Vol 18 No 10, October 2009
1674-1056/2009/18(10)/4182–05

Chinese Physics B

© 2009 Chin. Phys. Soc.
and IOP Publishing Ltd

Effects of signal modulation and coloured cross-correlation of coloured noises on the diffusion of a harmonic oscillator*

Liu Li(刘 立)[a], Zhang Liang-Ying(张良英)[b], and Cao Li(曹 力)[a]†

[a] School of Physics, Huazhong University of Science and Technology, Wuhan 430074, China

[b] Department of Physics & Electronics Information Technology, Xiangfan University, Xiangfan 441053, China

(Received 14 April 2009; revised manuscript received 27 April 2009)

The diffusion in a harmonic oscillator driven by coloured noises $\xi(t)$ and $\eta(t)$ with coloured cross-correlation in which one of the noises is modulated by a biased periodic signal is investigated. The exact expression of diffusion coefficient d as a function of noise parameter, signal parameter, and oscillator frequency is derived. The findings in this paper are as follows. 1) The curves of d versus noise intensity D and d versus noises cross-correlation time τ_3 exist as two different phases. The transition between the two phases arises from the change of the cross-correlation coefficient λ of the two Ornstein–Uhlenbeck (O-U) noises. 2) Changing the value of τ_3, the curves of d versus Q, the intensity of colored noise that is modulated by the signal, can transform from a phase having a minimum to a monotonic phase. 3) Changing the value of signal amplitude A, d versus Q curves can transform from a phase having a minimum to a monotonic phase. The above-mentioned results demonstrate that a like noise-induced transition appears in the model.

Keywords: coloured cross-correlation, cross-correlation coefficient, diffusion coefficient, signal modulated noise, noise-induced transition

PACC: 0540

1. Introduction

An increasing interest in noise-induced phenomena in non-equilibrium systems has been aroused in the past decade. Directed transport in Brownian motors (ratchets),[1,2] noise cross-correlated effects,[3−5] signal modulated effects,[6−8] stochastic resonance,[9−13] and noise-induced transition[14−17] are a few examples in this field. One of the objects, which has received special attention in this context, is the noise-driven harmonic oscillator.[18−22] The harmonic oscillator is the simplest toy model for different phenomena in nature and as such is the typical theoreticians' paradigm for various fundamental conceptions. Recently, Gitterman investigated a harmonic oscillator with multiplicative noise and found that stochastic resonance appeared in an under-damped, forced linear oscillator with multiplicative coloured noise.[20] In 2004, a harmonic oscillator with a fluctuating damping parameter was studied by Gitterman.[21] In 2007, Cao *et al* studied the signal modulation effect and noise cross-correlation effect of harmonic oscillator diffusion.[6] However, due to the model studied in Ref.[6] is subjected to an intensive restraint, the correlation times of the two O-U noises τ_1 and τ_2 have the same value, the cross-correlation time τ_3, that is, $\tau_1 = \tau_2 = \tau_3$, consequently all the effects arising from different values of τ_1, τ_2, and τ_3 are hidden. Particularly, the effects of τ_3 on the oscillator diffusion should be very important and quite different from those of τ_1 and τ_2. In the present paper, this restraint is removed.

2. The model

We consider a harmonic oscillator driven by two coloured (O-U) noises with coloured cross-correlation described by the Langevin equation (LE)

$$\ddot{x} = -\tilde{\Omega}^2 x + (i + A\cos\Omega t)\xi(t) + \eta(t), \qquad (1)$$

where i denotes the bias intensity, A and Ω are the amplitude and the frequency of the signal, respectively. A key point of Eq.(1) is that one of the noises, $\xi(t)$, is modulated by a biased signal. The statistical properties of the two noises are assumed to be

$$\langle\xi(t)\rangle = \langle\eta(t)\rangle = 0,$$

*Project supported by the National Natural Science Foundation of China (Grant No 10275025).

†Corresponding author. E-mail: dajinwu@public.wh.hb.cn

$$\langle \xi(t)\xi(t+t')\rangle = \frac{Q}{\tau_1}\exp\left[-\frac{|t'|}{\tau_1}\right],$$

$$\langle \eta(t)\eta(t+t')\rangle = \frac{D}{\tau_2}\exp\left[-\frac{|t'|}{\tau_2}\right],$$

$$\langle \xi(t)\eta(t+t')\rangle = \langle \eta(t)\xi(t+t')\rangle = \frac{\lambda\sqrt{DQ}}{\tau_3}\exp\left[-\frac{|t'|}{\tau_3}\right].$$

Here, Q and D denote the intensities of noises, λ is the noise cross-correlation coefficient, $-1 \le \lambda \le 1$, and τ_1, τ_2 and τ_3 are the correlation times and the cross-correlation time of the noises. The difference between the above model and the model in Ref.[6] lies in the fact that the values of τ_1, τ_2 and τ_3 can now be changed independently but they are no longer restricted in a one-dimensional parameter subspace, in which $\tau_1 = \tau_2 = \tau_3 = \tau$. With this new model, we can study the effects of τ_1, τ_2 or τ_3 on oscillator diffusion separately. Another purpose of the present paper is to study the effect of noise cross-correlation coefficient λ on oscillator diffusion under the new conditions.

3. Exact diffusion coefficient

It is well known that the diffusion coefficient d is defined as

$$d = \lim_{t\to\infty}\frac{\langle x(t)^2\rangle}{t}. \tag{2}$$

To obtain diffusion coefficient d we need the solution of Eq.(1). With the initial condition (x_0, v_0), this solution is given by

$$x(t) = \left\{x_0 - \int_0^t \frac{\sin\tilde{\Omega}t_1}{\tilde{\Omega}}[(i + A\cos\Omega t_1)\xi(t_1) + \eta(t_1)]\mathrm{d}t_1\right\}\cos\tilde{\Omega}t + \left\{\frac{v_0}{\tilde{\Omega}} + \int_0^t \frac{\cos\tilde{\Omega}t_1}{\tilde{\Omega}}[(i + A\cos\Omega t_1)\xi(t_1) + \eta(t_1)]\mathrm{d}t_1\right\}\sin\tilde{\Omega}t. \tag{3}$$

So the variance $\langle x(t)^2\rangle$ reads

$$\langle x(t)^2\rangle = \left\langle\left[\left\{x_0 - \int_0^t \frac{\sin\tilde{\Omega}t_1}{\tilde{\Omega}}[(i + A\cos\Omega t_1)\xi(t_1) + \eta(t_1)]\mathrm{d}t_1\right\}\cos\tilde{\Omega}t + \left\{\frac{v_0}{\tilde{\Omega}} + \int_0^t \frac{\cos\tilde{\Omega}t_1}{\tilde{\Omega}}[(i + A\cos\Omega t_1)\xi(t_1) + \eta(t_1)]\mathrm{d}t_1\right\}\sin\tilde{\Omega}t\right]^2\right\rangle. \tag{4}$$

Diffusion coefficient d can be obtained by calculating variance (4) with the statistical properties of noises. The detail for this calculation is too long and should not be given here. In the following we summarize the types of the terms contained in variance (4). Variance (4) contains 16 terms. Each term is an average value with respect to the noise statistically. The results are divided into four types of terms: the terms which decay exponentially with time; the terms which change periodically with time; the terms which exhibit linear divergence with time; and the terms that are independent of time. Except for the third type of terms, the terms of other types do not contribute to the diffusion coefficient. The exact expression of oscillator diffusion coefficient d for the model is given as

$$d = \frac{1}{2\tilde{\Omega}^2}\left\{\frac{QA^2}{2\tau_1^2}\left[\frac{1}{1/\tau_1^2 + (\tilde{\Omega}+\Omega)^2} + \frac{1}{1/\tau_1^2 + (\tilde{\Omega}-\Omega)^2}\right] + \frac{2D/\tau_2^2}{1/\tau_2^2 + \tilde{\Omega}^2} + \frac{2i^2Q/\tau_1^2}{1/\tau_1^2 + \tilde{\Omega}^2} + \frac{\dfrac{4\lambda i\sqrt{DQ}}{\tau_3^2}}{1/\tau_3^2 + \tilde{\Omega}^2}\right\}. \tag{5}$$

4. Three important special cases

4.1. Case 1. $\tau_3 = 0$

This is the case of white cross-correlation of two O–U noises.[3] Now expression (5) reduces to

$$d = \frac{1}{2\tilde{\Omega}^2}\left\{\frac{QA^2}{2\tau_1^2}\left[\frac{1}{1/\tau_1^2 + (\tilde{\Omega}+\Omega)^2} + \frac{1}{1/\tau_1^2 + (\tilde{\Omega}-\Omega)^2}\right] + \frac{2D/\tau_2^2}{1/\tau_2^2 + \tilde{\Omega}^2} + \frac{2i^2Q/\tau_1^2}{1/\tau_1^2 + \tilde{\Omega}^2} + 4\lambda i\sqrt{DQ}\right\}. \tag{6}$$

4.2. Case 2. $\tau_1 = \tau_2 = \tau_3 = \tau$

When $\tau_1 = \tau_2 = \tau_3 = \tau$, expression (5) recovers the result of our previous work[6]

$$d = \frac{QA^2}{4\tilde{\Omega}^2}\left[\frac{1}{1+\tau^2(\tilde{\Omega}+\Omega)^2} + \frac{1}{1+\tau^2(\tilde{\Omega}-\Omega)^2}\right] + \frac{D + i^2Q + 2i\lambda\sqrt{DQ}}{\tilde{\Omega}^2}\left(\frac{1}{1+\tau^2\tilde{\Omega}^2}\right). \quad (7)$$

4.3. Case 3. $A = 0$

In this case, expression (5) is simplified into

$$d = \frac{1}{2\tilde{\Omega}^2}\left(\frac{2D/\tau_2^2}{1/\tau_2^2+\tilde{\Omega}^2} + \frac{2i^2Q/\tau_1^2}{1/\tau_1^2+\tilde{\Omega}^2} + \frac{\frac{4\lambda i\sqrt{DQ}}{\tau_3^2}}{1/\tau_3^2+\tilde{\Omega}^2}\right). \quad (8)$$

It is obvious that under $A = 0$, the condition for $d = 0$ can be derived. Due to the fact that the signs of the first two terms in expression (8) are positive, the third term of Eq.(8) must be negative, otherwise the value of d cannot be zero. This leads to a conclusion that to make $d = 0$, λ and i must be different in sign. We let $\lambda = -1$ and assume $i > 0$. From expression (8), a sufficient and necessary condition for $d = 0$ is obtained as

$$\sqrt{\frac{Q}{D}} = \frac{1+\tau_1^2\tilde{\Omega}^2}{i(1+\tau_3^2\tilde{\Omega}^2)} \pm \sqrt{\frac{(1+\tau_1^2\tilde{\Omega}^2)^2}{i^2(1+\tau_3^2\tilde{\Omega}^2)^2} - \frac{1+\tau_1^2\tilde{\Omega}^2}{i^2(1+\tau_2^2\tilde{\Omega}^2)}}. \quad (9)$$

5. Discussion and conclusions

5.1. Oscillator diffusion coefficient affected by τ_1, τ_2 and τ_3

In Fig.1 we show the influence of noise correlation time on oscillator diffusion coefficient d. Figures 1(a) and 1(b) show a common feature of $d \sim Q$ curve, a non-monotonic behaviour, in which a minimum occurs. No matter how small the value of τ_1 or τ_2 is, a minimum still exists there. However, as can be seen from Fig.1(c), an increase in noise coloured cross-correlation time τ_3 leads the curve to transform from a non-monotonic phase to a monotonic phase. This is quite different from the influence of correlation time τ_1 or τ_2 on the oscillator diffusion coefficient d.

Figure 2 shows the influence of noise correlation time on $d \sim \Omega$ curve. Figures 2(b) and 2(c) show a common feature of $d \sim \Omega$ curve, a non-monotonic behaviour, in which a maximum occurs. No matter how small the value of τ_2 or τ_3 is, the minimum still exists there. However, as can be seen from Fig.2(a), a reduction of noises correlation time τ_1 leads the curve to transform from a non-monotonic phase to a monotonic phase. This is quite different from the influence of correlation time τ_2 or τ_3 on oscillator diffusion coefficient d. However, a common feature of the $d \sim \Omega$ curves in Fig.2 implicates that the high frequency signal tends to suppress the oscillator diffusion. This provides a method to control the oscillator diffusion.

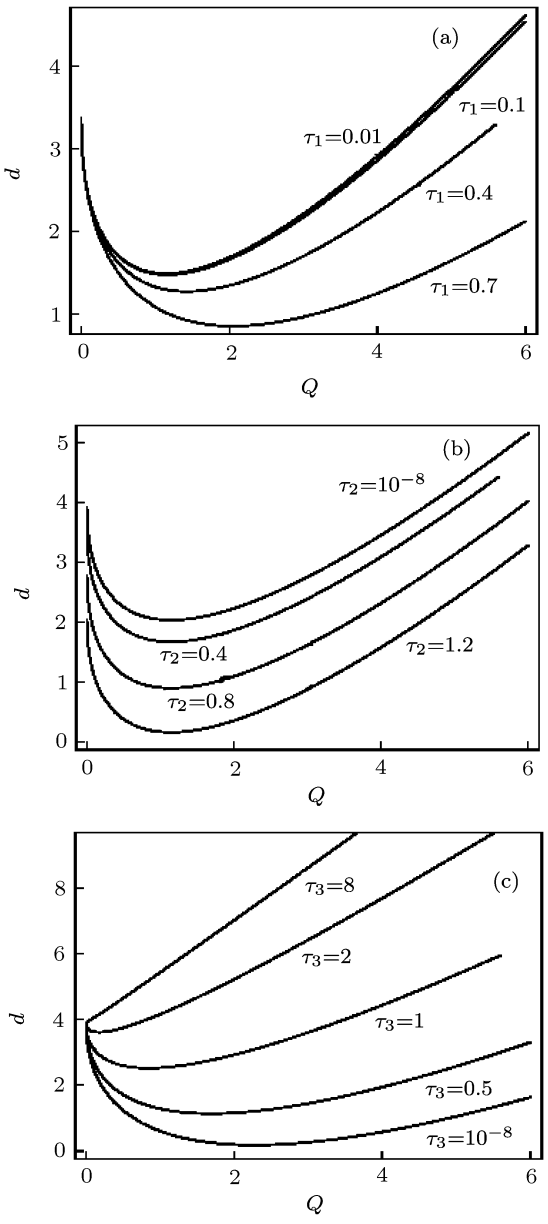


Fig.1. Curves for diffusion coefficient d as a function of noise intensity Q for different values of noise correlation time. The parameters used are $A = 0.31$, $\Omega = 1.8$, $i = 1$, $\tilde{\Omega} = 0.8$, $D = 2.5$, $\lambda = -1$. Panel (a) is for $\tau_2 = 0.5$ and $\tau_3 = 0.8$, panel (b) for $\tau_1 = 0.001$ and $\tau_3 = 0.8$, and panel (c) for $\tau_1 = 0.001$ and $\tau_2 = 0.001$.

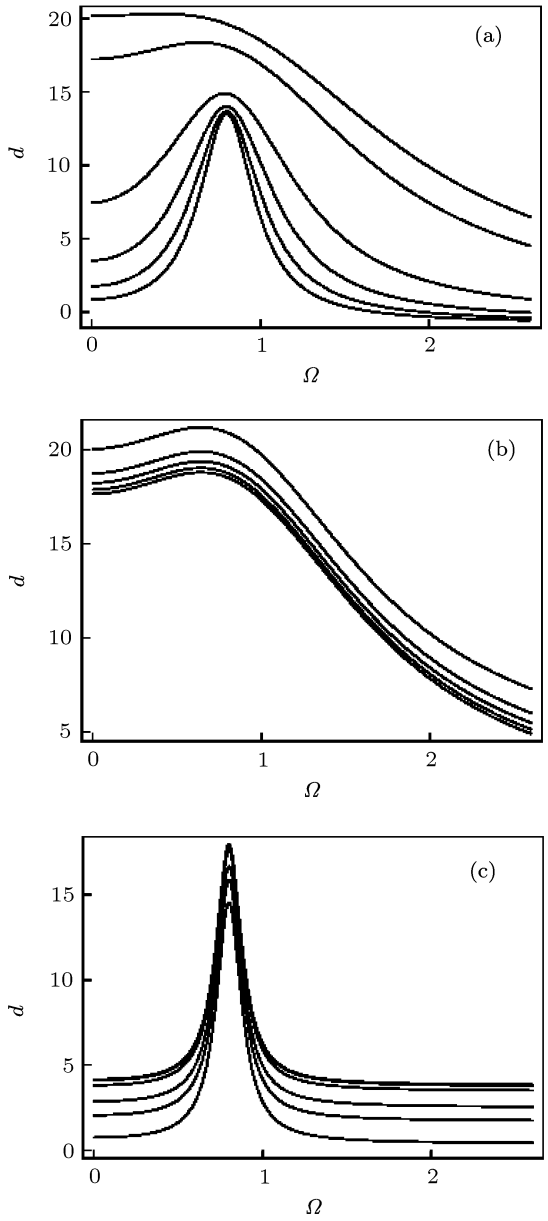


Fig.2. Curves for diffusion coefficient d as a function of signal frequency Ω for different values of noise correlation time. The parameters used are $A = 6$, $Q = 1$, $i = 1$, $\tilde{\Omega} = 0.8$, $D = 2.5$, $\lambda = -1$. In panel (a) $\tau_2 = 1$ and $\tau_3 = 0.9$, and the curves from the bottom to the top correspond to the values of $\tau_1 = 5$, 4, 3, 2, 1, and 0.8, respectively. In panel (b) $\tau_1 = 1$ and $\tau_3 = 3.9$, and the curves from the top to the bottom correspond to the values of $\tau_2 = 1$, 2, 3, 4 and 100, respectively. In panel (c) $\tau_1 = 11$ and $\tau_2 = 0.001$, and the curves from the bottom to the top correspond to the values of $\tau_3 = 0.8$, 1.4, 2, 4, 7, and 9, respectively.

5.2. Effects of cross-correlation coefficient λ on d

The values of diffusion coefficient d as a function of τ_3 for different values of λ give two forms of $d \sim \tau_3$ curves. When $\lambda > 0$, $d \sim \tau_3$ curve monotonically decreases while when $\lambda < 0$, $d \sim \tau_3$ curve monotonically increases as shown in Fig.3. The sign of noise cross-correlation coefficient λ greatly affect the feature of $d \sim \tau_3$ curve.

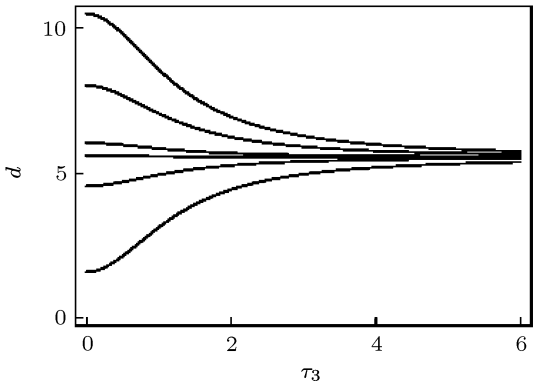


Fig.3. Curves for diffusion coefficient d as a function of cross-correlation time τ_3 for different values of the cross-correlation coefficient λ. The parameters used are $A = 0.31$, $Q = 1$, $i = 1$, $\Omega = 1.8$, $\tilde{\Omega} = 0.8$, $D = 2.5$, $\tau_1 = 0.001$, and $\tau_2 = 0.001$, the curves from the top to the bottom correspond to the values of $\lambda = 1, 0.5$, 0.1, 0.01, –0.2, and –0.8, respectively.

Figure 4 shows the curves for diffusion coefficient d as a function of signal biased intensity i for different values of cross-correlation coefficient λ. The $d \sim i$ curves exhibit two forms. One has a minimum located in $i < 0$ region if $\lambda > 0$, the other has a minimum located in $i > 0$ region if $\lambda < 0$.

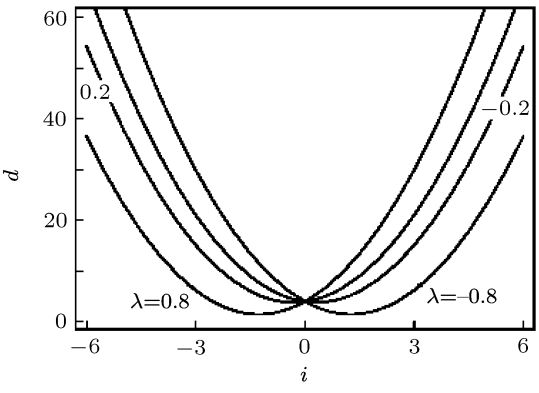


Fig.4. Curves for diffusion coefficient d as a function of signal bias intensity i for different values of the cross-correlation coefficient λ. The parameters used are $A = 0.31$, $\Omega = 1.8$, $Q = 1$, $\tilde{\Omega} = 0.8$, $D = 2.5$, $\tau_1 = 0.001$, $\tau_2 = 0.001$, and $\tau_3 = 10^{-8}$.

Figure 5 shows the influence of noise cross-correlation coefficient λ on $d \sim D$ curves. An increase in the value of λ leads the curve to transform from a non-monotonic phase (a minimum) to a monotonic

phase.

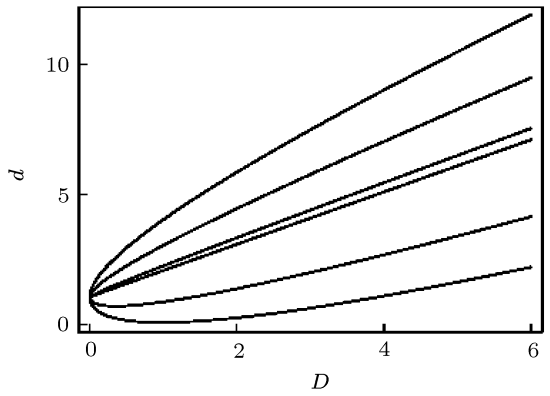


Fig.5. Curves for diffusion coefficient d as a function of noise intensity D for different values of the cross-correlation coefficient λ. The parameters used are $A = 0.31$, $Q = 1$, $i = 1$, $\Omega = 1.8$, $\tilde{\Omega} = 1$, $\tau_1 = 0.001$, $\tau_2 = 0.001$, and $\tau_3 = 0.1$, the curves from the bottom to the top correspond to the values of $\lambda = -1$, –0.6, 0.01, 0.1, 0.5, 1, respectively.

5.3. The control of oscillator diffusion coefficient by changing signal amplitude

According to the signal modulation of noise, we can control the oscillator diffusion coefficient d by changing signal amplitude A. In Fig.6 we plot the $d \sim Q$ curves for different values of the signal amplitude A. In a small signal region, $d \sim Q$ curves have a minimum. In a large signal region, $d \sim Q$ curves become monotonic.

In summary, we find the following points. (i) The $d \sim D$ curves or $d \sim \tau_3$ curves exist as two different phases. The transition between the two phases arise from the change of the cross-correlation coefficient λ of the two O-U noises. (ii) Changing the value of correlation time τ_1 of the colored noise that is modulated by the signal, the $d \sim \Omega$ curves can transform from a phase having a maximum to a monotonic phase. In addition, by changing the value of τ_3, the curves for d versus Q can transform from a phase having a minimum to a monotonic phase. (iii) Changing the value of signal amplitude A, $d \sim Q$ curves can transform from a phase having a minimum to a monotonic phase. The above-mentioned results demonstrate that a like noise-induced-transition appears in the model and the oscillator diffusion can be controlled by the signal modulation of noise and the coloured cross-correlation of noises. Particularly, the high frequency signal tends to suppress the oscillator diffusion.

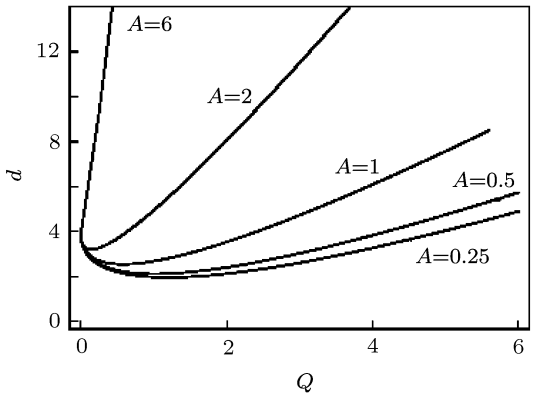


Fig.6. Curves for diffusion coefficient d as a function of noise intensity Q for different values of signal amplitude A. The parameters used are $\Omega = 1.8$, $i = 1$, $\tilde{\Omega} = 0.8$, $\lambda = -1$, $D = 2.5$, and $\tau_1 = 0.1$.

References

[1] Reimann H 2002 *Phys. Rep.* **361** 57
[2] Cao L and Wu D J 2000 *Phys. Rev.* E **62** 7478
[3] Jia Y and Li J R 1997 *Phys. Rev. Lett.* **78** 994
[4] Cao L and Wu D J 2006 *Phys. Rev.* A **73** 023802
[5] Li Y H and Mei D C 2008 *Chin. Phys.* B **17** 76
[6] Cao L and Wu D J 2007 *Int. J. Mod. Phys.* B **21** 1721
[7] Du L C and Mei D C 2009 *Chin. Phys.* B **18** 946
[8] Dykman M I, Luchinsky D G, McClintock P V E, Stein N D and Stocks N G 1992 *Phys. Rev.* A **46** R1713
[9] Chen D Y and Wang Z L 2009 *Acta Phys. Sin.* **58** 1403 (in Chinese)
[10] Liang G Y 2003 *Chin. Phys.* **12** 377
[11] Cao L and Wu D J 2007 *Physica* A **376** 191
[12] Jin G X, Zhang L Y and Cao L 2009 *Chin. Phys.* B **18** 952
[13] Zhang L Y, Jin G X and Cao L 2008 *Acta Phys. Sin.* **57** 4706 (in Chinese)
[14] Horsthemke W and Lefever R 1984 *Noise-Induced Transitions* (Berlin: Springer)
[15] Mei D C, Xie C W and Zhang L 2004 *J. Euro. Phys.* B **41** 107
[16] Zhang L, Cao L and Wu D J 2008 *Phys. Rev.* A **77** 015801
[17] Denisov S I, Vitrenko A N and Horsthemke W 2003 *Phys. Rev.* E **68** 046132
[18] Mankin R, Laas K and Laas T 2008 *Phys. Rev.* E **78** 031120
[19] Fulinski A and Gora P F 2000 *J. Stat. Phys.* **101** 483
[20] Gitterman M 2003 *Phys. Rev.* E **67** 057103
[21] Gitterman M 2004 *Phys. Rev.* E **69** 041101
[22] Denisov S I, Vitrenko A N, Horsthemke W and Hanggi P 2006 *Phys. Rev.* E **73** 036120