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Synchronization transition of limit-cycle system with homogeneous phase shifts*

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The behaviors of coupled oscillators, each of which has periodic motion with random natural frequency in the absence of coupling, are investigated when phase shifts are considered. In the system of coupled oscillators, phase shifts are the same between different oscillators. Synchronization and synchronization transition are revealed with different phase shifts. Phase shifts play an important role for this kind of system. When the phase shift $\alpha < 0.5\pi$, the synchronization state can be attained by increasing the coupling, and the system cannot reach the synchronization state while $\alpha \geq 0.5\pi$. A clear scaling between complete synchronization critical coupling strength K_{pc} and $\alpha - 0.5\pi$ is found.

Keywords: nonlinear dynamics, coupled limit-cycle system, synchronization transition, phase shift
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1. Introduction

The behaviors of systems of coupled oscillators have attracted more interests in recent years.[1] Many systems such as the coupled laser system, Josephson junction arrays, biological and chemical oscillators have exhibited this kind of behavior.[2−9] Winfree supposed that each oscillator was coupled to the collective rhythm generated by the whole population, analogous to a mean-field approximation in physics.[10] Using numerical simulations and analytical approximations, Winfree discovered that such oscillator populations could exhibit the temporal analog of a phase transition. Kuramoto himself began working on collective synchronization in 1975 and improved Winfree's model in a brief note.[11] In this note the Kuramoto model was reported. The Kuramoto model consists of a population of N coupled phase oscillators $\theta_i(t)$ having natural frequencies ω_i distributed with a given probability $g(\omega)$, and whose dynamics are governed by[12]

$$\dot{\theta}_i = \omega_i + \sum_{j=1}^{N} K_{ij} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N. \quad (1)$$

K_{ij}, θ_i, and $\dot{\theta}_i$ are the coupling strength between the i-th oscillator and the j-th oscillator, the i-th instantaneous phase and the corresponding phase velocity respectively. Thus each oscillator tries to run independently at its own frequency, while the coupling tends to synchronize it to all the others. By making a suitable choice of rotating frame, we can transform Eq.(1) into an equivalent system of oscillators whose natural frequencies have zero mean. Many models for the coupling, such as global coupling, nearest coupling, hierarchical coupling, random long-range coupling, or even state-dependent interactions are discussed.[12−14] Kuramoto and his co-workers studied the model whose coupling function was slightly modified to account for an enhancement of the oscillators frequencies due to their interaction.[9,15]

With the mean-field coupling among phase oscillators, each oscillator is coupled by all of others. Thus the model changes as follows[2,16] by taking $K_{ij} = K/N > 0$ in Eq.(1)

$$\dot{\theta}_i = \omega_i + \frac{K}{N} \sum_{j=1}^{N} \sin(\theta_j - \theta_i), \quad i = 1, \ldots, N. \quad (2)$$

K, θ_i, and $\dot{\theta}_i$ are the coupling strength, the i-th instantaneous phase and the corresponding phase velocity respectively. The complex-valued order parameter

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was defined by Kuramoto

$$r\,\mathrm{e}^{\mathrm{i}\psi} = \frac{1}{N}\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\theta_j},$$

where $r(t)$ with $0 \le r(t) \le 1$ measures the coherence of the oscillator population, and $\psi(t)$ is the average phase.

In the limit of infinitely many oscillators, $N \to \infty$, when $K \to 0$, the oscillators rotate at angular frequencies given by their own natural frequencies, and the oscillators are not synchronization. This state is called the incoherent state. Alternatively, in the limit of strong coupling, $K \to \infty$, the oscillators become synchronized to each other, and $r = 1$. This state is called the coherent state.

For intermediate coupling, some oscillators are locked, and some are rotating out of synchrony with the locked oscillators. This state is called partial synchronization. Thus synchronization in the mean-field Kuramoto model is revealed by the nonzero order parameter.

When the coupling is sufficiently weak, the oscillators run incoherently, whereas beyond a certain threshold, collective synchronization emerges spontaneously.

It is well known that there is a critical coupling $K = K_\mathrm{c}$. For $K < K_\mathrm{c}$, the system has the trivial stationary solution $r = 0$, corresponding to an angular distribution of oscillators having equal probability in the circle. The oscillators run incoherently, and hence the trivial solution is called the inherent solution or simply incoherence.

For $K = K_\mathrm{c}$, the system has the trivial solution $r = 0$ corresponding to incoherence, the angular distribution of oscillators $\rho = 1/2\pi$. However, for $K = K_\mathrm{c}$, it also has a second branch of solutions to the partially synchronized phase. This branch bifurcates continuously from $r = 0$ at the value $K = K_\mathrm{c}$ obtained analytically by setting $r = 0$, and $K_\mathrm{c} = 2/[\pi g(0)]$.

2. Model of limit-cycle system with homogeneous phase shifts

Quite a few investigations have been carried out for this modified model of the Kuramoto model.[17,18] Even though many studies on the Kuramoto model and its extended studies have been carried out, it is still instructive to further study the synchronization of the system of coupled oscillators with homogeneous phase shifts. When we consider the effects of the phase shifts, the Kuramoto model becomes as follows:

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}\sin(\theta_j - \theta_i - \alpha), \tag{3}$$

where $i = 1, 2, \ldots, N$, and K, θ_i, and $\dot{\theta}_i$ are the same as previously defined in Eq.(2). $\alpha \in [0, \pi]$ is the phase shift. In particular, we are interested in the characteristic features on the motions of individual oscillators, i.e., the synchronization and synchronization transition which is induced by the phase shifts. We can see that α plays a significant role in affecting the synchronization and dynamics of nonlinear systems.

For the case $\alpha = 0$, the model returns to the Kuramoto model without considering the phase shift effects, and the phase-locking dynamics have been fully investigated. Under this circumstance, the interaction between any two oscillators presents an attractive attribute. With $\alpha = \pi$, the interaction is a kind of repulsive force. For the intermediate state, neither the attractive interaction nor the repulsive attribute are exhibited by the phase shift. For the case $\alpha \neq 0$, it has been reported that phase shifts can induce desynchronization and also lead to chaotic motions of the system.[18] Here we will investigate the synchronization and dynamics of a coupled oscillators system with $\alpha \neq 0$ systematically.

For simplicity, the model we adopted is the following array of sinusoidally coupled phase oscillators with different natural frequencies $\{\omega_i\}$ and with globally coupling

$$\dot{\theta}_i = \omega_i + \frac{K}{N}\sum_{j=1}^{N}[\sin(\theta_j - \theta_i - \alpha) + \sin\alpha],\ \ i = 1, 2, \ldots, N, \tag{4}$$

where K, θ_i, and ω_i, and $\dot{\theta}_i$ are the same as those in Eq.(3) respectively. Without losing generality we scale ω_i such that

$$\sum_{j=1}^{N}\omega_j = 0.$$

To further investigate the synchronization, one may define an average frequency of the i-th oscillator as

$$\bar{\omega}_i = \lim_{T\to\infty}\frac{1}{T}\int_0^T \dot{\theta}_i(t)\mathrm{d}t. \tag{5}$$

Synchronization between different oscillators, in the sense of $\bar{\omega}_i = \bar{\omega}_j$, $i \neq j$, can be observed in the region where strict phase locking of $\bar{\theta}_i = 0$ is broken.

In order to get a general idea about the global behavior of the system, we first define the following two macroscopic quantities $\langle\omega(t)\rangle$ and Ω

$$\langle\omega(t)\rangle = \frac{1}{N}\sum_{j=1}^{N}\dot{\theta}_i(t),$$
$$\Omega = \frac{1}{N}\sum_{j=1}^{N}|\dot{\theta}_i(t) - \langle\omega(t)\rangle|. \quad (6)$$

It is clear that for phase locking $\Omega = 0$ because each θ_i is locked to a fixed value, and $\langle\omega(t)\rangle$ is the same. When Ω is closed to 0, the coherence of the system is good, otherwise incoherence enhances.

According to the mean-field theory, we define the order parameter

$$r = \frac{1}{N}\left|\sum_{j=1}^{N}\mathrm{e}^{\mathrm{i}\theta_j}\right|.$$

Obviously, Ω and r are time-dependent. Then we further define two quantities R and $\bar{\Omega}$

$$R = \lim_{T\to\infty}\int_0^T r(t)\mathrm{d}t,$$
$$\bar{\Omega} = \lim_{T\to\infty}\int_0^T \Omega(t)\mathrm{d}t. \quad (7)$$

Here R and $\bar{\Omega}$ measure the coherence of the oscillator population. When $R = 1$ and $\bar{\Omega} = 0$, the coherence of the system is best.

3. Synchronization and synchronization transition

From the investigation, like the case for $\alpha = 0$, we can find that the synchronization and synchronization transition are observed in the system of the coupled phase oscillators with homogeneous phase shifts. To give a certain small phase shift, with the increasing of coupling, the onset of synchronization and phase-locking can be observed. In Fig.1, we plot $\bar{\Omega}$ vs. K for $N = 100$ with different phase shifts. In all the cases, natural frequencies are randomly chosen from a normal Gaussian distribution. When phase shifts are small, for different phase shifts, with increasing of coupling strength, $\bar{\Omega}$ are almost the same. The $\bar{\Omega}$ are not separated from each other. When coupling strength reaches a certain threshold value, corresponding to the small phase shifts, for example, $\alpha = 0.1\pi$, 0.2π, 0.3π, and 0.4π respectively, the $\bar{\Omega}$ become small with increasing of coupling strength. This indicates that for the system with small phase shifts, synchronization can be attained by increasing the coupling. While for the big $\alpha = 0.5\pi$, 0.6π, 0.7π, 0.8π, 0.9π, and π, $\bar{\Omega}$ becomes large with the increasing of coupling. Synchronization cannot be attained. The system is incoherent even if the coupling is sufficiently large.

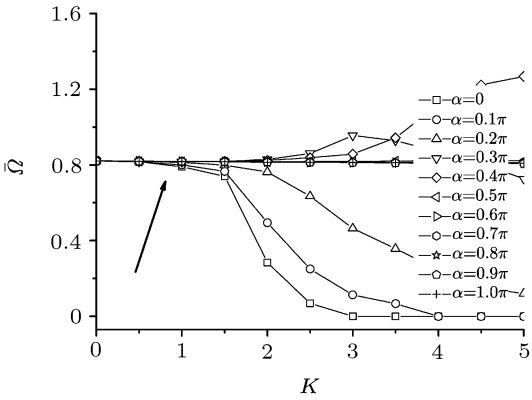


Fig.1. $\bar{\Omega}$ defined in Eq.(7) with different phase shifts, $\bar{\Omega}$ is developed with K.

In Fig.2, we plot R vs. K for $N = 100$ with different phase shifts $\alpha = 0.1\pi, 0.2\pi, 0.3\pi, \ldots, \pi$, respectively. The arrow denotes the onset of the synchronization. For $\alpha = 0.1\pi$, 0.2π, 0.3π, and 0.4π, we can find that R increases when the coupling strength is increased. The system can attain synchronization by increasing the coupling. While for $\alpha = 0.5\pi$, 0.6π, 0.7π, 0.8π, 0.9π, and π, the system exhibits incoherent state.

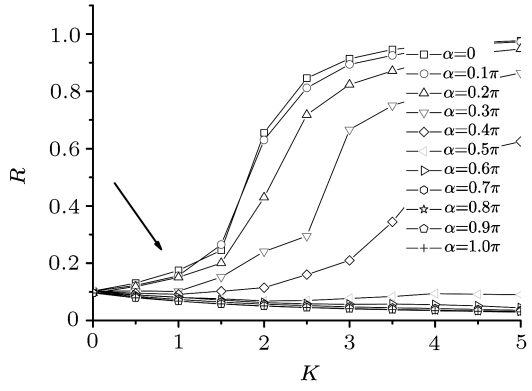


Fig.2. R defined in Eq.(7). With different phase shifts, R developed with K.

From Figs.1 and 2, we can see that phase shifts lead to more incoherence. When phase shifts are small, the system can be attained by increasing the coupling. But when phase shifts are bigger than 0.5π, the synchronization of the system cannot be reached by increasing the coupling.

Figure 3(a) further gives the $\bar{\omega}_i$ vs. K by varying K from $K = 0$ to $K = 8$ when phase shift equals 0.2π,

and the number of oscillators is 100. From this figure, similar synchronization bifurcation tree reported in Ref.[19] can be observed. But when coupling is small, the $\bar{\omega}_i$ lines parallel each other. Only the coupling is larger than a certain threshold value, the paralleling lines loss and all the $\bar{\omega}_i$ are equal. The system reaches phase-locking state when the coupling is sufficiently large.

Figure 3(b) gives $\bar{\omega}_i$ vs. K by varying K from $K = 0$ to $K = 50$ for $\alpha = 0.5\pi$ and the number of oscillators $N = 100$. We can find that whenever the coupling strength increases, the synchronization bifurcation tree cannot be observed. The lines of $\bar{\omega}_i$ are always parallel, and all $\bar{\omega}_i$ increase with the same quantity.

From the further investigation, we can find that the system can reach synchronization when $\alpha < 0.5\pi$ with large enough coupling strength and the system cannot reach synchronization when $\alpha \geq 0.5\pi$ although the coupling is very large. From Fig.3(a), we can see that the synchronization bifurcation tree has parallel lines when coupling strength is small. It manifests that lines of $\bar{\omega}_i$ have the same increasing value with the increasing of coupling. Then going back to Eq.(4), we can get the following equation according to the mean-field theory

$$\dot{\theta}_i = \omega_i + Kr\sin(\psi - \theta_i - \alpha) + K\sin\alpha. \qquad (8)$$

When the coupling strength is small enough, for example, smaller than the onset of synchronization value, the incoherence exists, $r = 0$. Equation (8) changes into the following form

$$\dot{\theta}_i = \omega_i + K\sin\alpha, \quad i = 1, 2, \ldots, N,$$

From this equation, we can understand parallel lines in Fig.3(a) clearly.

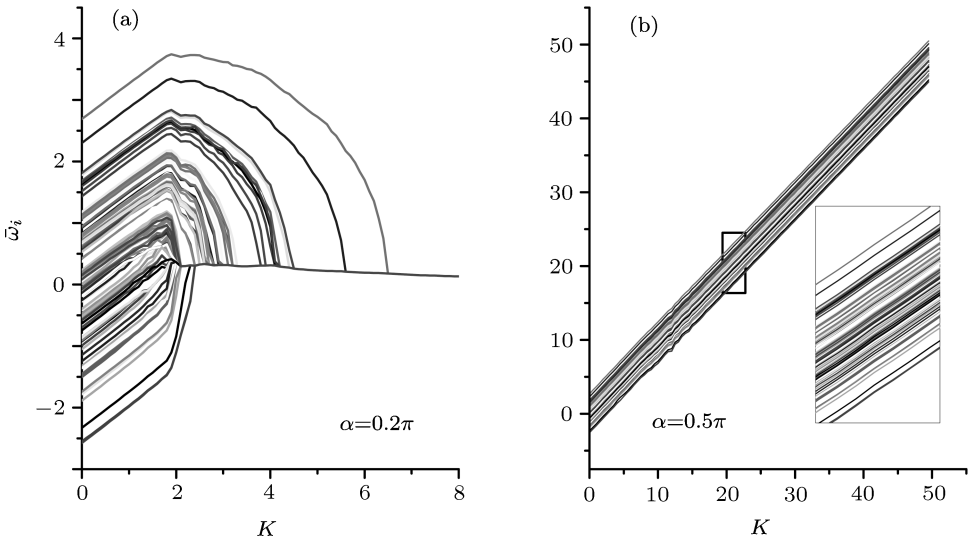


Fig.3. $\bar{\omega}_i$, defined in Eq.(5), change with the coupling strength. (a) for $\alpha = 0.2\pi$, (b) for $\alpha = 0.5\pi$, the inset shows the paralleled lines in details.

It has previously been found that the synchronization is an accessible state for a smaller phase shift with large enough coupling strength, while it is not attainable for larger phase shifts. To clarify the synchronization transition, we select large enough coupling strength which can make the system attain the synchronization state without phase shifts. In Fig.4 we plot R vs. phase shifts at different large enough coupling strengths for $N = 200$. Phase shifts vary from zero to π. For $K = 5.0$, $K = 10$, and $K = 20$, when phase shifts α are small, R is near to 1, with the increasing of α, R reduces step by step. This manifests that the synchronization of the system reduces. When α is equal or larger than 0.5π, R reduces immediately to zero. It means that the synchronization vanishes.

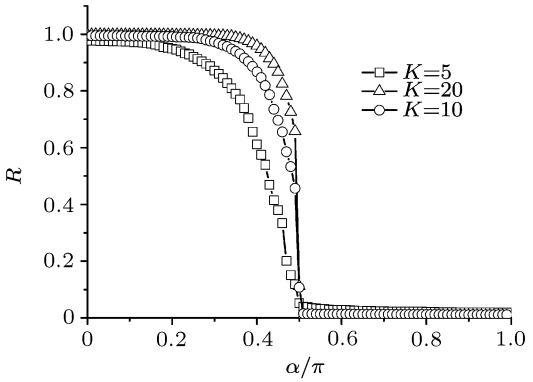


Fig.4. R, defined in Eq.(7) vs. phase shifts, with different coupling strengths.

The figure tells us that the system reaches the synchronization state without phase shifts. With increasing of phase shifts the synchronization state is gradually destroyed. When phase shifts are larger than 0.5π, the synchronization state is lost with different large enough coupling strengths. For the system, $\alpha = 0.5\pi$ is a phase transition point. When $\alpha < 0.5\pi$, the system can attain the synchronization state by increasing the coupling strength while the synchronization state loses when $\alpha \leq 0.5\pi$ even if the coupling strength is big enough.

$\alpha = 0.5\pi$ is a critical point for phase shift qualitatively from previous research. Synchronization transition is further investigated. In this paper, the synchronization transition point of the system for different phase shifts is defined. For different phase shifts, K_{pc} is the point of phase-locking. Phase shift is unchangeable in the coupling process. When $K = K_{pc}$, $\bar{\omega}_i = \bar{\omega}_j$ if $i \neq j$, the K_{pc} can be achieved for different phase shifts. In Fig.5(a), we present the motion of K_{pc} vs. α for $N = 100$. For $\alpha = 0$, K_{pc} is small, and K_{pc} becomes large as phase shift increases. With the increasing of the phase shift, K_{pc} becomes larger and larger. As α gets closer to 0.5π, the K_{pc} becomes larger. When $\alpha = 0.5\pi$, $K_{pc} \to \infty$. In Fig.5(b), we present the relation between K_{pc} and $0.5\pi - \alpha$ for $N = 100$, in which the data have been re-scaled logarithmically. The scaling between K_{pc} and $0.5\pi - \alpha$ is found. The power law distribution can be achieved

$$K_{pc} \propto \beta(\alpha_{pc} - \alpha)^{-\gamma}.$$

Here $\alpha_c = 0.5\pi$, and $\beta = 6.0$, $\gamma = 1.9$.

In conclusion, we have numerically presented an interesting finding for a limit-cycle system with homogeneous phase shifts. The phase shift plays an important role in the system. In the investigation, synchronization and desynchronization have been presented, and synchronization transition can be observed.

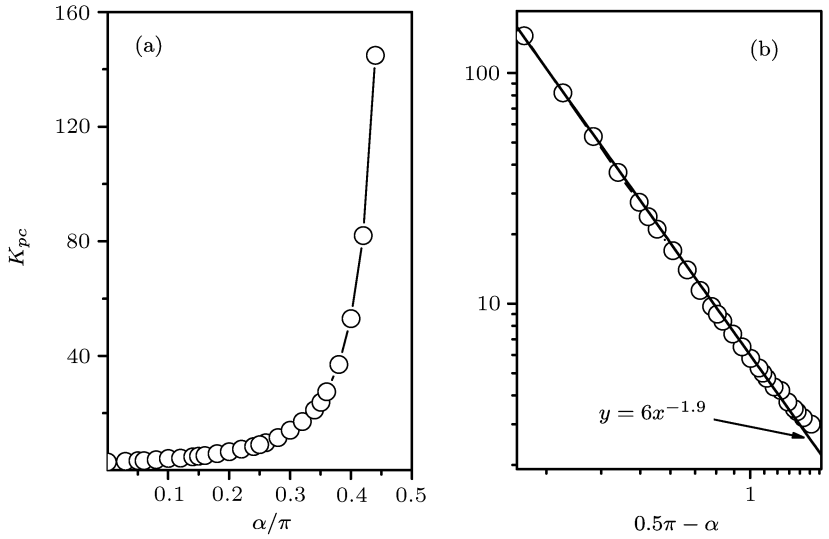


Fig.5. (a) K_{pc}, defined in Eq.(8) vs. phase shifts. (b) K_{pc} vs. phase, the data have re-scaled logarithmically.

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