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# Critical rotation of an anharmonically trapped Bose–Einstein condensate*

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We consider rotational motion of an interacting atomic Bose-Einstein condensate (BEC) with both two- and three-body interactions in a quadratic-plus-quartic and harmonic-plus-Gaussian trap. By using the variational method, the influence of the three-body interaction and the anharmonicity of the trap on the lowest energy surface mode excitation and the spontaneous shape deformation (responsible for the vortex formation) in a rotating BEC is discussed in detail. It is found that the repulsive three-body interaction helps the formation of the vortex and reduces the lowest energy surface mode frequency and the critical rotational frequency of the system. Moreover, the critical rotational frequency for the vortex formation in the harmonic-plus-Gaussian potential is lower than that in the quadratic-plus-quartic potential.

**Keywords:** Bose–Einstein condensate, vortex, anharmonic potential
**PACC:** 0365, 0530, 0570

## 1. Introduction

Recently, the existence and formation of quantized vortices in Bose-Einstein condensate (BEC) have been studied extensively both experimentally[1−6] and theoretically.[7−13] In particular, the related phenomena of a rotating BEC, such as a lattice of singly quantized vortices, an array of giant vortices with a hole in the center, have been investigated theoretically.[14−19] By applying external rotating to the system, a macroscopic quantum fluid exhibits interesting counter-intuitive phenomena. When the rotational frequency is greater than one-half of the lowest energy surface mode excitation frequency, the cloud would appear in a spontaneous shape deformation. Furthermore, the vortex would appear when the rotational frequency is increased to a critical value. This has been predicted in an anharmonically trapped BEC theoretically[20,21] and detected at the ENS experiment.[8] However, until now, theoretical studies of the vortex formation in BEC are mainly focused on the two-body interactions. We know that in low temperature and density, where the interatomic distance is much greater than the distance scale of atom-atom interactions, two-body *s*-wave scattering should be important and three-body interactions can be neglected. But, if the atom density is high, for example, in the case of BEC in optical lattices, three-body interactions can start to play an important role.[22] As reported in Ref.[23] that even for a small strength of the three-body force, the region of stability for the condensate can be extended considerably.

Motivated by this observation, we examine the vortex formation in a rotating BEC confined in a quadratic-plus-quartic and harmonic-plus-Gaussian potential when both two- and three-body interactions are taken into account. Using the variational method, we find that the three-body interaction and the anharmonicity of the trap have significant effect on the lowest energy surface mode and the spontaneous shape deformation (responsible for vortex formation) in a rotating BEC. It is found that the repulsive three-body interaction depresses the lowest energy surface mode frequency and the critical rotational frequency of the system. It can help the vortex formation of the system. Additionally, we also find that the critical rotational frequency for a vortex formation in the harmonic-plus-Gaussian potential is lower than that in the quadratic-plus-quartic potential.

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## 2. A rotating BEC confined in a quadratic-plus-quartic potential

In this Section, we analyse the quadrupole mode frequency ($\omega_{\rm q}$), the stability of BEC, and the formation of vortex in a quadratic-plus-quartic potential.

### 2.1. Governing equations and variational analyses

The dynamics of two-dimensional BEC with two- and three-body interactions at low temperature are governed by the time-dependent Gross-Pitaevskii equation[20−22]

$$\mathrm{i}\hbar\frac{\partial\psi(\boldsymbol{r})}{\partial t} = \Big[-\frac{\hbar^2}{2m}\nabla^2 + V(\boldsymbol{r}) + g_1|\psi(\boldsymbol{r})|^2 + g_2|\psi(\boldsymbol{r})|^4 - \Omega_0 L_z\Big]\psi(\boldsymbol{r}), \tag{1}$$

where the parameter $V(\boldsymbol{r}) = \frac{1}{2}m[(\omega_x^2x^2 + \omega_y^2y^2 + \omega_z^2z^2) + \lambda\omega_0^2(x^2+y^2)^2]$ is the quadratic-plus-quartic potential with the small anharmonic term $\lambda$. We assume $\omega_x^2 = \omega_0^2(1+\epsilon)$ and $\omega_y^2 = \omega_0^2(1-\epsilon)$, where $\epsilon = (\omega_x^2-\omega_y^2)/(\omega_x^2+\omega_y^2)$ is a very small deformation parameter decided by the rotating deformed potential. Also, $g_1$ and $g_2$ are the parameters of the two- and three-body interactions, $g_1$ is proportional to the two-body scattering length $a$, and is given by $g_1 = 4\pi a\hbar^2/m$. For $^{87}$Rb atoms, $g_1 \doteq 5\hbar\times10^{-11}$ cm$^3$/s and $g_2 \doteq 4\hbar\times10^{-26}$ cm$^6$/s.[22] $L_z = xp_y - yp_x$ is the $z$ component of the angular momentum operator, and $\Omega_0$ is the trap rotating frequency. When $\epsilon = 0$, it preserves the circular symmetry of the system. $a_z = \sqrt{\hbar/m\omega_z}$.

For simplicity, we study the quasi-two-dimensional system. The wave function in the $z$ direction is separable and is given by: $\psi_z = (\sqrt{\pi}a_z)^{-1/2}\mathrm{e}^{-z^2/2a_z^2}$. We can obtain the effective two-dimensional (2D) Gross-Pitaevskii equation by integrating out the $z$ component. One can write down the Lagrangian density corresponding to the quasi-2D system as follows:

$$\pounds = \frac{\mathrm{i}\hbar}{2}\left(\psi\frac{\partial\psi^*}{\partial t} - \psi^*\frac{\partial\psi}{\partial t}\right) + \left(\frac{\hbar^2}{2m}|\nabla\psi|^2 + V(x,y)|\psi|^2 + \frac{\tilde{g}_1}{2}|\psi|^4 + \frac{\tilde{g}_2}{3}|\psi|^6 - \Omega_0\psi^*L_z\psi\right), \tag{2}$$

where $\tilde{g}_1 = g_1/\sqrt{2\pi}a_z$, $\tilde{g}_2 = g_2/\sqrt{3}\pi a_z^2$. We start with the time-dependent variational method[24] to study the properties of a rotating BEC confined in a quadratic-plus-quartic potential. Using a Gaussian ansatz for the condensate wave function:[20,21]

$$\Psi(X,Y,t) = C(t)\mathrm{e}^{-(1/2)[\alpha(t)X^2+\beta(t)Y^2-2\mathrm{i}\gamma(t)XY]}, \tag{3}$$

here, $C(t) = (N\sqrt{D}/\pi a_0^2)^{1/2}$ is the normalization constant and $D = \alpha_1\beta_1 - \gamma_2^2$, N is the total number of atoms. The new coordinates $X$ and $Y$ are the dimensionless variables, $X = x/a_0$, $Y = y/a_0$, where $a_0 = \sqrt{\hbar/m\omega_0}$ is the oscillator length and $2\omega_0^2 = \omega_x^2 + \omega_y^2$ is the mean frequency. Further, $\alpha(t) = \alpha_1(t) + \mathrm{i}\alpha_2(t)$, $\beta(t) = \beta_1(t) + \mathrm{i}\beta_2(t)$ and $\gamma(t) = \gamma_1(t) + \mathrm{i}\gamma_2(t)$ are the time-dependent dimensionless complex variational parameters. The $\alpha_1$ and $\beta_1$ are inverse square of the condensate widths in $x$ and $y$ directions, respectively.

Integrating the Lagrangian density over the space coordinates by substituting Eq.(3) into Eq.(2), we obtain the variational Lagrangian $L$

$$\begin{aligned}\frac{L}{N\hbar\omega_0} = \frac{1}{4D}\Big\{&-(\beta_1\dot{\alpha_2} + \alpha_1\dot{\beta_2} + 2\gamma_2\dot{\gamma_1}) \\ &+ (\alpha_1+\beta_1)D + (\alpha_2^2+\gamma_1^2)\beta_1 \\ &+ (\beta_2^2+\gamma_1^2)\alpha_1 + 2(\alpha_2+\beta_2)\gamma_1\gamma_2 \\ &+ (\alpha_1+\beta_1) + \epsilon(\beta_1-\alpha_1) \\ &+ \frac{\lambda(3\alpha_1^2+2\alpha_1\beta_1+3\beta_1^2+4\gamma_2^2)}{2D} + PD^{3/2} \\ &+ QD^2 + 2\Omega[\gamma_2(\alpha_2-\beta_2) + \gamma_1(\alpha_1-\beta_1)]\Big\},\end{aligned} \tag{4}$$

where $P = 2\sqrt{2/\pi}(N-1)a/a_z$ and $Q = 4\tilde{g}_2N^2/9\hbar\omega_0\pi^2a_0^4$ are the dimensionless parameters that indicate the strength of the two- and three-body interactions and it can be positive or negative depending on the sign of the $s$-wave scattering length $a$.

The variational energy of the rotating condensate at equilibrium is given as

$$\begin{aligned}\frac{E}{N\hbar\omega_0} = \frac{1}{4}\Big[&(\alpha_1+\beta_1) + \gamma_1^2\left(\frac{1}{\alpha_1}+\frac{1}{\beta_1}\right) \\ &+ \left(\frac{1}{\alpha_1}+\frac{1}{\beta_1}\right) + \epsilon\left(\frac{1}{\alpha_1}-\frac{1}{\beta_1}\right) \\ &+ \frac{\lambda}{2}\left(\frac{3}{\alpha_1^2}+\frac{2}{\alpha_1\beta_1}+\frac{3}{\beta_1^2}\right) + P\sqrt{\alpha_1\beta_1} \\ &+ Q\alpha_1\beta_1 + 2\Omega\gamma_1\left(\frac{1}{\beta_1}-\frac{1}{\alpha_1}\right)\Big],\end{aligned} \tag{5}$$

here, $\Omega = \Omega_0/\omega_0$. We have used the fact that $\alpha_2 = \beta_2 = \gamma_2 = 0$ to obtain the above equation.

By minimizing the energy with respect to the variational parameters, we can get the equilibrium value of the variational parameters, $\alpha_{10}$, $\beta_{10}$, and $\gamma_{10}$,

$$\alpha_{10}^2\left(1+\frac{P}{2}\sqrt{\frac{\beta_{10}}{\alpha_{10}}}+Q\beta_{10}\right)=\gamma_{10}^2-2\Omega\gamma_{10}+1+\epsilon+\lambda\left(\frac{3}{\alpha_{10}}+\frac{1}{\beta_{10}}\right), \tag{6}$$

$$\beta_{10}^2\left(1+\frac{P}{2}\sqrt{\frac{\alpha_{10}}{\beta_{10}}}+Q\alpha_{10}\right)=\gamma_{10}^2+2\Omega\gamma_{10}+1-\epsilon+\lambda\left(\frac{1}{\alpha_{10}}+\frac{3}{\beta_{10}}\right), \tag{7}$$

and

$$\gamma_{10}=\Omega\frac{(\beta_{10}-\alpha_{10})}{(\beta_{10}+\alpha_{10})}, \tag{8}$$

Equations (6)–(8) describe how the shape of the condensate changes due to the rotation. The relation between the phase parameter and the variational widths without three-body interaction is obtained in Refs.[25] and [26]. A new parameter $\eta$ can be defined as

$$\eta=\frac{\langle x^2\rangle}{\langle y^2\rangle}=\frac{\beta_{10}}{\alpha_{10}}=\frac{(\Omega+\gamma_{10})}{(\Omega-\gamma_{10})}, \tag{9}$$

where $\eta$ is the ratio of the widths along the $x$ and $y$ directions which describes the symmetry breaking of a rotational BEC. Equations (9) implies that $\Omega>\gamma_{10}$ since $\eta$ is always positive. When $\eta$ is always one, there is no shape deformation and hence a vortex cannot appear.

The average angular momentum per particle is given by

$$\frac{\langle L_z\rangle}{N\hbar}=\frac{1}{2}\gamma_{10}\left(\frac{1}{\beta_{10}}-\frac{1}{\alpha_{10}}\right)=\frac{\Omega(1-\eta)^2}{2\eta(\alpha_{10}+\beta_{10})}. \tag{10}$$

This relation explicitly shows how the angular momentum is transferred to the trapped BEC due to the spontaneous shape deformation with the rotation frequency. It has been measured experimentally[27,28] that when average angular momentum per particle is 1 ($\langle L_z\rangle=N\hbar$), the system starts deforming, it starts transferring angular momentum and the vortex starts to nucleate. Therefore, one can estimate the critical angular frequency for a single nucleated vortex from Eq.(10), $\Omega_{\rm c}=2\eta(\alpha_{10}+\beta_{10})/(1-\eta)^2$.

## 2.2. Lowest energy surface mode excitation frequency

In order to calculate the quadrupole mode frequency of BEC confined in a quadratic-plus-quartic potential, we expand the Lagrangian in the following way: $\alpha_1=\alpha_{10}+\delta\alpha_1$, $\beta_1=\beta_{10}+\delta\beta_1$, $\alpha_2=\delta\alpha_2$, $\beta_2=\delta\beta_2$, $\gamma_1=\delta\gamma_1$, and $\gamma_2=\delta\gamma_2$. We keep only the second order deviations from their equilibrium values, then we get the following equation

$$\begin{aligned}L=\frac{1}{4}\Bigg\{&\frac{\delta\beta_1\delta\dot{\beta}_2}{\beta_{10}^2}+\frac{\delta\alpha_1\delta\dot{\alpha}_2}{\alpha_{10}^2}-2\frac{\delta\gamma_2\delta\dot{\gamma}_1}{\alpha_{10}\beta_{10}}-\frac{\delta\dot{\alpha}_2}{\alpha_{10}}-\frac{\delta\dot{\beta}_2}{\beta_{10}}+2\alpha_{10}+2\beta_{10}+\frac{\delta\alpha_2^2}{\alpha_{10}}+\frac{\delta\beta_2^2}{\beta_{10}}\\
&+\frac{(\alpha_{10}+\beta_{10})}{\alpha_{10}\beta_{10}}\left(\frac{\delta\alpha_1^2}{\alpha_{10}^2}+\frac{\delta\beta_1^2}{\beta_{10}^2}+\frac{\delta\alpha_1\delta\beta_1}{\alpha_{10}\beta_{10}}+\frac{\delta\gamma_2^2}{\alpha_{10}\beta_{10}}\right)-\frac{(\delta\alpha_1+\delta\beta_1)}{\alpha_{10}\beta_{10}}\left(\frac{\delta\alpha_1}{\alpha_{10}}+\frac{\delta\beta_1}{\beta_{10}}\right)\\
&+\epsilon\frac{(\beta_{10}-\alpha_{10})}{\alpha_{10}\beta_{10}}\left(\frac{\delta\alpha_1^2}{\alpha_{10}^2}+\frac{\delta\beta_1^2}{\beta_{10}^2}+\frac{\delta\alpha_1\delta\beta_1}{\alpha_{10}\beta_{10}}+\frac{\delta\gamma_2^2}{\alpha_{10}\beta_{10}}\right)-\frac{\epsilon(\delta\beta_1-\delta\alpha_1)}{\alpha_{10}\beta_{10}}\left(\frac{\delta\alpha_1}{\alpha_{10}}+\frac{\delta\beta_1}{\beta_{10}}\right)\\
&+\frac{\lambda}{2(\beta_{10}\alpha_{10})^2}\left[2\delta\alpha_1\delta\beta_1+\left(9\frac{\beta_{10}^2}{\alpha_{10}^2}+2\frac{\beta_{10}}{\alpha_{10}}\right)\delta\alpha_1^2+\left(9\frac{\alpha_{10}^2}{\beta_{10}^2}+2\frac{\alpha_{10}}{\beta_{10}}\right)\delta\beta_1^2+2\left(4+3\frac{\alpha_{10}}{\beta_{10}}\right.\right.\\
&\left.\left.+3\frac{\beta_{10}}{\alpha_{10}}\right)\delta\gamma_2^2\right]+P\sqrt{\alpha_{10}\beta_{10}}\left(2-\frac{\delta\gamma_2^2}{2\alpha_{10}\beta_{10}}-\frac{1}{8}\left(\frac{\delta\alpha_1}{\alpha_{10}}-\frac{\delta\beta_1}{\beta_{10}}\right)^2\right)+Q(3\alpha_{10}\beta_{10}+\delta\alpha_1\delta\beta_1\\
&-\delta\gamma_2^2)+\frac{2\Omega}{\alpha_{10}\beta_{10}}(\delta\alpha_2-\delta\beta_2)\delta\gamma_2\Bigg\}.\end{aligned} \tag{11}$$

Using the Euler–Lagrangian equation of motion, we get the following two coupled equations of $\delta\alpha_1$ and $\delta\beta_1$

$$\delta\ddot{\alpha}_1=\left[\frac{P}{2}\alpha_{10}^2\sqrt{\frac{\beta_{10}}{\alpha_{10}}}-4(1+\epsilon)-2\lambda\left(\frac{9}{\alpha_{10}}+\frac{2}{\beta_{10}}\right)\right]\delta\alpha_1-\left[\frac{P}{2}\alpha_{10}^2\sqrt{\frac{\alpha_{10}}{\beta_{10}}}+2\lambda\frac{\alpha_{10}}{\beta_{10}^2}+2Q\alpha_{10}^3\right]\delta\beta_1, \tag{12}$$

$$\delta\ddot{\beta}_1=\left[\frac{P}{2}\beta_{10}^2\sqrt{\frac{\alpha_{10}}{\beta_{10}}}-4(1-\epsilon)-2\lambda\left(\frac{2}{\alpha_{10}}+\frac{9}{\beta_{10}}\right)\right]\delta\beta_1-\left[\frac{P}{2}\beta_{10}^2\sqrt{\frac{\beta_{10}}{\alpha_{10}}}+2\lambda\frac{\beta_{10}}{\alpha_{10}^2}+2Q\beta_{10}^3\right]\delta\alpha_1. \tag{13}$$

Here, we set $\Omega = 0$. For an isotropic trap, $\epsilon = 0$, and we also set $\delta\alpha_1 = -\delta\beta_1$ to calculate the lowest energy quadrupole mode frequency which is given by

$$\omega_{\rm q}^2 = 4 + \frac{20\lambda}{R_0} - PR_0^2 - 2QR_0^3, \tag{14}$$

where $R_0$ is the equilibrium radius of the system, and it can be obtained from the real solution of the cubic equation: $\left(1 + \frac{P}{2} + QR_0\right) R_0^3 - R_0 - 4\lambda = 0$. The above quadrupole frequency $\omega_{\rm q}$ is valid for all interaction (repulsive, attractive) strengths with a small anharmonic term ($\lambda$). For $Q = 0$, the stability of the system is discussed in Ref.[20]. Here, for simplicity, we discuss the stability of the repulsive two- and three-body BEC. For $P = 1$, $Q > 0$, when $\lambda \geq 0$, a stable BEC can exist; when $\lambda < 0$, there would be unstable in BEC. Figure 1 shows the quadrupole mode frequency $\omega_{\rm q}$ for negative values of $\lambda$ with $P = 1.0 > 0$. We can find that with the decreasing of $\lambda$ the BEC can diffuse. For fixed $P$ and $\lambda$, it is found that there is a critical value of $Q$ ($Q_{\rm c}$), when $Q > Q_{\rm c}$, BEC will diffuse. Such as, for $P = 1.0$, $\lambda = -0.04$, when $Q > Q_{\rm c} = 18.6$, the system is diffuse.

In Fig.2, we plot the quadrupole mode frequency $\omega_{\rm q}$ vs. $P$ and $Q$. It is clear that $\omega_{\rm q}$ depends on the nonlinear strengths $P$, $Q$ and parameter $\lambda$. Figures 2(a) and 2(b) show that $\omega_{\rm q}$ decreases as $P$, $Q$ increase and after a critical value of $P_{\rm c}$, $Q_{\rm c}$, $\omega_{\rm q}$ starts increasing slowly. In particular, for the different values of anharmonic term $\lambda$, the critical values of interaction parameters are different. For example, for $\lambda = 0.005$, $P_{\rm c} = 33.35$, $Q_{\rm c} = 66.55$, for $\lambda = 0.05$, $P_{\rm c} = 6.7$, $Q_{\rm c} = 6.4$. When $P < P_{\rm c}$, $Q < Q_{\rm c}$, $\omega_{\rm q}$ decreases as $P$, $Q$ increase and when $P > P_{\rm c}$, $Q > Q_{\rm c}$, $\omega_{\rm q}$ starts increasing. Figure 2(c) shows that the influence of the three-body interaction is very small when two-body interaction becomes much bigger which can play an important role in the BEC.

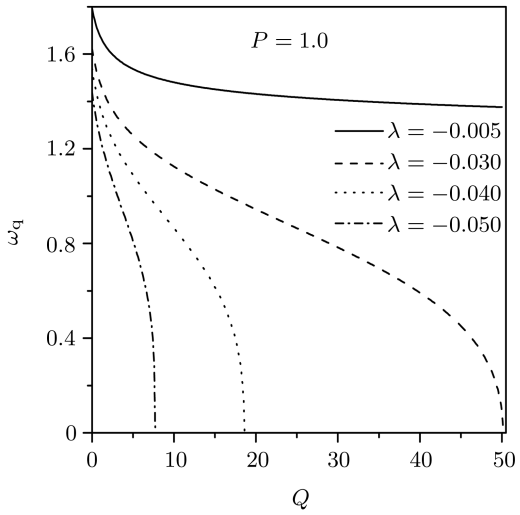



**Fig.1.** Quadrupole mode frequency $\omega_{\rm q}$ vs. interaction parameter $Q$ for $\lambda = -0.005, -0.03, -0.04, -0.05$ and $P = 1$.

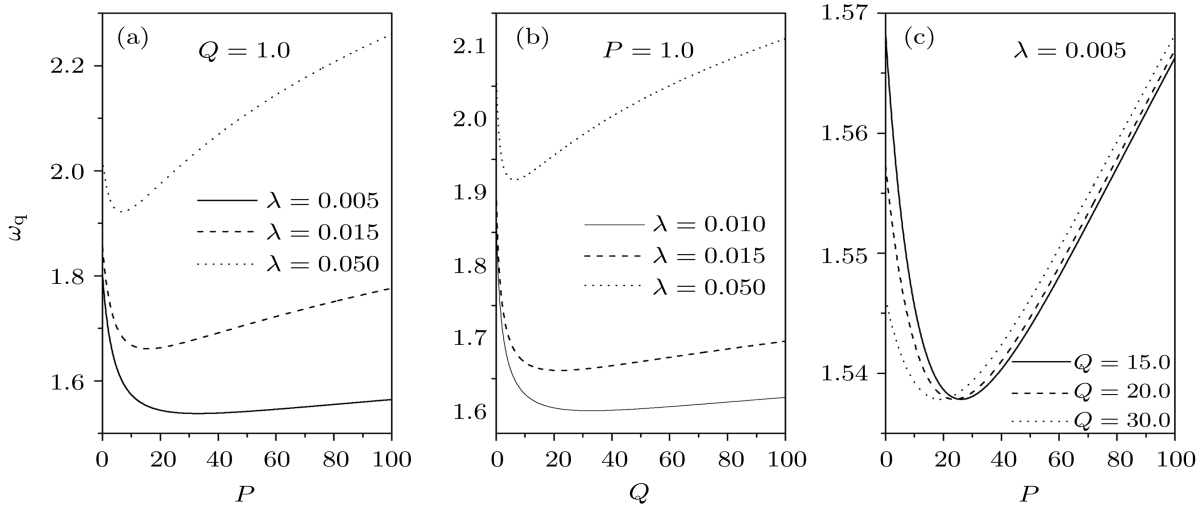



**Fig.2.** Plots of quadrupole mode frequency $\omega_{\rm q}$ vs. $P$ and $Q$ for different cases.

For $\Omega \neq 0$, the centrifugal term $-\Omega L_z$ shifts the quadrupole mode frequency by $\pm 2\Omega$.[20] The lowest energy surface mode excitation frequency is given by

$$\omega_{\rm q} = \sqrt{4 + \frac{20\lambda}{R_0} - PR_0^2 - 2QR_0^3} - 2\Omega. \tag{15}$$

### 2.3. Vortex formation in a quadratic-plus-quartic potential

In this Subsection, the effect of the quartic potential and three-body interaction on the possibility of a vortex formation in a harmonically trapped BEC will be discussed. Using three coupled Eqs.(6)–(8), we get the following two coupled polynomial equations

$$0 = (\alpha_{10}^3\beta_{10}^3 + 2\alpha_{10}^4\beta_{10}^2 + \alpha_{10}^5\beta_{10})\left(1 + \frac{P}{2}\sqrt{\frac{\beta_{10}}{\alpha_{10}}}\right) + Q(\alpha_{10}^5\beta_{10}^2 + 2\alpha_{10}^4\beta_{10}^3 + \alpha_{10}^3\beta_{10}^4)$$
$$-\alpha_{10}^3\beta_{10}(3\Omega^2 + 1 + \epsilon) + (\alpha_{10}\beta_{10}^3 + 2\alpha_{10}^2\beta_{10}^2)(\Omega^2 - 1 + \epsilon) - \lambda(5\alpha_{10}^2\beta_{10} + 7\alpha_{10}\beta_{10}^2 + \alpha_{10}^3 + 3\beta_{10}^3), \quad (16)$$
$$0 = (\alpha_{10}^3\beta_{10}^3 + 2\alpha_{10}^2\beta_{10}^4 + \alpha_{10}\beta_{10}^5)\left(1 + \frac{P}{2}\sqrt{\frac{\alpha_{10}}{\beta_{10}}}\right) + Q(\alpha_{10}^4\beta_{10}^3 + 2\alpha_{10}^3\beta_{10}^4 + \alpha_{10}^2\beta_{10}^5)$$
$$-\beta_{10}^3\alpha_{10}(3\Omega^2 + 1 - \epsilon) + (\beta_{10}\alpha_{10}^3 + 2\alpha_{10}^2\beta_{10}^2)(\Omega^2 - 1 + \epsilon) - \lambda(5\beta_{10}^2\alpha_{10} + 7\beta_{10}\alpha_{10}^2 + \beta_{10}^3 + 3\alpha_{10}^3). \quad (17)$$

For simplicity, we let $\epsilon = 0$, and solve those two coupled equations numerically. We find that there is a circular symmetry breaking of BEC confined in a quadratic-plus-quartic potential when the rotating frequency is greater than one-half of the quadrupole mode frequency $\omega_q$. However, when $\Omega$ is smaller than $\omega_q/2$, the circular symmetry breaking in BEC does not exist, because $\eta$ is always one, there is no spontaneous shape deformation in the system, the vortex cannot appear. Assume, $P = 5.0$, $Q = 1.0$ and $\lambda = 0.02$ and then we get $(\omega_q/2) = 0.87$, if we make $0 < \Omega = 0.75 < \omega_q/2$, $\eta$ has the only solution of $\eta = 1$, but, when we make $\Omega = 0.88 > \omega_q/2$, we will get three solutions of $\eta$ from Eqs.(16) and (17). One solution is $\eta = 1$ and the other two solutions are $\eta < 1$ and $\eta > 1$. It is clear that the spontaneous shape deformation exists in the system which is a circular symmetry breaking of BEC. In Fig.3, we plot $\eta$ vs. $\Omega$ for $\epsilon = 0$ and $\lambda = 0.011$. In this case, $\omega_q/2 = 0.89$ as well as $\eta$ is always one for $\Omega < 0.89$. When $\Omega > 0.89$, the system starts deforming itself, it starts transferring angular momentum to the system, and, when $\Omega = 0.99$, each particle gets a unit angular momentum, i.e., $L_z = N\hbar$, a single nucleated vortex will appear in the BEC. Figure 4 shows the critical rotation frequency ($\Omega_c$) of a single nucleated vortex v. anharmonic parameter $\lambda$ for $P = 1.0$ with different values of $Q$. We can find that $\Omega_c$ decreases with the increasing of $Q$ and increases with the increasing of $\lambda$. It implies that three-body interaction can help the vortex formation in a rotating BEC, even for small value of $Q$.

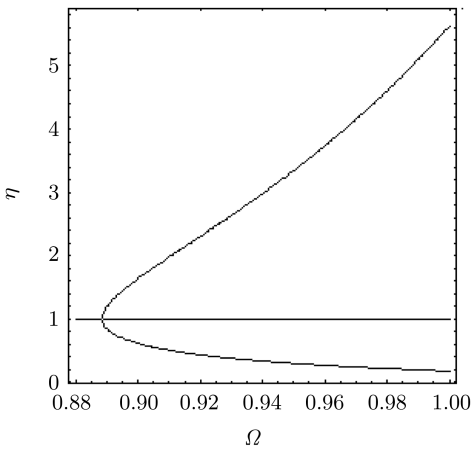



**Fig.3.** The variation of $\eta$ vs. rotational frequency $\Omega$ for $P = 1.0$, $Q = 1.0$, $\lambda = 0.011$, and $\epsilon = 0$.

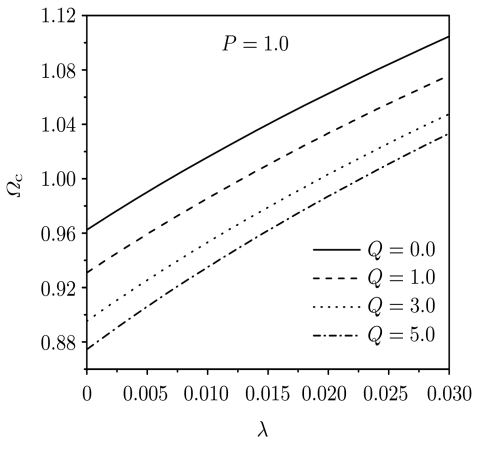



**Fig.4.** The critical rotation frequency $\Omega_c$ vs. anharmonic parameter $\lambda$ for $P = 1.0$, $\epsilon = 0$ with different values of $Q$ in a rotating BEC confined in quadratic-plus-quartic potential.

## 3. A rotating BEC confined in a harmonic-plus-Gaussian potential

It is distinguished from the case of rotating BEC confined in quadratic-plus-quartic potential, here, we use a harmonic-plus-Gaussian laser trap. We can show that by increasing the amplitude of the Gaussian laser trap,

a vortex appears in a slowly rotating BEC. The effective external potential is

$$V_{\rm trap} = \frac{1}{2}m\omega_0^2\left[(x^2+y^2)+\epsilon(x^2-y^2)+\frac{\omega_z^2}{\omega_0^2}z^2\right]+U(r), \tag{18}$$

where $U(r)=U_0{\rm e}^{-2r^2/w^2}$. At the ENS experimental set up,[29] the laser waist is $w=25$ $\mu$m and the amplitude of the laser beam is $U_0\sim 1.242\times10^{-30}$ J. Here, we set the harmonic trap frequency $\omega_0=120$ Hz and $\omega_z=1200$ Hz. In particular, the size of a condensate is the order of a few $\mu$m. Since $\sqrt{2}r/w<1$, we can expand the Gaussian laser potential,

$$U(r)=U_0{\rm e}^{-2r^2/w^2}\sim U_0\left(1-\frac{2r^2}{w^2}+\frac{2r^4}{w^4}\right).$$

The resulting potential can be rewritten as

$$V_{\rm trap}=\frac{1}{2}\hbar\omega_0\left[(\tilde{x}^2+\tilde{y}^2)(1-k)+\epsilon(\tilde{x}^2-\tilde{y}^2)+\frac{\omega_z^2}{\omega_0^2}\tilde{z}^2+\lambda(\tilde{x}^2+\tilde{y}^2)^2\right], \tag{19}$$

here, $k=4U_0/m\omega_0^2w^2$ and $\lambda=k(a_0/\omega)^2$. For different $k$, the effective potential is different. When $k<1$, $k=1$ and $k>1$, the effective potential is quadratic-plus-quartic potential, only quartic potential and quartic-minus-quadratic potential, respectively. If $k>1$, the structure of the potential becomes a Mexican hat. In principle, $k\geq1$ can be easily obtained in the current experimental set up at ENS by increasing $U_0$. Where the co-ordinates $\tilde{x}$, $\tilde{y}$ and $\tilde{z}$ are in units of the harmonic oscillator length $a_0=\sqrt{\hbar/m\omega_0}=2.489$ $\mu$m. Just as the access to Eq.(12), we can get the two equations of $\delta\alpha_1$ and $\delta\beta_1$:

$$\delta\ddot{\alpha_1}=\left[\frac{P}{2}\alpha_{10}^2\sqrt{\frac{\beta_{10}}{\alpha_{10}}}-4(1-k+\epsilon)-2\lambda\left(\frac{9}{\alpha_{10}}+\frac{2}{\beta_{10}}\right)\right]\delta\alpha_1-\left[\frac{P}{2}\alpha_{10}^2\sqrt{\frac{\alpha_{10}}{\beta_{10}}}+2\lambda\frac{\alpha_{10}}{\beta_{10}^2}+2Q\alpha_{10}^3\right]\delta\beta_1, \tag{20}$$

$$\delta\ddot{\beta_1}=\left[\frac{P}{2}\beta_{10}^2\sqrt{\frac{\alpha_{10}}{\beta_{10}}}-4(1-k-\epsilon)-2\lambda\left(\frac{2}{\alpha_{10}}+\frac{9}{\beta_{10}}\right)\right]\delta\beta_1-\left[\frac{P}{2}\beta_{10}^2\sqrt{\frac{\beta_{10}}{\alpha_{10}}}+2\lambda\frac{\beta_{10}}{\alpha_{10}^2}+2Q\beta_{10}^3\right]\delta\alpha_1. \tag{21}$$

The lowest energy quadrupole mode excitation frequency is given by

$$\omega_{\rm q}^2=4(1-k)+\frac{20\lambda}{R_0}-PR_0^2-2QR_0^3, \tag{22}$$

where $R_0$ can also be obtained from the real solution of the cubic equation: $[1+P/2+QR_0]R_0^3-(1-k)R_0-4\lambda=0$. For $Q=0$, the relation between $\omega_{\rm q}$ and the interaction parameter $P$ as well as $k$ is discussed in Ref.[21]. In Fig.5, we plot the quadrupole mode frequency $\omega_{\rm q}$ vs. $P$, $Q$ and $\lambda$. Figures 5(a) and 5(b) show that when $\lambda<0.01$ ($k<1$), $\omega_{\rm q}$ first decreases as $P$, $Q$ increase, but after a critical value of $P_{\rm c}$, $Q_{\rm c}$ (for $\lambda=0.005$, $Q_{\rm c}=32.4$; for $\lambda=0.009$, $P_{\rm c}=4.2$), $\omega_{\rm q}$ starts increasing slowly. When $\lambda\geq0.01$ ($k\geq1$), $\omega_{\rm q}$ only has the trend of increasing as $P$, $Q$ increase. Figure 5(c) shows that $\omega_{\rm q}$ decreases as $\lambda$ increases and after a minimum value of $\omega_{\rm q}$, $\omega_{\rm q}$ starts increasing. It is clear that when $\lambda$ is smaller, the three-body interaction decreases $\omega_{\rm q}$, $\omega_{\rm q}$ of the system with both two- and three-body interactions is smaller than that of the system with only two-body interaction. But, when $\lambda$ becomes much larger, the influence of the three-body interaction is smaller compared to the influence of $\lambda$, $\omega_{\rm q}$ starts increasing after a minimum value. In Fig.5(d), we show that the effect of the three-body interaction on $\omega_{\rm q}$ is reduced when the two-body interaction is sufficiently large.

For $\Omega\neq0$, the centrifugal term $-\Omega L_z$ shifts the quadrupole mode frequency by $-2\Omega$.[21] The quadrupole mode frequency with $m_z=-2$ is

$$\omega_{\rm q}=\sqrt{4(1-k)+\frac{20\lambda}{R_0}-PR_0^2-2QR_0^3}-2\Omega. \tag{23}$$

The effect of the harmonic-plus-Gaussian laser trap potential on the formation of vortex in rotating BEC will be discussed. The method which is used in analysing the rotating BEC confined in a quadratic-plus-quartic potential can also be used here. We can

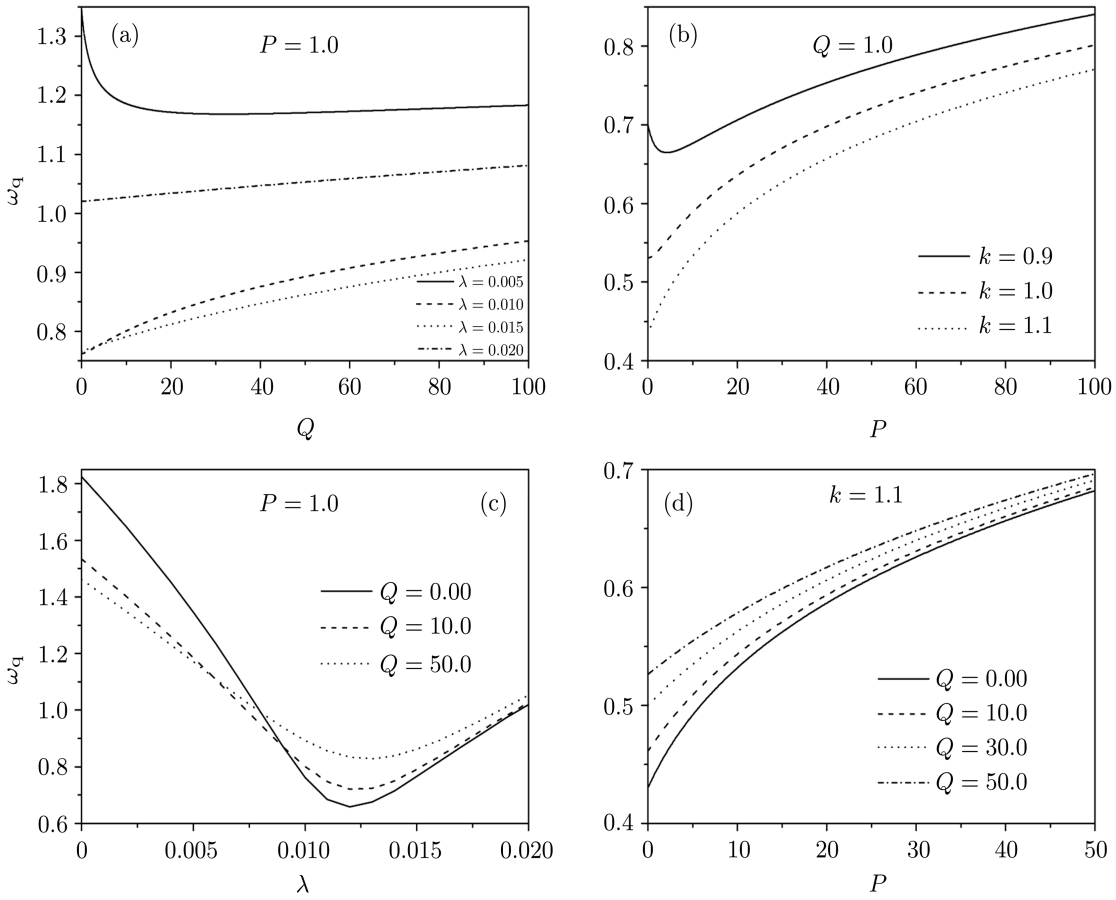



**Fig.5.** Plots of quadrupole mode frequency $\omega_q$ vs. $P, Q$ and $\lambda$ for different cases.

get the equilibrium equations of the system confined in harmonic-plus-Gaussian potential. And then, the following two coupled polynomial equations can also be obtained from equilibrium equations:

$$0 = (\alpha_{10}^3\beta_{10}^3 + 2\alpha_{10}^4\beta_{10}^2 + \alpha_{10}^5\beta_{10})\left(1 + \frac{P}{2}\sqrt{\frac{\beta_{10}}{\alpha_{10}}}\right) + Q(\alpha_{10}^5\beta_{10}^2 + 2\alpha_{10}^4\beta_{10}^3 + \alpha_{10}^3\beta_{10}^4) - \alpha_{10}^3\beta_{10}(3\Omega^2 - k + 1 + \epsilon)$$
$$+ (\alpha_{10}\beta_{10}^3 + 2\alpha_{10}^2\beta_{10}^2)(\Omega^2 - 1 + k + \epsilon) - \lambda(5\alpha_{10}^2\beta_{10} + 7\alpha_{10}\beta_{10}^2 + \alpha_{10}^3 + 3\beta_{10}^3), \quad (24)$$

$$0 = (\alpha_{10}^3\beta_{10}^3 + 2\alpha_{10}^2\beta_{10}^4 + \alpha_{10}\beta_{10}^5)\left(1 + \frac{P}{2}\sqrt{\frac{\alpha_{10}}{\beta_{10}}}\right) + Q(\alpha_{10}^4\beta_{10}^3 + 2\alpha_{10}^3\beta_{10}^4 + \alpha_{10}^2\beta_{10}^5) - \beta_{10}^3\alpha_{10}(3\Omega^2 - k + 1 - \epsilon)$$
$$+ (\beta_{10}\alpha_{10}^3 + 2\alpha_{10}^2\beta_{10}^2)(\Omega^2 - 1 + k + \epsilon) - \lambda(5\beta_{10}^2\alpha_{10} + 7\beta_{10}\alpha_{10}^2 + \beta_{10}^3 + 3\alpha_{10}^3). \quad (25)$$

In Fig.6, we plot the variation of $\eta$ with the rotational frequency $\Omega$ in a harmonic-plus-Gaussian potential for $\epsilon = 0$ and $k = 1.1$. In this case, $\omega_q/2 = 0.346$, $\Omega_c = 0.445$. When $\Omega > 0.346$, the spontaneous shape deformation exists in the rotating system and the vortex will form when $\langle L_z \rangle = N\hbar$ at $\Omega = 0.445$. For the harmonic-plus-Gaussian potential, when $k > 1$, the structure of the effective external potential is a Mexican hat, so one can produce a vortex in a slowly rotating BEC. Then, we present the relation between the critical rotational frequency ($\Omega_c$) and the anharmonic term $\lambda$ in Fig.7. It is clear that $\Omega_c$ decreases with $\lambda$ for different values of $Q$. When $\lambda < 0.01$, $\Omega_c$ is small compared to the system with only two-body interaction. When $\lambda > 0.01$, $\Omega_c$ is larger compared to the system with only two-body interaction. It implies that the three-body interaction helps to create a vortex when $\lambda < 0.01$. However, when $\lambda$ becomes larger enough, $\Omega_c$ starts increasing after a minimum value, because the structure of the potential is tighter and deeper which will play an important role in the vortex formation, and the influence of the three-body interaction is reduced.

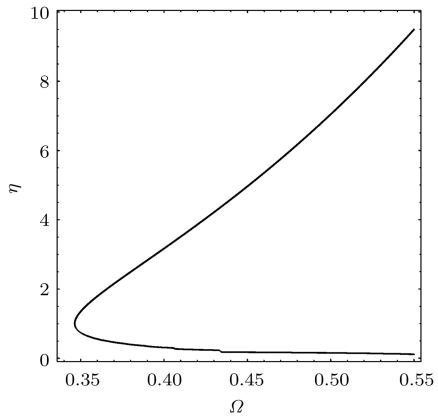



**Fig.6.** The variation of $\eta$ vs. rotational frequency $\Omega$ for $P = 1.0$, $Q = 1.0$, $\lambda = 0.011$ ($k = 1.1$), and $\epsilon = 0$.

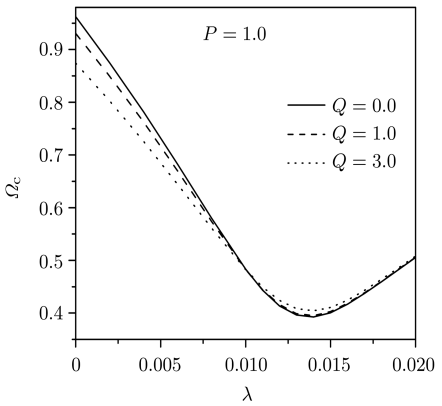



**Fig.7.** The critical rotation frequency $\Omega_{\rm c}$ vs. anharmonic parameter $\lambda$ for $P = 1.0$, $\epsilon = 0$ and different values of $Q$ in a rotating BEC confined in a harmonic-plus-Gaussian laser potential (Mexican hat structure).

## 4. Conclusions

In this paper, we have calculated the lowest energy surface mode excitation frequency and the critical rotational frequency ($\Omega_{\rm c}$) of a rotating BEC with both two- and three-body interactions confined in a quadratic-plus-quartic and harmonic-plus-Gaussian trap. We have shown that the three-body interaction depresses the lowest energy surface mode frequency and the critical rotational frequency of the system. The BEC shape deformation and consequently forming a quantized vortex in a rotating repulsive BEC depends on the strengths of the atom-atom interaction and the anharmonical potential. We find that the critical rotational frequency for a vortex formation in the harmonic-plus-Gaussian potential is lower than that in the quadratic-plus-quartic potential.

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