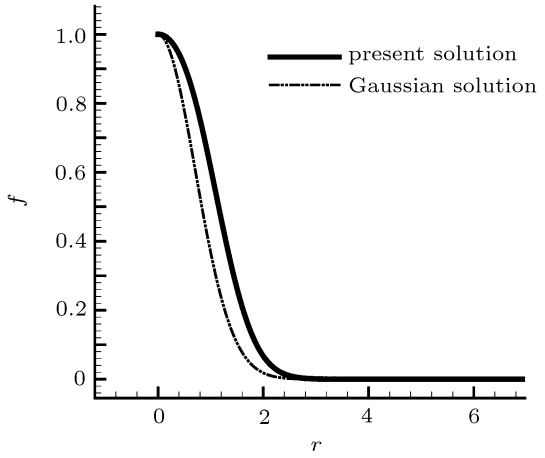


Fig.1. Comparison of the one-point velocity distribution solution.

4. Conclusions

The one and two-point velocity distributions of homogenous isotropic turbulence have been obtained by using the equations for the multi-point velocity distributions given by Lundgren[2] and Monin[3] and the cross-independence closure hypothesis given by Tatsumi.[4,5] The one-point velocity distribution is found to be the inertial normal distribution which depends upon the energy dissipation rate. This former conclusion is questionable due to the present findings. Under the hypothesis of cross-independence, one can also obtain the non-Guassian type distribution. The present study of the velocity distributions in homogeneous isotropic turbulence seems to provide us with possible ways to understand the insight of the origin of non-Gaussian character for isotropic turbulence.

Appendix: Solutions of the differential equation for one-point velocity distribution function

The detailed calculation is given as follows.[6] A lot of useful partial differential equations can be reduced to confluent hyper-geometric equations. The one-point velocity distribution function satisfies

$$\frac{\mathrm{d}^2 f}{\mathrm{d}\xi^2} + \left(\frac{2}{\xi} + \frac{a_1}{2}\xi\right)\frac{\mathrm{d}f}{\mathrm{d}\xi} + \frac{a_2}{2}f = 0 \tag{A1}$$

with the boundary condition

$$f(0) = 1,$$

$$f(\infty) = 0.$$

The complete solutions are given as follows:

When $\sigma = 3$,

$$f(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right).$$

When $\kappa = \sigma - 3/2$,

$$f(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right)\mathrm{F}\left(\frac{9}{4} - \sigma, \frac{3}{2}, \frac{a_1}{4}\xi^2\right).$$

When $\kappa = 3/2 - \sigma$,

$$f(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right)\mathrm{F}\left(\sigma - \frac{3}{4}, \frac{3}{2}, \frac{a_1}{4}\xi^2\right).$$

When $\sigma = 3/2$,

$$f(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right)\mathrm{F}\left(\frac{1}{4}, \frac{1}{2}, \frac{a_1}{4}\xi^2\right).$$

The detailed calculations are given as follows.

A lot of useful partial differential equations can be reduced to confluent hyper-geometric equations, $P_{k,m}(\varsigma)$ is the solution of Whittaker equation as that defined by

$$\frac{\mathrm{d}^2 W}{\mathrm{d}\varsigma^2} + \left[-\frac{1}{4} + \frac{\kappa}{\varsigma} + \frac{1/4 - m^2}{\varsigma^2}\right] W = 0, \tag{A2}$$

where

$$y(z) = z^{\beta}\mathrm{e}^{f(z)} P_{\kappa,m}(h(z)). \tag{A3}$$

After some reductions, the equation of $y(z)$ reads

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} - \left[\frac{h''}{h'} + \frac{2\beta}{z} + 2f'(z)\right]\frac{\mathrm{d}y}{\mathrm{d}z} + g \cdot y(z) = 0, \tag{A4}$$

where

$$g = (f')^2 - f'' + 2\beta\frac{f'}{z} + \frac{\beta(\beta+1)}{z^2} + \frac{h''}{h'}\left(\frac{\beta}{z} + f'\right) + g_1,$$

$$g_1 = \left(\frac{h'}{h}\right)^2\left(\frac{1}{4} - m^2 + \kappa h - \frac{h^2}{4}\right).$$

The solutions of the above equation could be deduced in terms of Whittaker function. We discussed this equation in the following special case:

$$f(z) = az^{\lambda}, \tag{A5}$$

$$h(z) = Az^{\lambda}. \tag{A6}$$

The equation under this condition reads as

$$\frac{\mathrm{d}^2 y}{\mathrm{d}z^2} + \left[\frac{1-\lambda-2\beta}{z} - 2\lambda\alpha z^{\lambda-1}\right]\frac{\mathrm{d}y}{\mathrm{d}z} + qy(z) = 0, \tag{A7}$$

where

$$q = \lambda^2\left(\alpha^2 - \frac{A^2}{4}\right)z^{2\lambda-2} + \lambda(2\alpha\beta + A\kappa\lambda)z^{\lambda-2} + \frac{\beta(\beta+\lambda) + \lambda^2(1/4 - m^2)}{z^2}.$$

The solution of this equation is

$$y(z) = z^{\beta}\mathrm{e}^{\alpha z^{\lambda}} P_{\kappa,m}\left(Az^{\lambda}\right). \tag{A8}$$

For isotropic turbulence, the corresponding parameters satisfied

$$1 - \lambda - 2\beta = 2, \tag{A9}$$

$$\lambda - 1 = 1, \tag{A10}$$

$$-2\lambda\alpha = \frac{a_1}{2}, \tag{A11}$$

$$\lambda^2\left(\alpha^2 - \frac{A^2}{4}\right) = 0, \tag{A12}$$

$$\beta(\beta+\lambda) + \lambda^2\left(\frac{1}{4} - m^2\right) = 0, \tag{A13}$$

$$\lambda(2\alpha\beta + A\kappa\lambda) = \frac{a_2}{2}. \tag{A14}$$

Hence, we have

$$\lambda = 2, \tag{A15}$$

$$\alpha = -\frac{a_1}{8}, \tag{A16}$$

$$\beta = -\frac{3}{2}, \tag{A17}$$

$$m = \pm\frac{1}{4}, \tag{A18}$$

$$A = \pm\frac{a_1}{4}, \tag{A19}$$

$$\kappa = \pm\left\{\frac{a_2}{a_1} - \frac{3}{2}\right\}. \tag{A20}$$

From the above analyses, we can introduce two parameters to classify turbulence, they are: $a_1, \sigma = a_2/a_1$.

If $2m$ is not an integer,

$$P_{k,m}(z) = \mathrm{e}^{-z/2} z^{1/2+m}\mathrm{F}\left(\tfrac{1}{2} + m - \kappa, 1 + 2m, z\right), \tag{A21}$$

$$P_{k,-m}(z) = \mathrm{e}^{-z/2} z^{1/2-m}\mathrm{F}\left(\tfrac{1}{2} - m - \kappa, 1 - 2m, z\right). \tag{A22}$$

For the case $\kappa = 0$, we must use the second Kummer formula,

$$P_{0,m}(z) = z^{1/2+m}{}_0\mathrm{F}_1\left(1 + m; \frac{z^2}{16}\right). \tag{A23}$$

By making use of the boundary condition, we could choose the rational parameters for isotropic turbulence. The solution of equation could be rewritten as

$$\begin{aligned} y(z) &= z^{\beta}\mathrm{e}^{\alpha z^{\lambda}} P_{\kappa,m}\left(Az^{\lambda}\right) \\ &= z^{\beta}\mathrm{e}^{\alpha z^{\lambda}} \cdot \mathrm{e}^{-Az^2/2}\left(Az^{\lambda}\right)^{1/2+m}\mathrm{F}\left(\frac{1}{2} + m - \kappa, 1 + 2m, Az^{\lambda}\right) \\ &= A^{1/2+m} \cdot \mathrm{e}^{(\alpha - A/2)z^2} z^{\beta+\lambda m+\lambda/2} \cdot \mathrm{F}\left(\frac{1}{2} + m - \kappa, 1 + 2m, Az^{\lambda}\right). \end{aligned} \tag{A24}$$

Let $A > 0$, this resulted in the definition of exponent.

If we choose $m = -\frac{1}{4}$, in the above solution, the exponent of z is

$$\beta + \lambda m + \frac{\lambda}{2} = -\frac{3}{2} + 2 \times \left(-\frac{1}{4}\right) + 1 = -2 < 0. \tag{A25}$$

The boundary condition $y(0)$ would be broken under this condition. So we only choose

$$m = \frac{1}{4}. \tag{A26}$$

Another condition must be satisfied, i.e.

$$\alpha + \frac{A}{2} = 0. \tag{A27}$$

The solution is

$$y(z) = \mathrm{e}^{-Az^2} \cdot \mathrm{F}\left(\frac{3}{4} - \kappa, \frac{3}{2}, Az^2\right). \tag{A28}$$

There is an important parameter k in the above solution, the following multiple values could be applied.

When $\kappa = \{a_2/a_1 - 3/2\}$,

$$y(z) = \mathrm{e}^{-Az^2} \cdot \mathrm{F}\left(\frac{9}{4} - \sigma, \frac{3}{2}, Az^2\right). \tag{A29}$$

When $\kappa = -\{a_2/a_1 - 3/2\}$,

$$y(z) = \mathrm{e}^{-Az^2} \cdot \mathrm{F}\left(\sigma - \frac{3}{4}, \frac{3}{2}, Az^2\right). \tag{A30}$$

We must treat the other special case $k = 0$, by using the secondary Kummer formula

$$P_{0,m}(z) = z^{1/2+m}{}_0\mathrm{F}_1\left(1 + m; \frac{z^2}{16}\right), \tag{A31}$$

where

$${}_0\mathrm{F}_1\left(1 + m; \frac{z^2}{16}\right) = \mathrm{e}^{-z/2}\mathrm{F}(m, 2m, z).$$

For this case, the solution of equation is

$$y(z) = \mathrm{e}^{-Az^2} \cdot \mathrm{F}\left(\frac{1}{4}, \frac{1}{2}, Az^2\right). \tag{A32}$$

Another reduced case for $\sigma = 3$, the solution is

$$f(\xi) = \mathrm{e}^{-a_1\xi^2/4}. \tag{A33}$$

At last, we have already obtained the following complete set of solutions of isotropic turbulence, depending on two parameters:

When $\sigma = 3$,

$$f(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right).$$

When $\kappa = \sigma - 3/2$,

$$f(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right)\mathrm{F}\left(\frac{9}{4} - \sigma, \frac{3}{2}, \frac{a_1}{4}\xi^2\right).$$

When $\kappa = 3/2 - \sigma$,

$$f(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right)\mathrm{F}\left(\sigma - \frac{3}{4}, \frac{3}{2}, \frac{a_1}{4}\xi^2\right).$$

When $\sigma = 3/2$,

$$f(\xi) = \exp\left(-\frac{a_1}{4}\xi^2\right)\mathrm{F}\left(\frac{1}{4}, \frac{1}{2}, \frac{a_1}{4}\xi^2\right).$$

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