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A control method for congested traffic in the coupled map car-following model*

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Based on the pioneer work of Konishi *et al*, a new control method is presented to suppress the traffic congestion in the coupled map (CM) car-following model under an open boundary. A control signal concluding the velocity differences of the two vehicles in front is put forward. The condition under which the traffic jam can be contained is analyzed. The results are compared with that presented by Konishi *et al* [*Phys. Rev.* 1999 E **60** 4000–4007]. The simulation results show that the temporal behavior obtained by our method is better than that by the Konishi's *et al.* method, although both the methods could suppress the traffic jam. The simulation results are consistent with the theoretical analysis.

Keywords: traffic flow, the coupled map car-following model, feedback control
PACC: 0560, 0550

1. Introduction

Nowadays, with the development of economics, traffic congestion becomes more and more serious, especially in big cities. Car-following models are proposed for the description of interacting driver-vehicle units. They are not only of great importance to an autonomous cruise control system, but also as important evaluation tools for intelligent transportation system strategies since the 1990s. Car-following theories have been given much research interest. They are based on the assumption that each driver reacts in some specific fashion to a stimulus from the vehicle ahead of him.

In 1961, Newell[1] presented a car-following model described by a differential equation of the first order. The optimal velocity (OV) function was firstly defined in Ref.[1]. In 1995, Bando *et al*[2] proposed a microscopic traffic model called the optimal velocity model (OVM). It was based on the idea that each vehicle has an optimal velocity, which depends on the following distance of the preceding vehicle. After that, much interest was paid in the optimal velocity model, moreover, Konishi *et al*[3] proposed the delayed-feedback control. The traffic system can run well with the delayed-feedback control.

Then Zhao and Gao[4] presented a similar control method for congested traffic induced by bottlenecks in the CM car-following model. Recently, based on applying the intelligent transportation system (ITS), cooperative driving models related to the microscopic car-following model and the macroscopic lattice hydrodynamic model were investigated in Refs.[5] and [6], where the ITS application meant that drivers could receive information of other vehicles on roads, and then adjust the velocities of their own vehicles. In the light of this information, it is possible to improve the stability of traffic flow and suppress the appearance of a traffic jam.

In order to improve the OVM and considering the ITS application, we put forward a new control method taking into account the two velocity differences $\Delta v_i(n)$ and $\Delta v_{i-1}(n)$, where $\Delta v_i(n) = v_{i-1}(n) - v_i(n)$, $\Delta v_{i-1}(n) = v_{i-2}(n) - v_{i-1}(n)$.

In this paper, there are three main sections. First, we introduce the CM car-following model and briefly its stability analysis. Second, we present the control method, and analyze the condition under which the traffic jam can be well suppressed. Third, we make the numerical simulation. We also compare our method

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with that presented by Konishi *et al.*[3]

2. CM car-following model and its stability analysis

2.1. CM car-following model

Our investigations are based on the CM car-following model proposed by Konishi *et al.* Let us give the description of this traffic model. The leading vehicle is described as

$$x_0(n+1) = v_0 T + x_0(n), \tag{1}$$

where $x_0(n) > 0$ is the position of the leading vehicle at time $t = nT$, $v_0 > 0$ is its speed, $T > 0$ is the sampling time and $x_0(0) > 0$ is the leading vehicle's initial position.

The following vehicle i is given by the following equation

$$x_i(n+1) = v_i(n)T + x_i(n) \quad (i = 1, 2, \ldots, N), \tag{2}$$

where $x_i(n) > 0$ is the position of the i-th vehicle at time $t = nT$, $v_i(n) > 0$ is its speed at time $t = nT$ and N is the total number of vehicles. The velocity of the following vehicles is governed by[3]

$$v_i(n+1) = \alpha_i[V_i^{\rm op}(y_i(n)) - v_i(n)]T + v_i(n), \tag{3}$$

where $\alpha_i > 0$ is the sensitivity of the i-th vehicle driver, and $V_i^{\rm op}(y_i(n))$ is the OV function, which only depends on the distance between the $(i-1)$-th and the i-th vehicles:

$$y_i(n) = x_{i-1}(n) - x_i(n). \tag{4}$$

The OV function has been given by[3]

$$V_i^{\rm op}(y_i(n)) = \frac{v_i^{\max}}{2}\left[1 + \bar{H}_{\rm sat}\left(2\frac{y_i(n) - \eta_i}{\zeta_i}\right)\right], \tag{5}$$

where the saturation function, $\bar{H}_{\rm sat}$ is described as

$$\bar{H}_{\rm sat} = \begin{cases} +1, & \text{if } \rho > +1; \\ \rho, & \text{if } -1 \le \rho \le +1; \\ -1, & \text{if } \rho < -1. \end{cases} \tag{6}$$

In Eq.(5) $v_i^{\max} > 0$ is the maximum speed, $\eta_i > 0$ is the neutral headway distance, and $\zeta_i > 0$ is the parameter. For guaranteeing the existence of a vehicle group, we assume that the speed v_0 of the leading vehicle is less than the maximum speed of all the following vehicles

$$v_0 < v_i^{\max} \quad (i = 1, 2, \ldots, N). \tag{7}$$

To avoid the collisions and backward motions, all the vehicles adopt the full-braking action

$$\text{if} \quad y_i(n) < y_i^{\min}, \tag{8}$$

then

$$x_i(n+1) = x_i(n), \ v_i(n+1) = 0. \tag{9}$$

This action implies that the i-th vehicle stops suddenly when the headway distance is less than the safety distance $y_i^{\min}$. From this, we imply that $y_i^{\min} < \eta_i - \zeta_i/2$.

2.2. Stability analysis

For simplicity, we regard our car-following model as a simple system. If the leading vehicle is running with the speed v_0, the dynamics of the following vehicle [i.e., i-th vehicle] can be given as

$$v_i(n+1) = \alpha_i[V_i^{\rm op}(y_i(n)) - v_i(n)]T + v_i(n), \tag{10}$$

$$y_i(n+1) = v_0 T - v_i(n)T + y_i(n). \tag{11}$$

The steady state of the above system (10) and (11) is

$$[v_i^*, y_i^*]^T = \left[v_0, \frac{v_0}{r_i} - \frac{\zeta_i}{2} + \eta_i\right]^T, \tag{12}$$

where $r_i = v_i^{\max}/\zeta_i$. Obviously, we can ascertain that the essential condition for the existence of the steady state is the condition $v_0 < v_i^{\max}$. According to the control theory, Konishi *et al* directly obtained the following essential condition for the above system to be stable. All the roots of

$$P_i(z) = z^2 + a_i z + b_i = 0 \tag{13}$$

are within a unit circle, where $a_i = \alpha_i T - 2$, $b_i = 1 - \alpha_i T + \alpha_i r_i T^2$. From this condition, the ranges of the parameters are given as

$$0 < r_i < \frac{1}{T}, \quad 0 < \alpha_i < \frac{4}{(2 - r_i T)T}. \tag{14}$$

When the preceeding vehicle is not running with the constant speed v_0, the dynamics of the following vehicle can be given as

$$v_i(n+1) = \alpha_i[V_i^{\rm op}(y_i(n)) - v_i(n)]T + v_i(n), \tag{15}$$

$$y_i(n+1) = v_{i-1}(n)T - v_i(n)T + y_i(n). \tag{16}$$

Also according to the control theory, Konishi *et al* directly derived the following condition of no traffic

jam assuming that all the vehicles satisfy the following condition

$$\max_{|z|=1} |G_i(z)| \leq 1, \quad (i = 1, 2, \ldots, N). \tag{17}$$

From this condition, the range of the parameters are given as

$$0 < r_i < \frac{1}{T}, \quad 0 < \alpha_i < \frac{4}{(2 - r_i T)T} \tag{18}$$

and

$$\frac{8 + \alpha_i T(\alpha_i T - 8)}{\alpha_i T^2(\alpha_i T - 6)} \leq r_i \leq \frac{\alpha_i}{2 + \alpha_i T}. \tag{19}$$

This condition means that the traffic system is robust. The robustness is to enhance the stability of the system, under which the amplitude of the transient disturbance decreases with the development of time even if the structure and parameters of the system are variational. Konishi *et al* put forward a control scheme for suppression of a traffic jam in the CM car-following model. A control signal term, which is proportional to the speed difference between the current time step and previous time step, is added to the system (15) and (16). Four feedback gains are designed to satisfy the two conditions given by expressions (13) and (17), which were elaborated in Ref.[3] in detail. In the next section, we will propose a simple control scheme to suppress the traffic jam.

3. The control method

The aim of our paper is to put forward a simple control scheme for the suppression of a traffic jam in the CM car-following model. In our model, a feedback control signal $u_i(n)$ is designated as follows:

$$u_i(n) = p\Delta v_i(n) + (1 - p)\Delta v_{i-1}(n), \tag{20}$$

where $p > 0$ is the feedback gain, which will be adjusted in the following section of this paper, and

$$\Delta v_i(n) = v_{i-1}(n) - v_i(n), \tag{21}$$

$$\Delta v_{i-1}(n) = v_{i-2}(n) - v_{i-1}(n). \tag{22}$$

The control signal term is added to the traffic system (15) and (16) as

$$v_i(n+1) = \alpha_i[V_i^{\mathrm{op}}(y_i(n)) - v_i(n)]T + v_i(n) + u_i(n), \tag{23}$$

$$y_i(n+1) = v_{i-1}(n)T - v_i(n)T + y_i(n). \tag{24}$$

In the control signal $u_i(n)$, we consider the velocity differences of the two vehicles in front. Some papers consider the velocity difference of one vehicle in front, we see that the $(i-2)$-th vehicle can also affect the i-th vehicle efficiently under some conditions, thus we consider the $(i-2)$-th vehicle and the $(i-1)$-th vehicle in front of the i-th vehicle at the time $t = nT$. We should note that the control signal term $u_i(n)$ acts only if the traffic state is unstable, and the signal vanishes in the stable system.

Around the steady state, the controlled system (15) and (16) can be described by

$$\begin{pmatrix} \delta v_i(n+1) \\ \delta y_i(n+1) \end{pmatrix} = \begin{pmatrix} 1 - \alpha_i T - p & \alpha_i r_i T \\ -T & 1 \end{pmatrix} \begin{pmatrix} \delta v_i(n) \\ \delta y_i(n) \end{pmatrix} + \begin{pmatrix} 2p - 1 & 1 - p \\ T & 0 \end{pmatrix} \begin{pmatrix} \delta v_{i-1}(n) \\ \delta y_{i-2}(n) \end{pmatrix}, \tag{25}$$

$\delta v_i(n) = \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} \delta v_i(n) \\ \delta y_i(n) \end{pmatrix}$, $\delta v_i(n) = v_i(n) - v_i^*$, $\delta y_i(n) = y_i(n) - y_i^*$. The transfer function from $\delta v_{i-1}(n)$ to $\delta v_i(n)$ is described by

$$\begin{aligned} \bar{G}_i(z) &= \begin{pmatrix} 1 & 0 \end{pmatrix} \begin{pmatrix} z - 1 + \alpha_i T + p & -\alpha_i r_i T \\ T & z - 1 \end{pmatrix}^{-1} \begin{pmatrix} 2p - 1 & 1 - p \\ T & 0 \end{pmatrix} \\ &= \frac{\begin{pmatrix} (z-1)(2p-1) + \alpha_i r_i T^2 & (z-1)(1-p) \end{pmatrix}}{\bar{P}_i(z)}, \end{aligned}$$

where $\bar{P}_i(z) = z^2 + \bar{a}_i z + \bar{b}_i$, $\bar{a}_i = \alpha_i T + p - 2$, $\bar{b}_i = 1 - \alpha_i T + \alpha_i r_i T^2 - p$. In order to suppress the traffic jam in the CM car-following model, the feedback gain p is designed such that

(i) $\bar{P}_i(z)$ is stable,

(ii) $\max_{|z|=1} |\bar{G}_i(z)| \leq 1, \ \ (i = 1, 2, \dots N)$.

Theorem 1 There is no traffic jam in the controlled systems (15) and (16),
(i) if

$$0 < \alpha_i r_i T^2 < 2, \tag{26}$$

$$-2 + 4\alpha_i T - \alpha_i^2 T^2 - \alpha_i r_i T^2 + \alpha_i^2 r_i T^3 \geq 0 \ \text{ and } \ \Delta_1 > 0, \tag{27}$$

then the feedback gain p is set as

$$\max\{1 - \alpha_i T + \alpha_i r_i T^2, R_{11}\} < p < \min\left\{\frac{4 - 2\alpha_i T + \alpha_i r_i T^2}{2}, R_{12}\right\}, \tag{28}$$

where

$$\begin{aligned} R_{11} &= (2 + 2\alpha_i T + \alpha_i r_i T^2 - \sqrt{\Delta_1})/8, \\ R_{12} &= (2 + 2\alpha_i T + \alpha_i r_i T^2 + \sqrt{\Delta_1})/8, \\ \Delta_1 &= (2 + 2\alpha_i T + \alpha_i r_i T^2)^2 - 16(-2 + 4\alpha_i T - \alpha_i^2 T^2 - \alpha_i r_i T^2 + \alpha_i^2 r_i T^3); \end{aligned}$$

(ii) if

$$0 < \alpha_i r_i T^2 < 2, \tag{29}$$

$$-2 + 4\alpha_i T - \alpha_i^2 T^2 - \alpha_i r_i T^2 + \alpha_i^2 r_i T^3 < 0, \tag{30}$$

then the feedback gain p is set as

$$1 - \alpha_i T + \alpha_i r_i T^2 < p < \min\left\{\frac{4 - 2\alpha_i T + \alpha_i r_i T^2}{2}, R_{12}\right\}; \tag{31}$$

(iii) if

$$0 < \alpha_i r_i T^2 < 4, \tag{32}$$

$$2 - \alpha_i^2 T^2 + 3\alpha_i r_i T^2 + \alpha_i^2 r_i T^3 \geq 0, \ \ \Delta_2 > 0, \tag{33}$$

then the feedback gain p is set as

$$\max\{-\alpha_i T + \alpha_i r_i T^2, R_{21}\} < p < \min\left\{1 - \alpha_i T + \alpha_i r_i T^2, \frac{4 - 2\alpha_i T + \alpha_i r_i T^2}{2}, R_{22}\right\}, \tag{34}$$

where

$$\begin{aligned} R_{21} &= (6 + 2\alpha_i T + \alpha_i r_i T^2 - \sqrt{\Delta_2})/8, \\ R_{22} &= (6 + 2\alpha_i T + \alpha_i r_i T^2 + \sqrt{\Delta_2})/8, \\ \Delta_2 &= (6 + 2\alpha_i T + \alpha_i r_i T^2)^2 - 16(2 - \alpha_i^2 T^2 + 3\alpha_i r_i T^2 + \alpha_i^2 r_i T^3); \end{aligned}$$

(iv) if

$$0 < \alpha_i r_i T^2 < 4, \tag{35}$$

$$2 - \alpha_i^2 T^2 + 3\alpha_i r_i T^2 + \alpha_i^2 r_i T^3 < 0, \tag{36}$$

then the feedback gain p is set as

$$-\alpha_i T + \alpha_i r_i T^2 < p < \min\left\{1 - \alpha_i T + \alpha_i r_i T^2, \frac{4 - 2\alpha_i T + \alpha_i r_i T^2}{2}, R_{22}\right\}; \tag{37}$$

(v) if

$$0 < \alpha_i r_i T^2 < 2, \tag{38}$$

$$4\alpha_i T - 5\alpha_i^2 T^2 - 4\alpha_i r_i T^2 + 8\alpha_i^2 r_i T^3 - 3\alpha_i^2 r_i^2 T^4 \geq 0, \tag{39}$$

then the feedback gain p is set as

$$p = 1 - \alpha_i T + \alpha_i r_i T^2, \ -\alpha_i T + \alpha_i r_i T^2 < p < \frac{4 - 2\alpha_i T + \alpha_i r_i T^2}{2}. \tag{40}$$

Proof See Appendix.

This theorem provides us with the ability to design the feedback gain p to suppress the traffic jam in the CM car-following model. Three main differences between our method and Konishi's *et al* method are obvious. First, our method has a simpler form than that of Konishi *et al*. In this method, there is only one parameter p to be designed. Second, from the real aspect of traffic, our method is more reasonable than that of Konishi *et al*. We represent the interaction between a vehicle and two vehicles in front. But more precisely, we consider that the interaction between the vehicles is different, so we use the parameter p. Third, four feedback gains in Konishi's *et al* method must be fixed at some exact values, while the feedback gain in our method can be set in a range.

4. Simulation results

In this Section, we will investigate whether the result of our theoretical analysis in Section 3 holds. Our simulations are based on the CM car-following model under open boundary conditions. The CM car-following model parameters are set as follows:[3]

$$\eta = 25.0 \text{ m}, \quad \zeta = 23.3 \text{ m},$$
$$v^{\max} = 33.6 \text{ m/s}, \quad \alpha = 2.0 \text{ s}^{-1},$$
$$v_0 = 20 \text{ m/s}, \quad T = 0.1 \text{ s},$$
$$y^{\min} = 7.02 \text{ m}, \quad r = v^{\max}/\zeta \approx 1.44 \text{ s}^{-1}.$$

It is assumed that all vehicles have the same parameters. The initial conditions are the steady state of the model. The initial positions and speeds are set as

$$x_i(0) = \sum_{j=i+1}^{N} y_j^*, \quad y_j(0) = y_j^*,$$
$$v_i(0) = v_i^*, \quad i = 1, 2, \ldots, N. \tag{41}$$

Consider a situation where the lead vehicle stops suddenly for four times:[3]
$v_0(n) = 0$, for nT=100–103, 120–123, 140–143, 160–163.
The external disturbance is used to examine the effects of the two methods on traffic flow. Now, let us set the feedback gain p in our method. By calculation, we obtain that these parameters fit three conditions (ii), (iii) and (v). Substituting the parameter values into expression (34), then $0.8288 < p < 0.8901$ is obtained. Substituting the parameter values into expression (37), then $0.4393 < p < 0.8288$ is obtained. Substituting the parameter values into expression (40) and then $p = 0.7712$ is obtained. For simplicity, the feedback gain p is chosen according to expression (32) and it is fixed at $p = 0.89$. In Konishi's *et al* method the procedure to design four feedback gains can be found in Ref.[3].

We consider 100 vehicles. Figures 1(a), 2(a) and 3(a) show the space-time plot of the running traffic models without control, having a controlled system with Konishi's *et al* method and that with our method respectively. The leading-vehicle stops at nT=100–103, 120–123, 140–143, and 160–163.

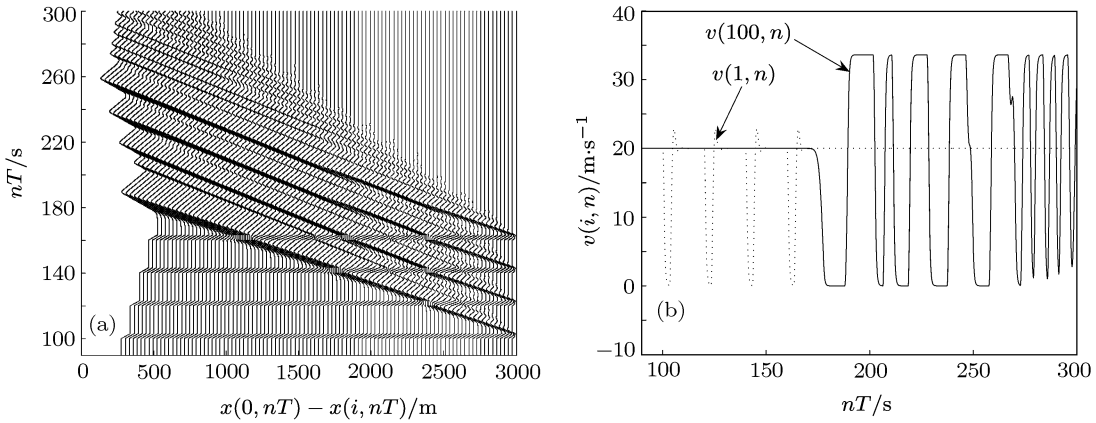


Fig.1. Numerical simulations for the free-running traffic model. The leading-vehicle stops for nT=100–103, 120–123, 140–143, and 160–163. (a) Space-time plot of the free-running traffic model. (b) Temporal velocity behavior of the first and 100-th vehicles.

As we see, the stop disturbance propagates backwards in the traffic model without control. From Figs.2(a) and 3(a), we observe that there is no oscillating behavior and traffic jam for having controlled signals in the system. Figures 1(b), 2(b) and 3(b) show the speed of the first and the 100-th vehicles of the running traffic model without control, having a controlled system with Konishi's *et al* method and that with our method. In Fig.1(b), we can see that the 100-th vehicle is running with the maximum speed $v_i^{\max}$, stopping or oscillating for a short time. From Figs.2(b) and 3(b), we can see that the 100-th vehicle is running smoothly for the amplitude of speed fluctuation decreases with the increase of vehicle number i. However, it should be noted that there are smaller oscillatings by our control method, compared with the system controlled by Konishi's *et al* method. These simulations show that both our method and that of Konishi *et al* can be used to suppress the traffic jam, and the traffic system exhibits better temporal behavior.

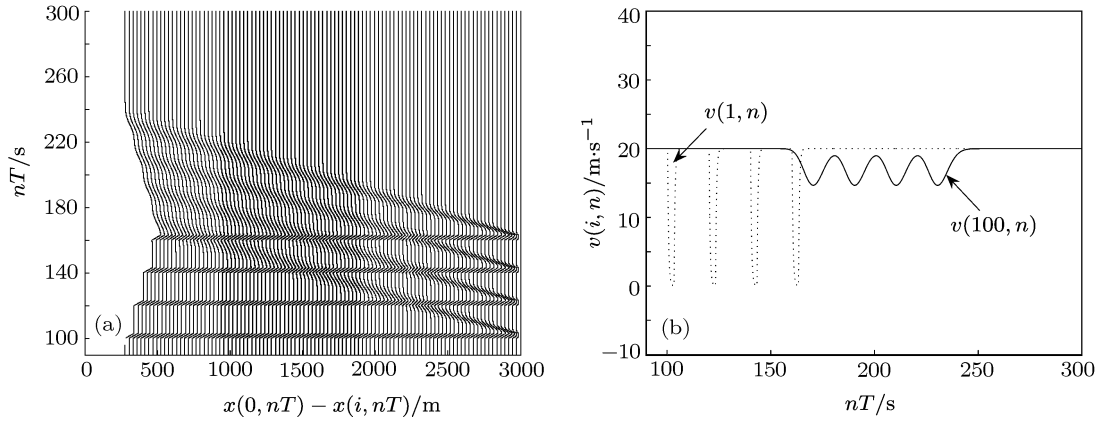


Fig.2. Numerical simulations for the system controlled by Konishi's *et al* method. The stop of the leading-vehicle is the same as in Fig.1. (a) Space-time plot of the system controlled by Konishi's *et al* method. (b) Temporal velocity behavior of the first and 100-th vehicles.

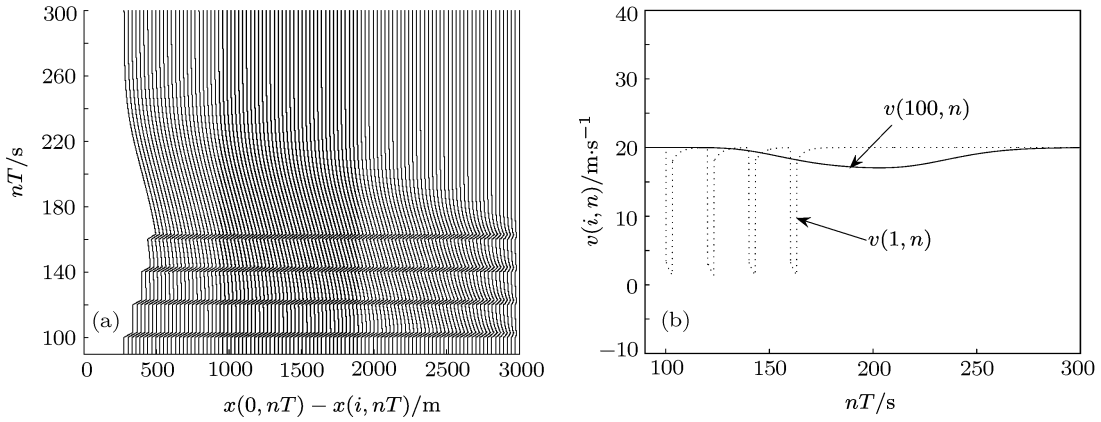


Fig.3. Numerical simulations for the system controlled by our proposed method. The stop of the leading-vehicle is the same as in Fig.1.(a) Space-time plot of the system controlled by our proposed method. (b) Temporal velocity behavior of the traffic model controlled by our proposed method.

For more details, we let the leading-vehicle stop suddenly for only time and observe the three vehicles: the first, the 25-th and the 50-th. We let the leading-vehicle stop for nT=100–102. Figures 4(a) and 5(a) show the space-time plot of the system controlled by Konishi's *et al* method and by our method respectively. Figures 4(b) and 5(b) show the temporal velocity behavior of three vehicles according to Figs.4(a) and 5(a) respectively.

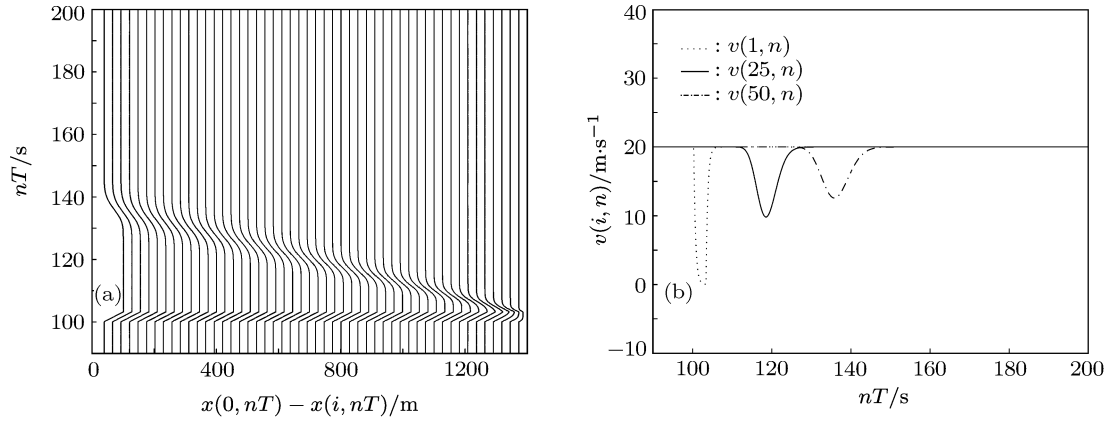


Fig.4. Numerical simulations for the system controlled by Konishi's *et al* method. The leading-vehicle stops at nT=100–102. (a) Space-time plot of the system controlled by Konishi's *et al* method. (b) Temporal velocity behavior of the first, 50-th, and 100-th vehicles.

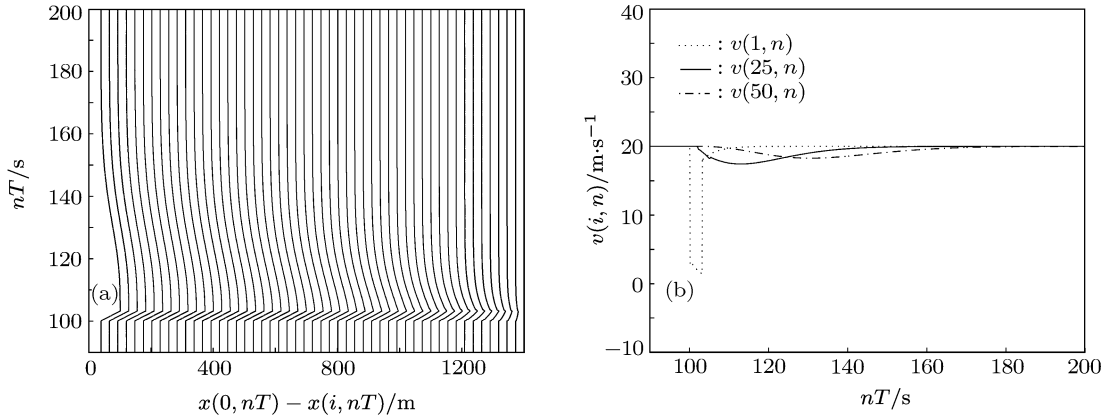


Fig.5. Numerical simulations for the system controlled by our proposed method. The leading-vehicle stops at nT=100–102. (a) Space-time plot of the system controlled by our proposed method. (b) Temporal velocity behavior of the first, 50-th, and 100th vehicles.

From Figs.4 and 5, we can see that our method is better than Konishi's *et al* method, for the traffic system exhibits better behavior by using our method.

5. Conclusions

This paper proposes a simple control method for the suppression of the traffic jam based on the pioneering work of Konishi *et al*.[3] The effect of our control signal on traffic congestion is studied with the CM car-following model.

We can describe three advantages of our model by theoretical analysis and numerical simulations as follows.

(i) The traffic jam can be suppressed by introducing the signal $u_i(n)$.

(ii) The stability of a traffic system becomes better by considering the interactions from the two vehicles in front.

(iii) The numerical simulation is consistent with the theoretical analysis.

Also, our controlling method is simple and feasible for real traffic and would be useful to any size of traffic system.

Appendix

Firstly, according to the stability criterion, $P_i(z)$ is stable if

$$\begin{cases} 1+a_i+b_i>0, \\ 1-b_i>0, \\ 1-a_i+b_i>0. \end{cases}$$

We can obtain

$$\alpha_i r_i T^2-\alpha_i T<p<(\alpha_i r_i T^2-2\alpha_i T+4)/2,$$

$$0<\alpha_i r_i T^2<4.$$

Then we consider

$$\max_{|z|=1}|G_i(z)| \le 1.$$

We can obtain

$$[(z-1)(2p-1)+\alpha_i r_i T^2][(\bar{z}-1)(2p-1)+\alpha_i r_i T^2]+(z-1)(\bar{z}-1)(1-p)^2 \ge |P_i(z)|^2.$$

It can be written as

$$\begin{aligned}&4b_i\cos^2\theta+[2(2p-1)^2-2(2p-1)\alpha_i r_i T^2+2(1-p)^2+2a_i(1+b_i)]\cos\theta+a_i^2+(1-b_i)^2\\&-2(2p-1)^2+2(2p-1)\alpha_i r_i T^2-\alpha_i^2 r_i^2 T^4-2(1-p)^2\ge 0,\end{aligned}$$

when $z=\cos\theta+\mathrm{i}\sin\theta$. That can be reduced to

$$\begin{aligned}&2(1-\cos\theta)[2+p^2-2\alpha_i T+\alpha_i^2T^2-\alpha_i^2 r_i T^3-p(2-2\alpha_i T+\alpha_i r_i T^2)+2(-1+p+\alpha_i T\\&-\alpha_i r_i T^2)\cos\theta-(2p-1)^2-(1-p)^2+(2p-1)\alpha_i r_i T^2]\ge 0,\end{aligned}$$

where $\theta\in[0,2\pi]$. Since

$$|\cos\theta|\le 1,\quad \text{for}\quad \forall\theta\in[0,2\pi].$$

We should consider

$$\begin{aligned}&2(-1+p+\alpha_i T-\alpha_i r_i T^2)\cos\theta+2+p^2-2\alpha_i T+\alpha_i^2T^2-\alpha_i^2 r_i T^3-p(2-2\alpha_i T+\alpha_i r_i T^2)\\&-(2p-1)^2-(1-p)^2+(2p-1)\alpha_i r_i T^2\ge 0.\end{aligned}\tag{A1}$$

As $|\cos\theta|<1$, expression (A1) can be described as

(a) $-2(-1+p+\alpha_i T-\alpha_i r_i T^2)+2+p^2-2\alpha_i T+\alpha_i^2T^2-\alpha_i^2 r_i T^3-p(2-2\alpha_i T+\alpha_i r_i T^2)-(2p-1)^2-(1-p)^2+(2p-1)\alpha_i r_i T^2\ge 0$,
if $2(-1+p+\alpha_i T-\alpha_i r_i T^2)>0$.

(b) $2(-1+p+\alpha_i T-\alpha_i r_i T^2)+2+p^2-2\alpha_i T+\alpha_i^2T^2-\alpha_i^2 r_i T^3-p(2-2\alpha_i T+\alpha_i r_i T^2)-(2p-1)^2-(1-p)^2+(2p-1)\alpha_i r_i T^2\ge 0$,
if $2(-1+p+\alpha_i T-\alpha_i r_i T^2)<0$.

(c) $2+p^2-2\alpha_i T+\alpha_i^2T^2-\alpha_i^2 r_i T^3-p(2-2\alpha_i T+\alpha_i r_i T^2)-(2p-1)^2-(1-p)^2+(2p-1)\alpha_i r_i T^2\ge 0$,
if $-1+p+\alpha_i T-\alpha_i r_i T^2=0$.

We can see that the above conditions (a)–(c) are equal to the conditions (i)–(v) in Theorem 1. For instance, from condition (a), we can obtain an inequality

$$\begin{aligned}&4p^2-(2+2\alpha_i T+\alpha_i r_i T^2)p-2\\&\quad+4\alpha_i T-\alpha_i^2T^2-\alpha_i r_i T^2+\alpha_i^2 r_i T^3\le 0.\end{aligned}$$

We denote

$$2+2\alpha_i T+\alpha_i r_i T^2$$

by B_1, and denote

$$-2+4\alpha_i T-\alpha_i^2T^2-\alpha_i r_i T^2+\alpha_i^2 r_i T^3$$

by C_1. Let

$$\Delta_1=B_1^2-16C_1.$$

The roots of the equation

$$\begin{aligned}&4p^2-(2+2\alpha_i T+\alpha_i r_i T^2)p-2\\&\quad+4\alpha_i T-\alpha_i^2T^2-\alpha_i r_i T^2+\alpha_i^2 r_i T^3=0\end{aligned}$$

are denoted by R_{11} and R_{12}. From the following conditions

$$\alpha_i r_i T^2-\alpha_i T<p<(\alpha_i r_i T^2-2\alpha_i T+4)/2,$$

$$p>1-\alpha_i T+\alpha_i r_i T^2,$$

we obtain that

$$1-\alpha_i T+\alpha_i r_i T^2<(\alpha_i r_i T^2-2\alpha_i T+4)/2,$$

which insures p could have some value. That is

$$0<\alpha_i r_i T^2<2.$$

In order to distinguish the signal R_{11} from R_{12}, we consider the signal of C_1, thus we obtain two conditions. If C_1 is positive, then the equation has two roots, one is positive and the other is not. Thus we obtain condition (i) in Theorem 1. If C_1 is negative, for the same reason, we obtain condition (ii) in Theorem 1. The conditions (iii), (iv) and (v) in Theorem 1 can be obtained from the same deduction, too.

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