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# Improving consensual performance of multi-agent systems in weighted scale-free networks*

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This paper studies consensus problems in weighted scale-free networks of asymmetrically coupled dynamical units, where the asymmetry in a given link is determined by the relative degree of the involved nodes. It shows that the asymmetry of interactions has a great effect on the consensus. Especially, when the interactions are dominant from higher- to lower-degree nodes, both the convergence speed and the robustness to communication delay are enhanced.

**Keywords:** consensus problems, multi-agent systems, scale-free networks
**PACC:** 0565

## 1. Introduction

Consensus is a fundamental problem for the distributed coordination of multi-agent systems in networks. It has received much attention from various fields of science and engineering. This is partly due to technological advances in communication, computation, and important practical applications, such as unmanned vehicles, automated highway systems, mini-satellites, and so on. Consensus problems make sense in the context of distributed systems and have a long history in the field of computer science.[1] The formal study of consensus problems in groups of experts originated in management science and statistics in the 1960s.[2] About two decades later, de Groot applied the ideas of the statistical consensus theory in the aggregation of information with uncertainty obtained by multiple sensors[3] and medical experts.[4] In networked multi-agent systems, consensus means reaching an agreement regarding a certain quantity of interest that depends on the state of all agents. A consensus algorithm (or protocol) is an interaction rule that specifies the information exchange between an agent and all of its neighbors.[5] The main relevant problem about consensus of agents is concerned with the synchronization of coupled oscillators, flocking and swarming, formation control, and the rendezvous problem, etc.[6−17]

The implementation of consensus for a multi-agent system strongly depends on the topology of the communication network established among agents. In Ref.[18], Olfati-Saber and Murray provided a feasible range of sampling rates leading to consensus in the case of discrete-time dynamic networks. In this way, the theoretical foundations of general consensus problems were established. To improve the speed of convergence towards consensus, they further proposed a method based on the addition of a few long links to a regular lattice, thus transforming it into a small-world network.[19] In Ref.[20], Xiao and Boyd transformed the fastest distributed linear averaging problem into a convex optimization problem by considering a particular per-step convergence optimization index. In addition, they proved that, when the network topology is symmetric, the problem of finding the fastest converging linear iteration can be cast as a semidefinite programming problem, and thus can be efficiently and globally solved. In Ref.[21], consensus problems in a heterogeneous influence network were investigated. It showed that, by decreasing the scaling exponent in the associated power-law distribution, the ability of the network to reach the direction consensus among its agents is significantly enhanced due to the leading roles played by a few hubs of agents.

Apart from the topological issue, it is important to realize that many complex networks are weighted

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and asymmetric, i.e., the interaction between two nodes is characterized not only by the existence of a link but a link with a varying weight assigned to it. In this paper, we investigate the collective behavior of multi-agent systems in an asymmetrically weighted scale-free network. The aim of the work is to investigate the ability of the network to achieve global consensus among agents in terms of the asymmetric coupling influence of the network. We find that the consensual performance, such as the convergence speed and the robustness to communication delay, is significantly improved when the direction of the dominant coupling is from higher- to lower-degree nodes.

The rest of this paper is organized as follows. In Section 2, we define consensus problems on networks and present the linear protocol. In Section 3, we introduce the asymmetrically weighted scale-free network model and analyse consensus in this network numerically. Finally, we give concluding remarks in Section 4.

## 2. Formulation of consensus problem

For a networked system under consideration which consists of $N$ agents, each agent is regarding as a node in a directed network $G$ and each edge $(v_i, v_j) \in \varepsilon(G)$ corresponds to an available information link from agent $i$ to agent $j$ at time $t$. Moreover, each agent updates its current state, based upon the information received from its neighbors.

The information states with agent dynamics are given by

$$\dot{x}_i = u_i, \quad i = 1, 2, \ldots, N, \tag{1}$$

where $x_i \in R^n$ denotes the information state of the $i$-th agent and $u_i \in R^n$ is the input control. For simplicity, we consider the linear dynamic system[1)]

$$u_i = \sum_{j \in N_i} a_{ij}(x_j(t) - x_i(t)), \quad x_i(0) \in R, \tag{2}$$

where $a_{ij}$ is the $(i, j)$ entry of the adjacency matrix of the associated communication at time $t$, and $N_i$ represents the set of agents whose information is available from the agent $\Sigma_i$ at the time $t$. The control $u_i$ drives $x_i$ to the average position of its neighbors. So the collective dynamics of the nodes in system (1) can be expressed by

$$\dot{x}(t) = -Lx(t), \tag{3}$$

where $x = [x_1(t), x_2(t), \ldots, x_n(t)]^T$ denotes the aggregated state vector of the multi-agent system, and $L = [l_{ij}] \in R^{N \times N}$ is the Laplacian matrix of the network. One can say that the nodes in the network have reached a consensus if $x_i = x_j$ for all $i, j \in \Sigma, i \neq j$. Whenever the states of a network are all in agreement, the common value of all nodes is called the group decision value. Based on system (3), one can obtain the following definitions of the consensual algorithm.[22]

**Definition 1** The set of agents $\Sigma$ is said to be consensual, if $\|x_i - x_j\| = 0$ for each $(i, j) : i, j = 1, 2, \ldots, N$ as $t \geq t_0$. The set of agents $\Sigma$ is said to asymptotically reach global consensus if for any $x_i(0) : i = 1, 2, \ldots, N, \|x_i - x_j\| \to 0$ as $t$ tends to infinity for each $(i, j) : i, j = 1, 2 \ldots, N$. The set of agents $\Sigma$ is said to be reachable for global consensus if there exists an information update strategy (protocol) for each $x_i : i = 1, 2, \ldots, N$ that achieves global consensus asymptotically for $\Sigma$.

## 3. Consensus in asymmetrically weighted scale-free networks

### 3.1. Network construction

In this Section, we consider scale-free networks of asymmetrically coupled dynamical systems. The topological information on the graph is contained in the connection matrix $L$. Specifically, $L = \{l_{ij}\}$ is defined as follows:

$$l_{ij} = \begin{cases} \dfrac{-a_{ij}w_{ij}}{\sum_k a_{ik}w_{ik}}, & i \neq j, \\ 1, & i = j, \end{cases} \tag{4}$$

where $A = \{a_{ij}\}$ is the irreducible adjacency matrix associated to the network topology, that is, $a_{ij} = a_{ji} = 1$ if there is connection between $i$ and $j$, 0 otherwise. To construct scale-free networks, we use the Barabási–Albert (BA) algorithm.[23] $W = \{w_{ij}\}$ is a positive-definite matrix containing the information about the weights over the network connections, with $w_{ij}$ giving a measure of the strength of the interaction between nodes $i$ and $j$. In Ref.[24], Hwang *et al* suggested a particularly suitable way of choosing the weights over the network connections, according to the presence of a given ordering among nodes of the network. Specifically, the nodes are ordered according to a given scalar property, $p$, in such a way that one has $i \leq j$ if $p(i) \leq p(j)$. Here, we consider

[1)] For more general, one can consider a nonlinear form, $u_i = \sum_{j \in N_i} a_{ij} F(x_j(t) - x_i(t))$, $x_i(0) \in R$.

a very natural way of choosing the weights based on an ordering of degree, i.e., $p(i) = k_i$, where $k_i$ is the degree of node $i$. Then $W = \{w_{ij}\}$ is defined by

$$w_{ij} = \frac{Q_{ij}}{\sum_k a_{ik} Q_{ik}}, \tag{5}$$

where $Q_{ij} = [(1-q)/2][(1+q)/2]$ if $k_i > k_j$ ($k_i < k_j$) and $q \in (-1, 1)$ is a parameter governing the coupling asymmetry within the network. Especially, $q = 0$ corresponds to the case of symmetric coupling. The limit of $q \to 1$ ($-1$) corresponds to the case of a directed network which only has connections from high-degree nodes to low-degree nodes (from low-degree nodes to high-degree nodes).

## 3.2. Algorithm

Suppose a network without isolated nodes and each node's neighboring set is fixed and is only determined by each node's initial state $x_i(0)$, $i = 1, 2, \ldots, N$, then the Laplacian matrix $L$ is a symmetric and irreducible constant matrix. In this case, the set of eigenvalues of $L$ can be ordered sequentially in an ascending way

$$0 = \lambda_1 \leq \lambda_2 \leq \lambda_3 \leq \ldots \leq \lambda_N \leq 2\Delta, \tag{6}$$

where $\Delta$ is the maximum degree of the network. The zero eigenvalue is known as the trivial eigenvalue of $L$. For a connected graph $G$, one has $\lambda_2 > 0$. The second smallest eigenvalue, $\lambda_2$, is called the connectivity of a graph. Fiedler has proved that $\lambda_2$ is directly related to the consensual speed of the network.[25] Therefore it is a measure of the speed of consensual algorithms.

If we consider time delay, the consensual algorithm takes the following form

$$\dot{x}_i(t) = \sum_{j \in N_i} a_{ij}(x_j(t-\tau) - x_i(t-\tau)), \quad i = 1, 2, \ldots, N \tag{7}$$

with a collective dynamics

$$\dot{x}(t) = -Lx(t-\tau). \tag{8}$$

Here we assume that the delay in all links is equal to $\tau$ (universal delay). The following lemma provides an upper bound on the delay such that stability of the network dynamics is maintained in presence of the delay.

**Lemma 1** For the networked system (8), a necessary and sufficient condition for stability is given by[18]

$$\tau < \tau_{\max} = \frac{\pi}{2\lambda_n}. \tag{9}$$

Thus $\lambda_n$ is a measure of robustness to time delay for reaching consensus in a network. The smaller the value of $\lambda_n$ is, the longer the delay time $\tau$ the network system can stand.

## 3.3. Numerical results

The network topology determines the eigenvalue distribution of its Laplacian matrix, which influences the performance of the linear consensual protocol and its robustness to time delay. In the following, we study these problems by simulations.

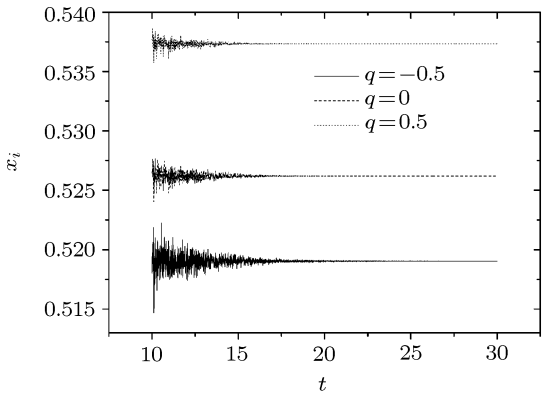



**Fig.1.** Three state trajectories of the network corresponding to $q = -0.5, 0$, and $0.5$, respectively. The network parameters are $N = 1000$ and $m_0 = m = 5$.

Figure 1 shows the evolution of network states $x_i(t)$ with different values of $q$ for the dynamic system (3). In each case, the states $x_i(t)$ asymptotically reach a common value as time $t$ increases. In fact, given any value of $q$, global consensus can be asymptotically achieved. The final consensus value is given by $x^* = \sum_{i=1}^{N} \gamma_i x_i(0)$, where $\gamma = [\gamma_1, \ldots, \gamma_n]^T$ with $\gamma_i \geq 0$ being a nonnegative left eigenvector of $L$ associated with eigenvalue 0. Notice that $\gamma_i \neq 0$ if agent $i$ has a directed path to all other agents in the information exchange, else $\gamma_i = 0$. Based on Ref.[14], the coefficients $\gamma_i$ satisfy the following linear equations:

$$\begin{cases} \gamma^T L = 0, \\ \sum_{j=1}^{N} \gamma_j = 1. \end{cases} \tag{10}$$

The speed of reaching a consensus is the key in the design of the network topology as well as analysis of the performance of a consensual algorithm. Here we study the speed of convergence of the linear algorithm. The state equation is described by Eq.(3). The disagreement of a group of nodes with state $x$ is defined by[18]

$$\varphi(t) = x^{\mathrm{T}} L x. \tag{11}$$

The initial states are uniformly distributed random numbers from the unit interval. In Fig.2 we plot the disagreement $\varphi(t)$ as a function of time $t$ for different values of $q$. Given any value of $q$, there exists a threshold $T_c(q)$ so that complete consensus can be reached (i.e., $\varphi(t) < 10^{-9}$ if $t \geq T_c$). To understand the influence of $q$ on the convergence speed, we plot the threshold $T_c$ as a function of $q$ in Fig.3.

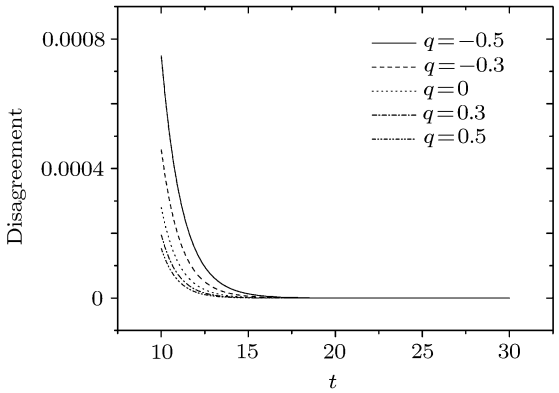



**Fig.2.** The disagreement $\varphi(t)$ as a function of time $t$ for different $q$. The network parameters are the same as those in Fig.1.

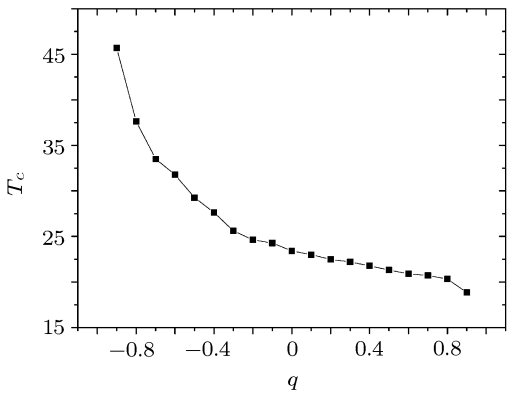



**Fig.3.** The convergence time $T_c$ as a function of $q$ in weighted scale-free networks. The network parameters are the same as those in Fig.1.

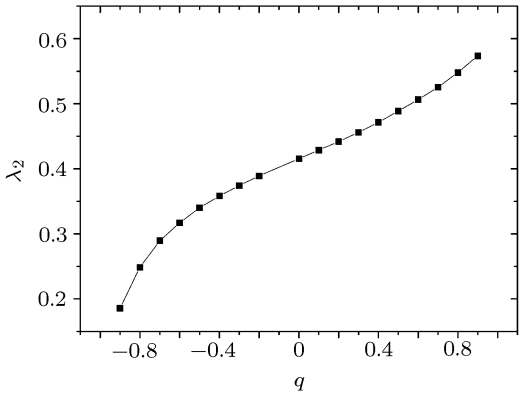



**Fig.4.** Behavior of the second smallest eigenvalue $\lambda_2$ as a function of $q$. The network parameters are the same as those in Fig.1.

Obviously, $T_c$ is very sensitive to $q$. As $q$ increases, $T_c$ decreases monotonically, indicating that it is much easier to achieve global consensus in the network with asymmetric interactions, directed from high-degree nodes to low-degree ones. In Fig.4 we present $\lambda_2$ as a function of $q$ to test the validity of the above claims. The increase of $\lambda_2$ confirms the need of a dominant interaction from hubs to nonhubs for improving the consensual speed.

Next we study the robustness of consensus problems to time delay in the node's communication. In Fig.5

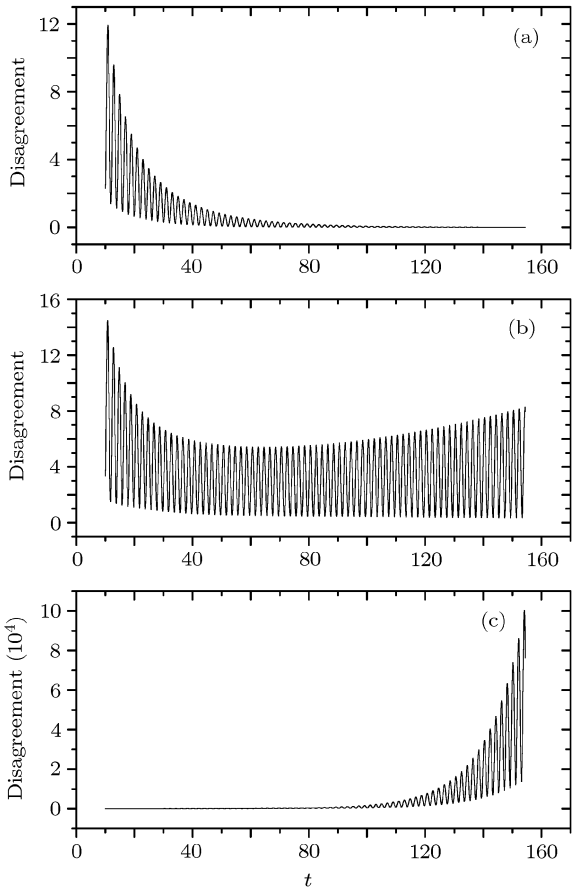



**Fig.5.** The disagreement $\varphi(t)$ as a function of time $t$ for $q = 0.5$ (a), 0 (b) and $-0.5$ (c). Parameters are $N = 1000$, $m_0 = m = 5$, and $\tau = 1.0$.

we plot the disagreement $\varphi(t)$ as a function of time $t$ with various $q$. The dynamical behaviors emerge differently for different $q$. For $q = 0.5$, $\varphi(t)$ decreases as $t$ increases. By $t \geq 120$, the whole network reaches global consensus (i.e., $\varphi(t) < 10^{-9}$). For $q = 0.0$, synchronous oscillations occur in the network. For $q = -0.5$, $\varphi(t)$ increases quickly as $t$ increases. When $t$ tends to infinity, the network not only fails to converge, but also diverges. All these results suggest that the asymmetric interactions, directed from hubs

to nonhubs in a heterogeneous network, play a key role during the evolution of the system. The stronger the strength of the coupling dominating in the direction from the high-degree nodes to the low-degree ones is, the longer the delay time $\tau$ the network system can stand. Figure 6 shows the behavior of $\lambda_N$ as a function of $q$. As $q$ increases, $\lambda_N$ decreases evidently. According to Lemma 1, we conclude that the network becomes more robust to time delay.

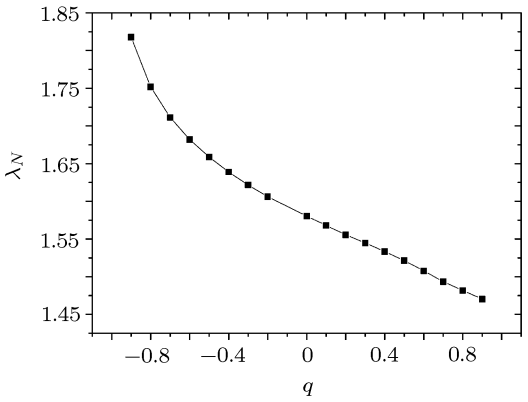



**Fig.6.** Behavior of the highest eigenvalue $\lambda_N$ as a function of $q$. Parameters are the same as those in Fig.5.

# 4. Conclusion

In this paper, we have studied consensus problems of multi-agent systems in weighted scale-free networks. Simulation results show that the hub of agents plays a leading role during achieving global consensus of the network. When the dominant direction of couplings is from the hub to the nonhub nodes, both the speed to reach consensus and robustness to the communication delay of the network are greatly enhanced. Our approach may provide new insights in the study of collective communication or coordination in distributed social networks.

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