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The (1+1)-dimensional spatial solitons in media with weak nonlinear nonlocality*

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We study the propagation of (1+1)-dimensional spatial soliton in a nonlocal Kerr-type medium with weak nonlocality. First, we show that an equation for describing the soliton propagation in weak nonlocality is a nonlinear Schrödinger equation with perturbation terms. Then, an approximate analytical solution of the equation is found by the perturbation method. We also find some interesting properties of the intensity profiles of the soliton.

Keywords: nonlocal nonlinear Schrödinger equation, weak nonlocality, spatial optical soliton
PACC: 4265, 4265S

1. Introduction

Spatial optical solitons, which are self-trapped optical beams, propagate without change through a nonlinear medium when the diffraction associated with a finite size of the wave is balanced by the self-focusing induced by the nonlinearity. The propagation of the nonlocal optical spatial solitons is described by the nonlocal nonlinear Schrödinger equation (NNLSE),[1−3] where the nonlinear term represents a nonlocal form with a symmetric and real-valued response kernel. Based on the degree of the nonlocality which is determined by the relative width of the response kernel and the optical beam, there are four kinds of nonlocality:[3,4] local, weakly nonlocal, general nonlocal and strongly nonlocal. In the present paper, we define the degree of nonlocality as $\alpha = w_m/w$, where w is the beam width of the light beam and w_m is the characteristic length of the response function. The smaller the value of α is, the weaker the degree of nonlocality will be. For the weakly nonlocal case in the present paper $\alpha < 0.1$ holds true.

In 1997, Snyder and Mitchell[1] simplified the NNLSE into a linear model, and they obtained an exact Gaussian-shaped stationary solution named an *accessible soliton*. Since then, the investigation of the spatial soliton propagation in nonlocal nonlinear media has become very popular theoretically[3−14] and experimentally[15−21] for their rich phenomena and potential applications in communication and signal processing. Several review papers on the development of the nonlocal spatial solitons have been presented. It is shown theoretically that stable spatial bright (dark) soliton states can exist in self-focused (self-defocused) weakly nonlocal media[5] and Gauss-function-like bright soliton states can exist in self-focused strongly nonlocal media.[1,12] Furthermore, Granot *et al*[6] have also thrashed out the sub-wavelength nonlocal spatial solitons. "Exact solutions" of solitons in weakly nonlocal nonlinear media were given by Królikowski and Bang[7] and they analysed the modulation instabilities[3,8] and the properties of soliton stabilization with arbitrary degree of nonlocality.[4] According to the analogy between parametric interaction and nonlocality, the quadratic soliton is shown to be equivalent to the soliton in nonlocal Kerr-type media.[9] For the local case, an NNLSE can be degenerated into a nonlinear Schrödinger equation,[3] and the profiles of these solitons turn out to be the hyperbolic secant functions.[22,23]

Praiseworthily, the investigations of the soliton in the weakly nonlocal Kerr-type media have been carried out ceaselessly. The main purpose of the present article is to search for the soliton solution by the perturbation method[24,25] in a weakly nonlocal Kerr-type medium. In Section 2, we show that the equation for

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describing the soliton propagation in weakly nonlocal medium is the nonlinear Schrödinger equation with perturbation terms. By employing the perturbative analysis method, we find an approximate analytical solution of the relevant equation in Section 3. And then, we make a comparison between the numerical simulations and the analytical results in Section 4. We also find an interesting phenomenon that the movement of the intensity profile which has a Gaussian-type response function cannot change so obviously along with the increase of the weak nonlocality. Finally, we draw some conclusions from our study in Section 5.

2. Nonlocal nonlinear Schrödinger equation and its normalization

Now we consider the propagation of an optical beam in a nonlocal Kerr-type medium by a NNLSE,[1−3] which reads in the (1+1)-dimensional [(1+1)-D] case

$$\mathrm{i}\frac{\partial\psi}{\partial z}+\frac{1}{2k}\frac{\partial^2\psi}{\partial x^2}+k\eta\psi\int R(x-x')I(x',z)\mathrm{d}x'=0, \quad (1)$$

where x is the transverse coordinate, z is the propagation coordinate, $\psi(x,z)$ is the paraxial beam, $k=\omega n_0/c$ with n_0 being the linear refractive index of the material, w and w_m are the beam width and the width of the response function, η is the material constant ($\eta>0$ and $\eta<0$ correspond to the focusing and defocusing material respectively), $I(x,z)=|\psi(x,z)|^2$, and R is a symmetrical real non-linear spatial response of the medium, with $\int R(x)\mathrm{d}x=1$.

In the present paper, we discuss only the focusing case, so the refractive index change Δn induced by a beam with intensity $I(x,z)$ can be written in a general form as

$$\Delta n(I)=\int R(x-x')I(x',z)\mathrm{d}x'. \quad (2)$$

In the case where the nonlocality is weak, the characteristic length of the response function $R(x)$ is narrow compared to the width of the beam, then the intensity $I(x,z)$ can be expanded by means of Taylor's expansion. Here, we expand $I(x',z)$ around the point $x'=x$ to the fourth order and obtain

$$\Delta n(I)=I+\gamma_1\frac{\partial^2 I}{\partial x^2}+\gamma_2\frac{\partial^4 I}{\partial x^4}+\ldots, \quad (3)$$

where the nonlocality parameters $\gamma_1,\gamma_2>0$ and are given by

$$\gamma_1=\int\frac{1}{2}R(x)x^2\mathrm{d}x, \quad \gamma_2=\int\frac{1}{24}R(x)x^4\mathrm{d}x. \quad (4)$$

As will be shown below, $\gamma_2\partial^4 I/\partial x^4$ is the fourth-order quantification which is much smaller than the term $\gamma_1\partial^2 I/\partial x^2$ in the weakly nonlocal case. It indicates that the effect of $\gamma_2\partial^4 I/\partial x^4$ on the soliton is considerably small in comparison with the effect of $\gamma_1\partial^2 I/\partial x^2$ in the case of weak nonlocality. The same result can also be obtained, i.e. the sixth-order term and other higher order terms acting on the soliton are far smaller than the effects of the two lower order terms. So in the following discussion, we neglect the higher order terms and adopt

$$\Delta n(I)=I+\gamma_1\frac{\partial^2 I}{\partial x^2}+\gamma_2\frac{\partial^4 I}{\partial x^4}. \quad (5)$$

Substituting expressions (4) and (5) into Eq.(1) yields

$$\mathrm{i}\frac{\partial\psi}{\partial z}+\frac{1}{2k}\frac{\partial^2\psi}{\partial x^2}+k\eta\psi\left(|\psi|^2+\gamma_1\frac{\partial^2|\psi|^2}{\partial x^2}+\gamma_2\frac{\partial^4|\psi|^2}{\partial x^4}\right)=0. \quad (6)$$

Taking the normalized transform,

$$\tau=\frac{x}{w_0}, \quad \xi=\frac{z}{w_0^2 k}, \text{ and } \Psi=w_0\eta^{1/2}\psi, \quad (7)$$

where w_0 is the initial beam width and by inserting expression (7) into Eq.(6), we can obtain

$$\mathrm{i}\frac{\partial\Psi}{\partial\xi}+\frac{1}{2}\frac{\partial^2\Psi}{\partial\tau^2}+\Psi\left(|\Psi|^2+\frac{\gamma_1}{w_0^2}\frac{\partial^2|\Psi|^2}{\partial\tau^2}+\frac{\gamma_2}{w_0^4}\frac{\partial^4|\Psi|^2}{\partial\tau^4}\right)=0, \quad (8)$$

where

$$\frac{\gamma_1}{w_0^2}=\frac{\frac{1}{2}\int R(x)x^2\mathrm{d}x}{w_0^2}=\frac{1}{2}\frac{w_m^2}{w_0^2}, \quad (9)$$

$$\frac{\gamma_2}{w_0^4}=\frac{\frac{1}{24}\int R(x)x^4\mathrm{d}x}{w_0^4}=\frac{\langle x^4\rangle}{24w_0^4}. \quad (10)$$

Here, we introduce the kurtosis parameter to simplify expression (10). In one transversal dimension, i.e. in the x-direction, the kurtosis parameter can be expressed as[26]

$$K=\frac{\langle x^4\rangle}{(\langle x^2\rangle)^2}, \quad (11)$$

where $\langle x^4\rangle$ and $\langle x^2\rangle$ are the fourth and second intensity moments, respectively. Substituting expression (11) and the definitions of the fourth and second intensity moments into expression (10) yields

$$\frac{\gamma_2}{w_0^4}=\frac{K(\langle x^2\rangle)^2}{24w_0^4}=\frac{Kw_m^4}{24w_0^4}. \quad (12)$$

K is a number that transforms along propagation coordinate z. For the Gaussian beam of light, $K = 3$. When $K > 3$, is in contrast to the Gaussian beam, the section of light becomes abrupt. When $K < 3$, the section of light becomes smoother than the Gaussian beam. In the weakly nonlocal case, the degree of nonlocality should be less than 0.1, so $w_m^2 \ll w_0^2$ and $w_m^4 \ll w_0^4$. We introduce

$$\mu_1 = \frac{\gamma_1}{w_0^2}, \quad \mu_2 = \frac{\gamma_2}{w_0^4}, \tag{13}$$

from the forenamed argumentation, we make certain that μ_1 and μ_2 are both small in value (less than 0.01). They are the necessary conditions required for the perturbative analysis method, then equation (8) can be written as

$$\begin{aligned} &\mathrm{i}\frac{\partial \Psi}{\partial \xi} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial \tau^2} + \Psi|\Psi|^2 \\ &= -\left(\mu_1 \frac{\partial^2 |\Psi|^2}{\partial \tau^2} + \mu_2 \frac{\partial^4 |\Psi|^2}{\partial \tau^4}\right)\Psi. \end{aligned} \tag{14}$$

From the discussion above, we know that the terms on the right-hand side of the above equation can be seen as perturbation. Thus the equation for describing the soliton propagation in a weakly nonlocal medium can be seen as a nonlocal nonlinear Schrödinger equation with perturbation terms.

3. Perturbative method and solution

Consider the following perturbed NNLSE equation:

$$\mathrm{i}\frac{\partial \Psi}{\partial \xi} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial \tau^2} + \Psi|\Psi|^2 = \mathrm{i}\,\varepsilon(\Psi), \tag{15}$$

where $\varepsilon(\Psi)$ is a small perturbation that depends on Ψ, Ψ^* and their derivatives.[25] From the discussion above, we make sure that the perturbation method can be used to find the analytical solution of the soliton.

First, introduce

$$\varepsilon = \mathrm{i}\left(\mu_1 \frac{\partial^2 |\Psi|^2}{\partial \tau^2} + \mu_2 \frac{\partial^4 |\Psi|^2}{\partial \tau^4}\right)\Psi. \tag{16}$$

Since μ_1 and μ_2 are very small in value, the term of ε can be viewed as a small perturbation. Thus equation (14) can be written as

$$\mathrm{i}\frac{\partial \Psi}{\partial \xi} + \frac{1}{2}\frac{\partial^2 \Psi}{\partial \tau^2} + \Psi|\Psi|^2 = \mathrm{i}\,\varepsilon. \tag{17}$$

The perturbation technique attempts to obtain a set of four ordinary differential equations for four soliton parameters.[25] The solution of the perturbed NNLSE equation can be written as

$$\Psi(\xi,\tau) = \eta(\xi)\,\mathrm{sech}[\eta(\xi)(\tau - q(\xi))]\exp[\mathrm{i}\,\phi(\xi) - \mathrm{i}\,\delta(\xi)\tau], \tag{18}$$

where the profile $\eta = \eta(\xi)$ is real and symmetric, ξ is the propagation coordinate, τ is transverse coordinate, η, q, ϕ and δ represent amplitude, excursion of the beam to propagation coordinate, soliton phase and deflexion, respectively.

Substitute $\varepsilon(\Psi)$ into the following ordinary differential equations:[24,25]

$$\begin{aligned} \frac{\mathrm{d}\eta}{\mathrm{d}\xi} &= \mathrm{Re}\int \varepsilon(\Psi)\,\Psi^*(\tau)\mathrm{d}\tau \\ &= \mathrm{Re}\int [2\,\mathrm{i}\,\eta^6(-2 + \cosh[2\eta(-q+\tau)]) \\ &\quad \times \mathrm{sech}[\eta(-q+\tau)]^6\mu_1 + 66\,\mathrm{i}\,\eta^8\,\mathrm{sech}[\eta(-q+\tau)]^8\mu_2 \\ &\quad - 52\,\mathrm{i}\,\eta^8\cosh[2\eta(-q+\tau)]\,\mathrm{sech}[\eta(-q+\tau)]^8\mu_2 \\ &\quad + 2\,\mathrm{i}\,\eta^8\cosh[4\eta(-q+\tau)]\,\mathrm{sech}[\eta(-q+\tau)]^8\mu_2]\mathrm{d}\tau, \end{aligned} \tag{19}$$

$$\begin{aligned} \frac{\mathrm{d}\delta}{\mathrm{d}\xi} &= -\mathrm{Im}\int \varepsilon(\Psi)\tanh[\eta(\tau - q)]\,\Psi^*(\tau)\mathrm{d}\tau \\ &= -\mathrm{Im}\int [2\,\mathrm{i}\,\eta^6(-2 + \cosh[2\eta(-q+\tau)]) \\ &\quad \times \mathrm{sech}[\eta(-q+\tau)]^6\mu_1\tanh[\eta(-q+\tau)] \\ &\quad + \mathrm{i}\,\eta^8\,\mathrm{sech}[\eta(-q+\tau)]^9(92\sinh[\eta(-q+\tau)] \\ &\quad - 27\sinh[3\eta(-q+\tau)] + \sinh[5\eta(-q+\tau)])\mu_2]\mathrm{d}\tau, \end{aligned} \tag{20}$$

$$\begin{aligned} \frac{\mathrm{d}q}{\mathrm{d}\xi} &= -\delta + \frac{1}{\eta^2}\mathrm{Re}\int \varepsilon(\Psi)(\tau - q)\,\Psi^*(\tau)\mathrm{d}\tau \\ &= -\delta + \frac{1}{\eta^2}\mathrm{Re}\int [2\,\mathrm{i}\,\eta^6(-q+\tau) \\ &\quad \times (-2 + \cosh[2\eta(-q+\tau)])\,\mathrm{sech}[\eta(-q+\tau)]^6\mu_1 \\ &\quad - 2\,\mathrm{i}\,\eta^8(-q+\tau)(33 - 26\cosh[2\eta(-q+\tau)] \\ &\quad + \cosh[4\eta(-q+\tau)])\,\mathrm{sech}[\eta(-q+\tau)]^8\mu_2]\mathrm{d}\tau, \end{aligned} \tag{21}$$

$$\begin{aligned} \frac{\mathrm{d}\phi}{\mathrm{d}\xi} &= \mathrm{Im}\int \varepsilon(\Psi)[\eta^{-1} - (\tau - q)\tanh[\eta(\tau - q)]] \\ &\quad \times \Psi^*(\tau)\mathrm{d}\tau + \frac{1}{2}(\eta^2 - \delta^2) + q\frac{\mathrm{d}q}{\mathrm{d}\xi} \\ &= \frac{1}{\eta}\mathrm{Im}\int \varepsilon(\Psi)\,\Psi^*\mathrm{d}\tau \\ &\quad - \mathrm{Im}\int \varepsilon(\Psi)\tau\tanh[\eta(\tau - q)]\,\Psi^*\mathrm{d}\tau \\ &\quad + q\,\mathrm{Im}\int \varepsilon(\Psi)\tanh[\eta(\tau - q)]\,\Psi^*\mathrm{d}\tau \\ &\quad + \frac{1}{2}(\eta^2 - \delta^2) + q\frac{\mathrm{d}q}{\mathrm{d}\xi}. \end{aligned} \tag{22}$$

Finally, the different equations for the parameters can be obtained as

$$\frac{\mathrm{d}\eta}{\mathrm{d}\xi} = \frac{16}{15}\mathrm{Im}\left(\mu_1\eta\xi^5 + \frac{64}{21}\eta^7\mu_2\right) = 0 \Rightarrow \eta = \eta(0), \quad (23)$$

$$\frac{\mathrm{d}\delta}{\mathrm{d}\xi} = 0 \Rightarrow \delta = \delta(0) = 0, \quad (24)$$

$$\frac{\mathrm{d}q}{\mathrm{d}\xi} = -\delta + \frac{1}{\eta^2}\mathrm{Re}\left(\frac{16}{3}\,\mathrm{i}\,\eta^5\tau - \frac{96}{15}\mathrm{i}^5\eta\tau\mu_1\right) \Rightarrow q = -\delta = 0, \quad (25)$$

$$\frac{\mathrm{d}\phi}{\mathrm{d}\xi} = \frac{1}{2}\eta^2 - \frac{4}{3}\eta^4\mu_1 + \frac{16}{3}\eta^6\mu_2 \Rightarrow \phi = \left(\frac{1}{2}\eta^2 - \frac{4}{3}\eta^4\mu_1 + \frac{16}{3}\eta^6\mu_2\right)\xi, \quad (26)$$

where $\eta(0) = \eta$, $q(0) = 0$, $\phi(0) = 0$, and $\delta(0) = 0$. From the results, we discover that only soliton phase ϕ is affected by the perturbation ξ. The expression is given as

$$\phi = \left(\frac{1}{2}\eta^2 - \frac{4}{3}\eta^4\mu_1 + \frac{16}{3}\eta^6\mu_2\right)\xi. \quad (27)$$

With the parameters obtained above, we obtain the approximate analytical soliton solution in a weakly nonlocal nonlinear Kerr-type medium as

$$\Psi = \eta\,\mathrm{sech}(\eta\tau)\exp\left[\mathrm{i}\left(\frac{1}{2}\eta^2 - \frac{4}{3}\eta^4\mu_1 + \frac{16}{3}\eta^6\mu_2\right)\xi\right]. \quad (28)$$

Meanwhile, we also find that the normalized amplitude is a constant which does not vary with the weak nonlocality degree.

4. Comparison between analytical results and simulation results

To verify our analytical solutions, we employ the split-step Fourier method (SSFM).[25] We take NNLSE (1) as the evolution equation with the initial conditions $\psi(x, 0)$ to numerically simulate the propagation of an optical beam in a nonlocal medium whose response function is of the Gaussian-type.

As clearly depicted in Fig.1, when the nonlocal degree is equal to 0.1, we have almost the local case. It can be explained as that in the weakly nonlocal case, the evolution property of the soliton is similar to that of the local soliton. So it is reasonable that the equation for describing the soliton propagation in weak nonlocality can be expressed by the nonlinear Schrödinger equation with perturbation terms.

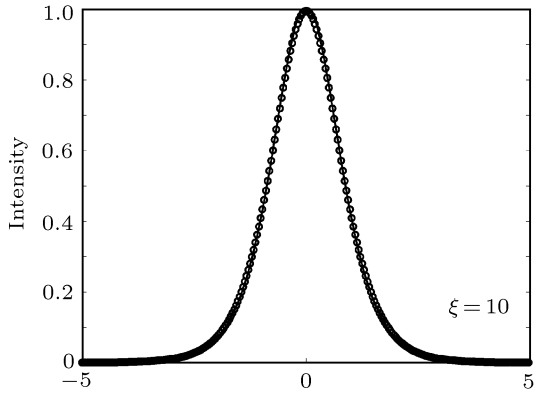


Fig.1. Profiles of the normalized intensity profiles for (1+1)-D soliton. The solid curve is for the local case and the circle curve is for the case where nonlocality is equal to 0.1.

Figure 2 shows a comparison between the cases with two different values of the nonlocality degree α.

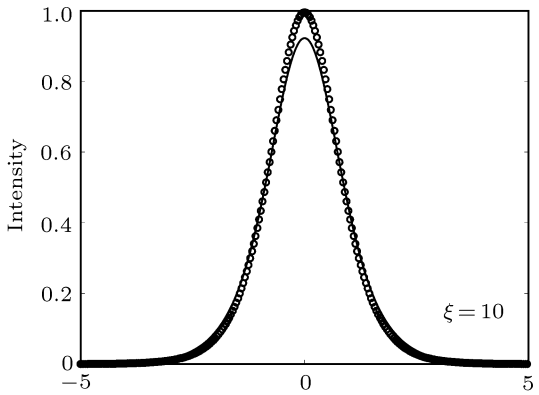


Fig.2. Profiles of the normalized intensity profiles for different values of degree of nonlocality α of (1+1)-D soliton: α=0.25 (solid curve) and 0.1 (circle curve) on the assumption that the response of the material is of the Gaussian-type.

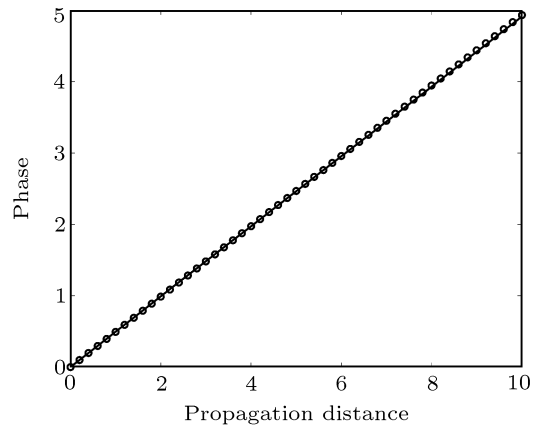


Fig.3. Comparison between analytical phase (circle curve) and exact numerical phase (solid curve) for (1+1)-D beam propagation with Gaussian-shaped response function when the nonlocality is 0.1.

Evidently, from the figures above, we know that the soliton width turns to expand along with the increase of the nonlocality degree. That means that the weak nonlocality smooths out the refractive index profile thereby leading to the broadening of the beam. Furthermore, the numerical simulations for the cases where $\alpha = 0.25$ and $\alpha > 0.25$ reveal that the profiles are too close to distinguish them.

Figures 3 and 4 show a comparison of phase evolution between analytical solution and exact numerical solution in different weakly nonlocal cases.

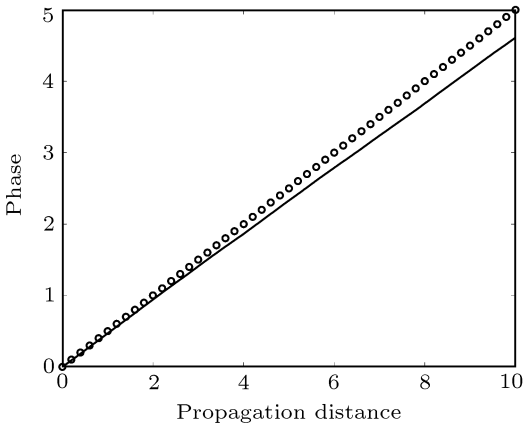


Fig.4. Comparison of analytical phase (circle curve) with exact numerical phase (solid curve) for (1+1)-D beam propagation with Gaussian-shaped response function when the nonlocality is 0.25.

As mentioned above, we arrive at the conclusion that our analytical results are consistent with our simulations. Thus, our method is correct and successful to achieve the (1+1)-D soliton solution in a weakly nonlocal nonlinear Kerr-type medium.

5. Conclusion

Based on the perturbation method, we find an approximate analytical solution of the NNLSE in a weakly nonlocal Kerr-type medium. Starting from the governing NNLSE, through the normalized transformation, we find that the equation for describing the soliton propagation in a weak nonlocality can be written as the nonlinear Schrödinger equation with perturbation terms. Then, we obtain the approximate analytical solutions, which are confirmed by the numerical simulations. Our analysis develops the understanding of the beam propagation in a weakly nonlocal nonlinear Kerr-type medium, that is, we can use the perturbation method to find the soliton solution. Furthermore, the movement of the intensity profile and the phase variation of the soliton solution can enhance the vision to us. The soliton properties are still worthy of studying in future research.

References

[1] Snyder A W and Mitchell D J 1997 *Science* **276** 1538
[2] Mitchell D J and Snyder A W 1999 *J. Opt. Soc. Am.* B **16** 236
[3] Królikowski W, Bang O, Rasmussen J J, Wyller J 2001 *Phys. Rev.* E **64** 016612
[4] Bang O, Królikowski W, Wyller J and Rasmussen J J 2002 *Phys. Rev.* E **66** 046619
[5] Abe S and Ogura A 1998 *Phys. Rev.* E **57** 6066
[6] Granot E, Sternklar S, Isbi Y, Malomed B and Lewis A 1999 *Opt. Commun.* **166** 121
[7] Królikowski W and Bang O 2001 *Phys. Rev.* E **63** 016610
[8] Wyller J, Królikowski W, Bang O and Rasmussen J J 2002 *Phys. Rev.* E **66** 066615
[9] Nikolov N I, Neshev D and Królikowski W Z 2003 *Phys. Rev.* E **68** 036614
[10] Guo Q, Luo B, Yi F, Chi S and Xie Y 2004 *Phys. Rev.* E **69** 016602
[11] Xie Y and Guo Q 2004 *Opt. Quantum Electron* **36** 1335
[12] Ouyang S G, Guo Q and Hu W 2006 *Phys. Rev.* E **74** 036622
[13] Królikowski W, Bang O, Nikolov N I, Neshev D, Wyller J, Rasmussen J J and Edmundson D 2004 *J. Opt.* B: *Quantum Semiclass. Opt.* **6** S288
[14] Cao L G, Guo Q, Hu W, Lu D Q, Yang P B and Zhu Y Q 2008 *Acta Phys. Sin.* **57** 6365 (in Chinese)
[15] Peccianti M, Rossi A, Assanto G, Luca A, Umeton C and Khoo I C 2000 *Appl. Phys. Lett.* **77** 7
[16] Peccianti M, Conti C, Assanto G 2002 *Appl. Phys. Lett.* **81** 3335
[17] Liu J S 2007 *Chin. Phys.* **16** 165
[18] Conti C, Peccianti M and Assanto G 2003 *Phys. Rev. Lett.* **91** 073901
[19] Peccianti M, Conti C and Assanto G 2003 *Phys. Rev.* E **68** R025602
[20] Conti C, Peccianti M and Assanto G 2004 *Phys. Rev. Lett.* **92** 113902
[21] Song Y J, Hu M L, Liu B W, Chai L and Wang Q Y 2008 *Acta Phys. Sin.* **57** 6425 (in Chinese)
[22] Stegeman G I and Segev M 1999 *Science* **286** 1518
[23] Stegeman G I, Christodoulides D N and Segev M 2000 *IEEE J. Sel. Top. Quantum Electron.* **6** 1419
[24] Hasegawa G P and Kodama Y 1995 *Soliton in Optical Communications* (New York: Oxford University Press)
[25] Agrawal G P 2001 *Nonlinear Fiber Optics*, third ed. (London: Academic) p51, p167
[26] Martinez-Herrero R, Piquero G and Mejias P M 1995 *Opt. Commun.* **115** 225