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Effects of periodic modulation on the nonlinear Landau–Zener tunneling*

Wu Li-Hua(吴利华) and Duan Wen-Shan(段文山)†

College of Physics and Electronic Engineering, Northwest Normal University, Lanzhou 730070, China

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We study the Landau-Zener tunneling of a nonlinear two-level system by applying a periodic modulation on its energy bias. We find that the two levels are splitting at the zero points of the zero order Bessel function for high-frequency modulation. Moreover, we obtain the effective coupling constant between two levels at the zero points of the zero order Bessel function by calculating the final tunneling probability at these points. It seems that the effective coupling constant can be regarded as the approximation of the higher order Bessel function at these points. For the low-frequency modulation, we find that the final tunneling probability is a function of the interaction strength. For the weak inter-level coupling case, we find that the final tunneling probability is more disordered as the interaction strength becomes larger.

Keywords: nonlinear two-level system, Landau–Zener tunneling, tunneling probability, periodic modulation

PACC: 0365, 0155, 7335

1. Introduction

The Landau-Zener tunneling (LZT) model is a basic model in quantum mechanics[1] and has versatile applications, such as in the collision theory,[2] in the spin tunneling of nanomagnets,[3−5] Bose-Einstein condensates (BEC)[6−12] and quantum computing,[13] etc.

The LZT model has been extended to many versions by taking diverse physical conditions into account, for example, the LZT problem with a time-varied sweeping rate,[14] the LZT model with a fast noise mimic system's interaction with environment,[15] the LZT model with periodic modulation,[16−19] and so on. Among them, LZT in a nonlinear two-level system is one of the most interesting models and has attracted much recent attention.[20−23] In this model, the level energies depend on the occupation of the levels. It may arise in a mean-field treatment of a many-body system where the particles predominantly occupy two energy levels. For example, such a model arises in the study of the motion of a small polaron,[24] a BEC in a double-well potential[20,25−27] or in an optical lattice,[28−30] or for a small capacitance Joseph junction where the charging energy may be important. Recently, investigation of this model under the external periodic modulation has attracted considerable attention.[16,31] In the previous investigations, this topic is mainly limited to the linear case. However, in the practical situation of experiments, the interaction between the particles is nonlinear.

In this paper, we study the effects of periodic modulation on the LZT in a nonlinear two-level system. We emphasize the influence of a periodic modulation applied on the energy bias of the system. In the high-frequency modulation, our numerical results indicate that the two levels are splitting at the zero points of the zero order Bessel function J_0 and the LZT between the two levels is strongly suppressed by the modulation at these points. These results were demonstrated by a recent experiment.[32] We especially obtain the effective coupling constant between the two levels at the zero points of J_0 by calculating the final tunneling probability at these points. And we find that the effective coupling constant can be regarded as the approximation of the higher order Bessel function at these points.

In the low-frequency modulation, the final tunneling probability for the nonlinear case varies as the interaction strength varies. For the weak inter-level coupling limit, we find that the final tunneling prob-

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†Corresponding author. E-mail: Duanws@nwnu.edu.cn

ability is more disordered as the interaction strength becomes larger.

2. Nonlinear Landau–Zener tunneling model

We consider a nonlinear two-level system which is described by the dimensionless Schrödinger equation[21]

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a \\ b \end{pmatrix} = H(\gamma)\begin{pmatrix} a \\ b \end{pmatrix}, \tag{1}$$

where a and b are the probability amplitudes of particles in each of the two levels, respectively. The Hamiltonian is given by

$$H(\gamma) = \begin{pmatrix} \frac{\gamma}{2} + \frac{c}{2}(|b|^2 - |a|^2) & \frac{v}{2} \\ \frac{v}{2} & -\frac{\gamma}{2} - \frac{c}{2}(|b|^2 - |a|^2) \end{pmatrix}, \tag{2}$$

where v is the coupling constant between the two levels, γ is the energy bias, and c is the nonlinear parameter describing the interaction. The total probability $|a|^2 + |b|^2$ is conserved and is set to be 1.

In the following sections, we will discuss how a periodic modulation affects the nonlinear LZT. Without losing generality, we assume that the modulation is applied on the energy bias with amplitude A, frequency ω and time t, i.e., $\gamma = \alpha t + A\sin\omega t$. We call α the sweeping rate. We will study three typical cases according to different modulation frequency regimes.

3. High-frequency modulation

The first case we consider is the high frequency limit, i.e. $\omega \gg v$. In this case, the energy bias undergoes a high-frequency modulation. The numerical results of the final tunneling probability $|a|^2$ for different interaction strengths c from Eq.(1) are shown in Fig.1.

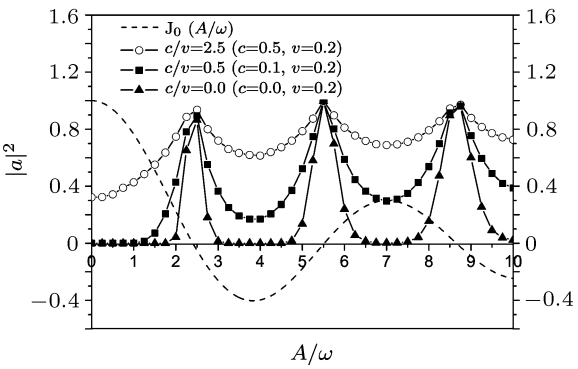


Fig.1. The final tunneling probability $|a|^2$ versus the dimensionless amplitude A/ω for the linear and nonlinear cases with the modulation frequency $\omega = 100$, the coupling constant $v = 0.2$ and the sweeping rate $\alpha = 0.001$. The dashed line is the zero order Bessel function $\mathrm{J}_0(A/\omega)$ as a function of A/ω.

We see that the final tunneling probability can be adjusted effectively by periodic modulation. Most strikingly, the final tunneling probability $|a|^2$ tends to 1 at the zero points of J_0 (J_0 is the zero order Bessel function).

The above observation can be understood as follows. Using the high-frequency approximation, the time-dependent system Eq.(1) is equivalent to a stationary one. Let us make the transformation $a = \mathrm{e}^{\mathrm{i}(A/2\omega)\cos\omega t}a'$, $b = \mathrm{e}^{-\mathrm{i}(A/2\omega)\cos\omega t}b'$.[17,18] Then we can obtain the following Schrödinger equation:

$$\mathrm{i}\frac{\mathrm{d}a'}{\mathrm{d}t} = \left[\frac{\alpha t}{2} + \frac{c}{2}(|b'|^2 - |a'|^2)\right]a' + \frac{v}{2}\mathrm{e}^{-\mathrm{i}(A/\omega)\cos\omega t}b', \tag{3}$$

$$\mathrm{i}\frac{\mathrm{d}b'}{\mathrm{d}t} = \left[-\frac{\alpha t}{2} - \frac{c}{2}(|b'|^2 - |a'|^2)\right]b' + \frac{v}{2}\mathrm{e}^{\mathrm{i}(A/\omega)\cos\omega t}a'. \tag{4}$$

Using the formula

$$\mathrm{e}^{\pm\mathrm{i}\,z\cos\omega t} = \sum_{n=-\infty}^{\infty}\mathrm{J}_n(z)(\pm\mathrm{i})^n\mathrm{e}^{\pm\mathrm{i}\,n\omega t}, \tag{5}$$

where $\mathrm{J}_n(z)$ is the n-th order Bessel function, by neglecting the contribution of those higher order Bessel functions to the integrals,[31,33] we obtain the Schrödinger equation as follows:

$$\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}t}\begin{pmatrix} a' \\ b' \end{pmatrix} = \begin{pmatrix} \frac{\alpha t}{2} + \frac{c}{2}(|b'|^2 - |a'|^2) & \frac{v}{2}\mathrm{J}_0(A/\omega) \\ \frac{v}{2}\mathrm{J}_0(A/\omega) & -\frac{\alpha t}{2} - \frac{c}{2}(|b'|^2 - |a'|^2) \end{pmatrix} \begin{pmatrix} a' \\ b' \end{pmatrix} \tag{6}$$

and find that the coupling constant becomes $V = v\mathrm{J}_0(A/\omega)$. At the zero points of $\mathrm{J}_0(A/\omega)$, V equals zero, which implies no coupling between the two levels, in other words, the two levels are splitting in the high-frequency modulation. Therefore, for the state of initial condition $|a|^2 = 1, |b|^2 = 0$ the final tunneling probability $|a|^2$ tends to 1 (Fig.1), i.e., there is almost no tunneling between the two levels. This result is confirmed by a recent experiment.[32] In this experiment, the authors observed the dynamical suppression of tunneling of a BEC in a strongly driven optical lattice. The strong driving was a sinusoidal shaking of the lattice corresponding to a time-varying linear potential, and the tunneling was measured by letting the BEC freely expand in the lattice. At the first zero point 2.4 and the second zero point 5.5 of $\mathrm{J}_0(A/\omega)$, strong suppression of tunneling was observed in this experiment. Namely, our study supports the result in Ref.[32].

However, we find that the final tunneling probability $|a|^2$ is not equal to 1 exactly at the zero points of $\mathrm{J}_0(A/\omega)$. This is because we have neglected those higher order Bessel functions.

To evaluate the effect of those higher order Bessel functions, we give the numerical result of the final tunneling probability $|a|^2$ as a function of the sweeping rate α from Eq.(1) for $c = 0.0$ and $v = 0.2$, which is described by the solid square in Fig.2. The result is for the case of the first zero point of $\mathrm{J}_0(A/\omega)$. We also obtain a fitting curve by choosing appropriate parameter v' from linear Landau-Zener formula[1] as follows:

$$\Gamma = \exp\left(-\frac{\pi v'^2}{2\alpha}\right), \tag{7}$$

which is described by the smooth line in Fig.2, where Γ is the final tunneling probability, v' is the effective coupling constant between the two levels and α is the sweeping rate.

We find that the effective coupling constant v' can be approximated as follows:

$$v' = 0.0333v, \tag{8}$$

where v is the coupling constant from Eq.(2), and 0.0333 is our simulated constant. We can use the effective coupling constant v' to approximate the effect of the higher order Bessel function at the first zero point of $\mathrm{J}_0(A/\omega)$.

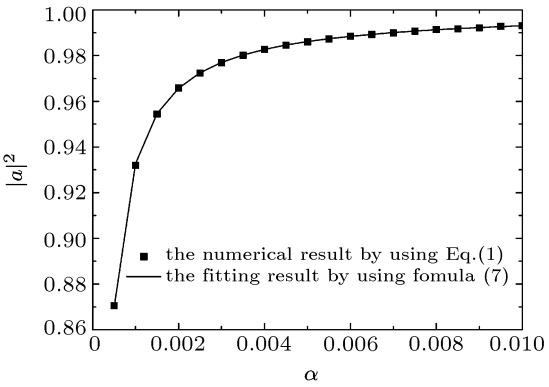


Fig.2. The final tunneling probability $|a|^2$ as a function of the sweeping rate α for the linear interaction ($c = 0$) with the modulation frequency $\omega = 100$ and the coupling constant $v = 0.2$ at the first zero point of $\mathrm{J}_0(A/\omega)$. The solid square describes the numerical result by using Eq.(1). The line describes the fitting result by using formula (7). We obtain the effective coupling constant $v' = 0.0333v$.

In order to find whether equation (8) can be an approximation of higher order Bessel function, we extend Eq.(8) to the nonlinear case and obtain the numerical results of the final tunneling probability $|a|^2$ as a function of the sweeping rate α for the following two cases:

(i) The high-frequency modulation with the coupling constant v ($v = 0.2$) and the energy bias $\gamma = \alpha t + A\sin\omega t$;

(ii) No modulation with the coupling constant $0.0333v$ ($v = 0.2$) and the energy bias $\gamma = \alpha t$.

The comparison between the two cases is shown in Fig.3 and harmony is obtained. It indicates that the system of high-frequency modulation with coupling constant v is equivalent to the system of no modulation but with coupling constant $0.0333v$ at the first zero point of $\mathrm{J}_0(A/\omega)$. Thus, the contribution of the higher order Bessel function can also be approximated by Eq.(8) for the nonlinear case at the first zero point of $\mathrm{J}_0(A/\omega)$.

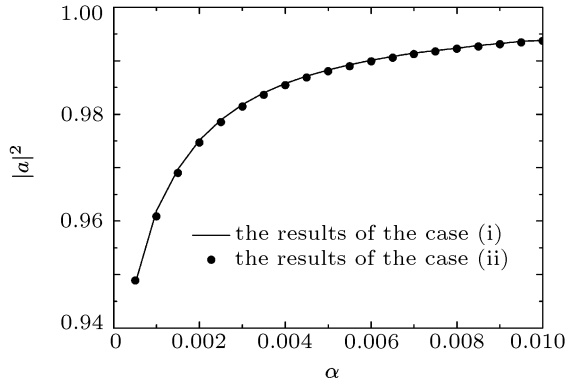


Fig.3. The numerical results for the case (i) and the case (ii) at the first zero point of $J_0(A/\omega)$. Here $c = 0.5$ and $v = 0.2$.

To determine whether the effective coupling constant v' is invariable at the first zero point of $J_0(A/\omega)$, we give the numerical results of the final tunneling probability $|a|^2$ as a function of the sweeping rate α for different modulation frequencies ω of the linear case. The results are shown in Fig.4. We find that the numerical results for $\omega = 80, 100, 150, 200$ are almost the same, so the effective coupling constant v' for $\omega = 80$, 100, 150, 200 are all the same, i.e., $v' = 0.0333v$. Accordingly, v' is invariable for different high frequencies at the first zero point of $J_0(A/\omega)$. Hence, we find that the contribution of the higher order Bessel function can be approximated by the effective coupling constant v' at the first zero point of $J_0(A/\omega)$. Furthermore, we find similar results for other zero points of $J_0(A/\omega)$. However, the effective coupling constants are different for different zero points of $J_0(A/\omega)$. We obtain $v' = 0.0068v$ at the second zero point, and $v' = 0.00096v$ at the third zero point.

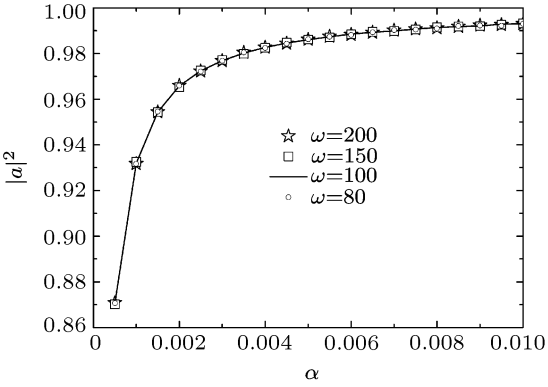


Fig.4. The final tunneling probability $|a|^2$ as a function of the sweeping rate α for different frequencies with $c = 0.0$ and $v = 0.2$ at the first zero point of $J_0(A/\omega)$.

Therefore, in the high-frequency periodic modulation, we conclude that the effective coupling constant v' can be considered as the approximation of the higher order Bessel function at the zero points of $J_0(A/\omega)$.

4. Low-frequency modulation

The second case we considered is that the periodic modulation varies much slower than the linear sweeping, i.e., $\omega/\sqrt{\alpha} \to 0$.[16] In this case, the numerical results of the final tunneling probability $|b|^2$ for the linear and nonlinear cases from Eq.(1) are shown in Fig.5.

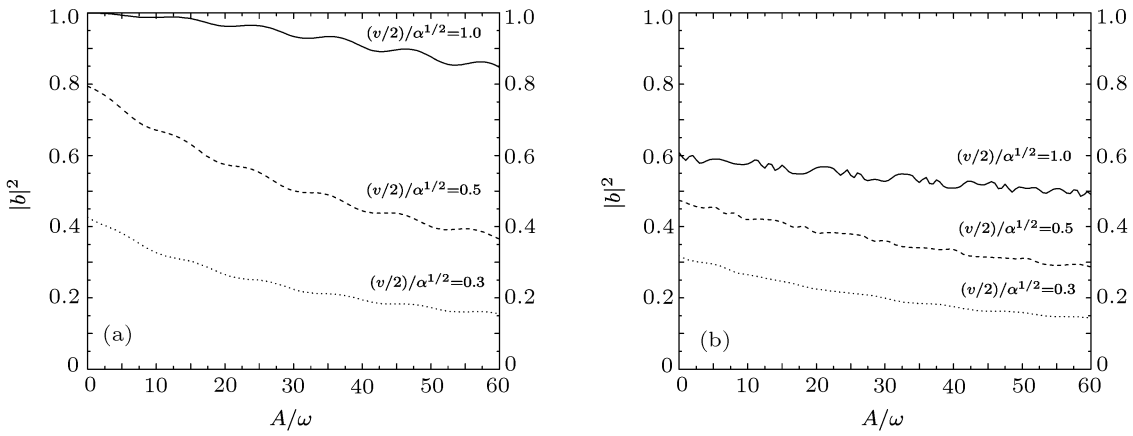


Fig.5. The final tunneling probability $|b|^2$ versus the dimensionless amplitude A/ω for different $(v/2)/\sqrt{\alpha}$ with the modulation frequency $\omega/\sqrt{\alpha} = 0.2$ ($\omega = 0.01$). Here (a) $c = 0.0$ (linear case) and (b) $c = 0.25$ (nonlinear case).

Comparing the linear case (Fig.5(a)) with the nonlinear case (Fig.5(b)), we find that the nonlinear tunneling probability $|b|^2$ is less than that of the linear case for different $(v/2)/\sqrt{\alpha}$ (i.e., different v). It is noted that the nonlinearity decreases the final tunneling probability $|b|^2$. It is also found that as the coupling constant v increases, the final tunneling probability $|b|^2$ increases for both the linear case and the nonlinear case.

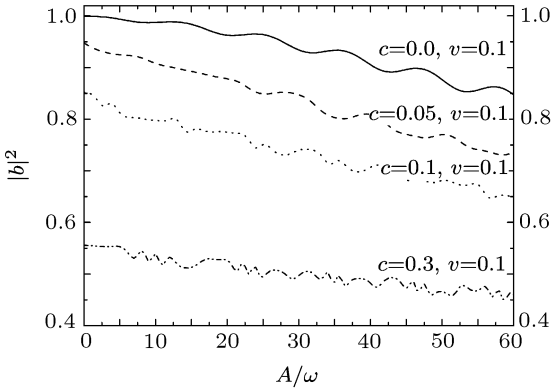


Fig.6. The final tunneling probability $|b|^2$ versus the dimensionless amplitude A/ω for different interaction strengths c with the modulation frequency $\omega/\sqrt{\alpha} = 0.2$ ($\omega = 0.01$).

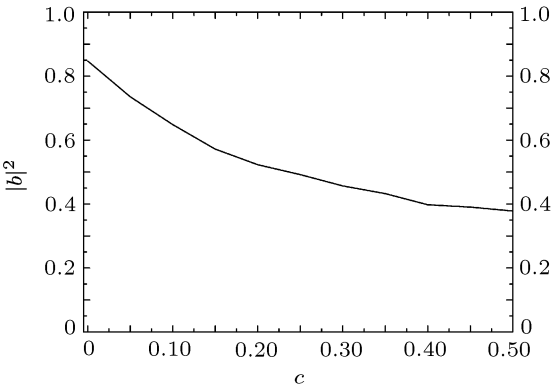


Fig.7. The final tunneling probability $|b|^2$ versus the interaction strength c.

To achieve further insight into the behavior of the nonlinear tunneling probability in the low-frequency modulation, we obtain the final tunneling probability $|b|^2$ as a function of A/ω for different interaction strengths c. The results are shown in Fig.6 for the case of $\omega/\sqrt{\alpha} = 0.2$ ($\omega = 0.01$) and $v = 0.1$. In Fig.6, the curve for the linear case is almost smooth but there are some small wiggles. The small wiggles appearing in the curve are because the numerical simulation must start at some finite time instead of the ideal limit: $t = -\infty$. Compared with the linear case, the curves for the nonlinear case become distortional. Furthermore, as the interaction strengths c increase the curves are more distortional. In order to further find the relationship between the final tunneling probability $|b|^2$ and the interaction strength c we give the numerical result of Fig.7 for the case of $\omega/\sqrt{\alpha} = 0.2$ ($\omega = 0.01$), $v = 0.1$ and $A/\omega = 60$. We, therefore, conclude that the final tunneling probability $|b|^2$ decreases as c increases.

5. Inter-level coupling

Now we consider the case that the inter-level coupling is very weak, i.e., $(v/2)/\sqrt{\alpha} \to 0$.[16] The linear case has been well studied previously. In the weak inter-level coupling case, the linear tunneling probability is suppressed in the resonance regimes, i.e., $\omega^2/\alpha = (n+1)\pi, (n+1/2)\pi$, and enhanced only in the off-resonance regimes, and more effectively at some discrete values of A/ω, i.e., $(n\pi, (n+1/2)\pi)$.[16]

In order to find the differences between the linear case and the nonlinear case, we give the comparisons of the two cases. We obtain the numerical results of the final tunneling probability $|b|^2$ for different interaction strengths c from Eq.(1) in the above regimes. The results are shown in Fig.8, ($\omega^2/\alpha = \pi$), and Fig.9 ($\omega^2/\alpha = \pi/2$).

From these results we find that the final tunneling probability $|b|^2$ for the nonlinear case is complicated and chaos may take place compared with the linear case $c = 0$. As the interaction strength c becomes larger, the final tunneling probability is more disordered and complicated. Although the final tunneling probability for the linear case is regular and periodic, it is disturbed for the nonlinear case. It seems that the chaos is caused by the strong resonance between the nonlinear Rabi oscillation and the periodic modulation.[17,18] We conclude that the effect of the nonlinearity to the final tunneling probability is very important.

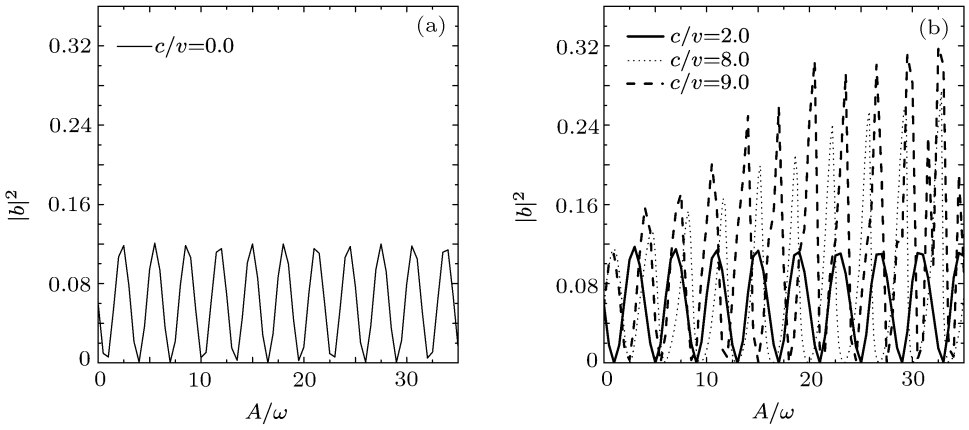


Fig.8. The final tunneling probability $|b|^2$ versus the dimensionless amplitude A/ω for different interaction strengths c with $\omega^2/\alpha = \pi$, $\omega = 0.1$ and $(v/2)/\sqrt{\alpha} = 0.1$. Here (a) $c/v = 0.0$ (linear case) and (b) $c/v = 2$, 8, 9 (nonlinear case).

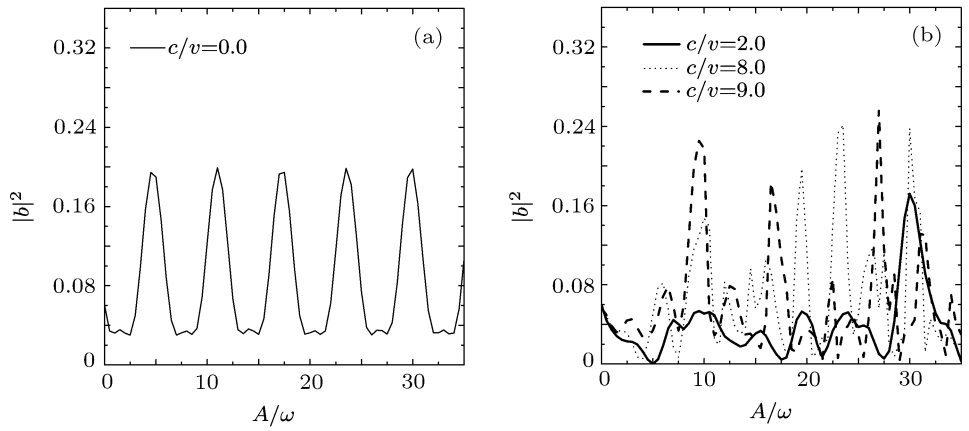


Fig.9. The final tunneling probability $|b|^2$ versus the dimensionless amplitude A/ω for different interaction strengths c with $\omega^2/\alpha = \pi/2$, $\omega = 0.1$ and $(v/2)/\sqrt{\alpha} = 0.1$. Here (a) $c/v = 0.0$ (linear case) and (b) $c/v = 2$, 8, 9 (nonlinear case).

6. Conclusion

In conclusion, the periodic modulation effect on the LZT in a nonlinear two-level system is investigated theoretically in this paper. In the high-frequency modulation, we find that the two levels are splitting at the zero points of the zero order Bessel function J_0 and the LZT between the two levels is strongly suppressed by the modulation at these points, which was observed in a recent experiment.[32] At the zero points of $J_0(A/\omega)$, we numerically obtain the effective coupling constant between the two levels by calculating the final tunneling probability, and demonstrate that the effective coupling constants can be regarded as the approximation of the higher order Bessel function at these points. In the low-frequency modulation, the final tunneling probability $|b|^2$ decreases as the interaction strength c increases, which indicates that the final tunneling probability varies as the interaction strength c varies. Whereas, for the weak inter-level coupling case, we observe a disturbed tunneling probability for the nonlinear case compared with the linear case. Moreover, the final tunneling probability is more disordered as the interaction strength c becomes larger.

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