Vol 18 No 10, October 2009
1674-1056/2009/18(10)/4094–06
Chinese Physics B
© 2009 Chin. Phys. Soc.
and IOP Publishing Ltd

New explicit and exact solutions of the Benney–Kawahara–Lin equation*

Xie Yuan-Xi(谢元喜)†

Department of Physics and Electronic Information, Hunan Institute of Science and Technology, Yueyang 414000, China

(Received 3 February 2009; revised manuscript received 28 March 2009)

In this paper, we present a combination method of constructing the explicit and exact solutions of nonlinear partial differential equations. And as an illustrative example, we apply the method to the Benney-Kawahara-Lin equation and derive its many explicit and exact solutions which are all new solutions.

Keywords: combination method, Benney–Kawahara–Lin equation, explicit and exact solution
PACC: 0340K, 0290

1. Introduction

To search for the explicit and exact solutions of nonlinear partial differential equations (NPDEs) by using different methods, is the goal of many researchers. Due to the complexity of nonlinear system, obtaining the explicit and exact solutions for a real nonlinear physical model equation is often difficult. Fortunately, a vast variety of powerful methods of constructing the explicit and exact solutions of NPDEs have been proposed and developed. They are the hyperbolic tangent function expansion method,[1−5] the homogeneous balance method,[6,7] the trial function method,[8−11] the auxiliary equation method,[12−19] the trial equation method,[20−21] the homotopy analysis method,[22−24] the variable coefficient algebraic method,[25] the (ω/g)-expansion method,[26] and so on. However, there exists no universal method of solving NPDEs. As a result, it is still a very challenging task to search for more powerful and efficient methods to solve NPDEs.

In the present paper, a combination method is proposed to explore the explicit and exact solutions of NPDEs. To illustrate the method, we employ it to solve the Benney–Kawahara–Lin equation, and we successfully find its many new explicit and exact solutions.

2. Description of the proposed approach

The basic idea of our approach is as follows. For a given NPDE, say, of two independent variables x and t

$$P(u, u_t, u_x, u_{xt}, \ldots) = 0, \tag{1}$$

generally speaking, on the left-hand side of Eq.(1) is a polynomial containing unknown function $u(x,t)$ and its various derivatives. In Eq.(1), the subscripts indicate partial derivatives. Now we consider a special case where equation (1) can be thought to be the combination of the following equations to some degree:

$$P_1(u, u_t, u_x, u_{xt}, \ldots) = 0, \tag{2a}$$
$$P_2(u, u_t, u_x, u_{xt}, \ldots) = 0, \tag{2b}$$
$$\ldots$$
$$P_n(u, u_t, u_x, u_{xt}, \ldots) = 0. \tag{2n}$$

That is to say, all these equations have the same nonlinear term; while the linear terms of Eq.(1) are equal to the sum of those of Eqs.(2a)–(2n). In this case, we can construct the solutions of Eq.(1) by the linear combination of those of Eqs.(2a)–(2n); in other words, we can assume that equation (1) has the following ansatz solution

$$u = a_0 + \sum_{j=1}^{n} a_j u_j, \tag{3}$$

*Project supported by the National Natural Science Foundation of China (Grant No 10672053).
†Corresponding author. E-mail: xieyuanxi66@126.com

where a_j $(j = 0, 1, 2, \ldots, n)$ are constants to be determined later, and u_j $(j = 1, 2, \ldots, n)$ are respectively the solutions of Eqs.(2a)–(2n). We call expression (3) the combination formula to find the solutions of NPDE (1). Thus we can easily find the solutions of Eq.(1) by the combination formula (3) if the solutions of Eqs.(2a)–(2n) are given. In the following, we seek the explicit and exact solutions to the Benney–Kawahara–Lin equation by the combination formula (3).

3. Solutions to the Benney–Kawahara–Lin equation by linear combination

The well-known Benney–Kawahara–Lin equation reads

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} + \delta\frac{\partial^5 u}{\partial x^5} = 0, \tag{4}$$

where α, β, γ and δ are arbitrary real constants. Equation (4) was deduced first by Benney and later by Kawahara and Lin, which is an important equation that describes the evolution of long waves in various problems in fluid dynamics. In purely dispersive form ($\alpha = \gamma = 0$), equation (4) reduces to the Kawahara equation which describes water waves with a surface tension.[27–29] In the purely dissipative form ($\beta = \delta = 0$), equation (4) reduces to the Kuramoto–Sivashinsky equation which originates from a weakly nonlinear and long-wave simplification of the Navier-Stokes equation and which has been used to describe different phenomena such as spatial patterns of the Belousov–Zhabotinsky reaction, surface-tension-driven convection in a liquid film and unstable flame fronts.[30–37] Though the Benney–Kawahara–Lin equation (4) is a very important NPDE, now the explicit and exact solutions are still unavailable because of the nonintegrability and of the difficulty in solving it. In what follows, we attempt to find the explicit and exact solutions under the condition $\alpha\beta\gamma\delta \neq 0$ by our combination method.

Evidently, the Benney–Kawahara–Lin equation (4) can be considered as the combination of the following Kuramoto–Sivashinsky equation and Kawahara equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \gamma\frac{\partial^4 u}{\partial x^4} = 0, \tag{5}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \beta\frac{\partial^3 u}{\partial x^3} + \delta\frac{\partial^5 u}{\partial x^5} = 0. \tag{6}$$

Furthermore, these three equations, i.e. equations (4)–(6), have all the same nonlinear term $(u\partial u/\partial x)$; while the linear terms of Eq.(4) $(\alpha\partial^2 u/\partial x^2 + \beta\partial^3 u/\partial x^3 + \gamma\partial^4 u/\partial x^4 + \delta\partial^5 u/\partial x^5)$ are equal to the sum of those of Eqs.(5) and (6). Thus we can construct the solutions of Eq.(4) by the combination formula (3) with the aid of the solutions of Eqs.(5) and (6); namely we may suppose that equation (4) has the following ansatz solution

$$u = a_0 + a_1 u_{\mathrm{KS}} + a_2 u_{\mathrm{K}}, \tag{7}$$

where a_0, a_1 and a_2 are constants to be determined later, and u_{KS} is the solution of Eq.(5), and u_{K} the solution of Eq.(6).

From Ref.[38] it follows that the Kuramoto–Sivashinsky equation (5) admits the following solution

$$u_{\mathrm{KS}} = c - \frac{270\alpha k}{19}\tanh(k\xi) + 120\beta k^3 \tanh^3(k\xi), \tag{8}$$

and that the Kawahara equation (6) admits the following solution

$$u_{\mathrm{K}} = c + \frac{144k^2}{13} - \frac{420k^2}{13}\operatorname{sech}^4(k\xi), \tag{9}$$

where

$$\xi = x - ct. \tag{10}$$

According to expressions (8) and (9) and employing Eq.(7), we assume that the Benney–Kawahara–Lin equation (4) has the following ansatz solution

$$\begin{aligned} u &= a_0 + a_1\left(c - \frac{270\alpha k}{19}\tanh(k\xi) + 120\beta k^3\tanh^3(k\xi)\right) + a_2\left(c + \frac{144k^2}{13} - \frac{420k^2}{13}\operatorname{sech}^4(k\xi)\right) \\ &= b_0 + b_1\tanh(k\xi) + b_2\tanh^2(k\xi) + b_3\tanh^3(k\xi) + b_4\tanh^4(k\xi), \end{aligned} \tag{11}$$

where

$$b_0 = a_0 + a_1 c + a_2\left(c - \frac{276k^2}{13}\right), \quad b_1 = -\frac{270a_1\alpha k}{19}, \quad b_2 = \frac{840a_2k^2}{13}, \quad b_3 = 120a_1\beta k^3, \quad \text{and} \quad b_4 = -\frac{420a_2k^2}{13}.$$

Inserting expression (11) into Eq.(4) engenders the following set of algebraic equations for the unknown constants b_0, b_1, b_2, b_3 and b_4:

$$-b_1ck - 2b_1k^3\beta + 16b_1k^5\delta + b_0b_1k + 2b_2k^2\alpha - 16b_2k^4\gamma + 6b_3k^3\beta - 120b_3k^5\delta + 24b_4k^4\gamma = 0, \tag{12}$$

$$\begin{aligned}-2b_1k^2\alpha + 16b_1k^4\gamma + b_1^2k - 2b_2ck - 16b_2k^3\beta + 272b_2k^5\delta \\ + 2b_0b_2k + 6b_3k^2\alpha - 120b_3k^4\gamma + 24b_4k^3\beta - 960b_4k^5\delta = 0,\end{aligned} \tag{13}$$

$$\begin{aligned}b_1ck + 8b_1k^3\beta - 135b_1k^5\delta - b_0b_1k - 8b_2k^2\alpha + 136b_2k^4\gamma + 3b_1b_2k \\ - 3b_3ck - 60b_3k^3\beta + 1848b_3k^5\delta + 3b_0b_3k + 12b_4k^2\alpha - 480b_4k^4\gamma = 0,\end{aligned} \tag{14}$$

$$\begin{aligned}2b_1k^2\alpha - 40b_1k^4\gamma - b_1^2k + 2b_2ck + 40b_2k^3\beta - 1232b_2k^5\delta - 2b_0b_2k + 2b_2^2k \\ - 18b_3k^2\alpha + 576b_3k^4\gamma + 4b_1b_3k - 4b_4ck - 152b_4k^3\beta + 7744b_4k^5\delta + 4b_0b_4k = 0,\end{aligned} \tag{15}$$

$$\begin{aligned}-6b_1k^3\beta + 240b_1k^5\delta + 6b_2k^2\alpha - 240b_2k^4\gamma - 3b_1b_2k + 3b_3ck + 114b_3k^3\beta \\ - 5808b_3k^5\delta - 3b_0b_3k + 5b_2b_3k - 32b_4k^2\alpha + 1696b_4k^4\gamma + 5b_1b_4k = 0,\end{aligned} \tag{16}$$

$$\begin{aligned}24b_1k^4\gamma - 24b_2k^3\beta + 1680b_2k^5\delta - 2b_2^2k + 12b_3k^2\alpha - 816b_3k^4\gamma - 4b_1b_3k \\ + 3b_3^2k + 4b_4ck + 248b_4k^3\beta - 19264b_4k^5\delta - 4b_0b_4k + 6b_2b_4k = 0,\end{aligned} \tag{17}$$

$$-120b_1k^5\delta + 120b_2k^4\gamma - 60b_3k^3\beta + 6600b_3k^5\delta - 5b_2b_3k + 20b_4k^2\alpha - 2080b_4k^4\gamma - 5b_1b_4k + 7b_3b_4k = 0, \tag{18}$$

$$-720b_2k^5\delta + 360b_3k^4\gamma - 3b_3^2k - 120b_4k^3\beta + 19200b_4k^5\delta - 6b_2b_4k + 4b_4^2k = 0, \tag{19}$$

$$-2520b_3k^5\delta + 840b_4k^4\gamma - 7b_3b_4k = 0, \tag{20}$$

$$-6270b_4k^5\delta - 4b_4^2k = 0. \tag{21}$$

Solving the above system of algebraic equations with the aid of Mathematica yields the following results:

$$b_0 = c + \frac{36\beta k^2}{7}, \quad b_1 = -\frac{420\alpha k}{13}, \quad b_2 = 0, \quad b_3 = \frac{420\alpha k}{13}, \quad b_4 = -1680k^4\delta, \quad k = \pm\frac{35\alpha}{26\beta}, \quad \text{and} \quad c = \text{arbitrary}, \tag{22}$$

where α, β, γ and δ satisfy constraint conditions $\alpha = 143\beta^2/1225\gamma$ and $\beta = 35\gamma^2/121\delta$.

Substituting expression (22) into Eq.(11) gives the following solution of the Benney–Kawahara–Lin equation (4):

$$u = c + \frac{36\beta k^2}{7} - \frac{420\alpha k}{13}\tanh(k\xi) + \frac{420\alpha k}{13}\tanh^3(k\xi) - 1680k^4\delta\tanh^4(k\xi). \tag{23}$$

From Ref.[38] it follows that the Kuramoto–Sivashinsky equation (5) has the following solution

$$u_{\mathrm{KS}} = c - \frac{270\alpha k}{19}\coth(k\xi) + 120\beta k^3\coth^3(k\xi), \tag{24}$$

and that the Kawahara equation (6) has the following solution

$$u_{\mathrm{K}} = c + \frac{144k^2}{13} - \frac{420k^2}{13}\operatorname{csch}^4(k\xi). \tag{25}$$

Based on expressions (24), (25) and Eq.(7), we presume that the Benney–Kawahara–Lin equation (4) has the following ansatz solution

$$u = a_0 + a_1\left(c - \frac{270\alpha k}{19}\coth(k\xi) + 120\beta k^3\coth^3(k\xi)\right) + a_2\left(c + \frac{144k^2}{13} - \frac{420k^2}{13}\operatorname{csch}^4(k\xi)\right)$$
$$= b_0 + b_1\coth(k\xi) + b_2\coth^2(k\xi) + b_3\coth^3(k\xi) + b_4\coth^4(k\xi), \tag{26}$$

where

$$b_0 = a_0 + a_1c + a_2\left(c - \frac{276k^2}{13}\right),\quad b_1 = -\frac{270a_1\alpha k}{19},\quad b_2 = \frac{840a_2k^2}{13},\quad b_3 = 120a_1\beta k^3,\quad \text{and}\quad b_4 = -\frac{420a_2k^2}{13}.$$

Applying the same procedure as before, we obtain the solution of the Benney–Kawahara–Lin equation (4) as follows:

$$u = c + \frac{36\beta k^2}{7} - \frac{420\alpha k}{13}\coth(k\xi) + \frac{420\alpha k}{13}\coth^3(k\xi) - 1680k^4\delta\coth^4(k\xi). \tag{27}$$

From Ref.[38] it follows that the Kuramoto–Sivashinsky equation (5) has the following solution

$$u_{\mathrm{KS}} = c + \frac{270\alpha k}{19}\tan(k\xi) - 120\beta k^3\tan^3(k\xi), \tag{28}$$

and that the Kawahara equation (6) has the following solution

$$u_{\mathrm{K}} = c - \frac{144k^2}{13} + \frac{420k^2}{13}\sec^4(k\xi). \tag{29}$$

In view of expressions (28) and (29) and using Eq.(7), we presume that the Benney–Kawahara–Lin equation (4) has the following ansatz solution

$$u = a_0 + a_1\left(c + \frac{270\alpha k}{19}\tan(k\xi) - 120\beta k^3\tan^3(k\xi)\right) + a_2\left(c - \frac{144k^2}{13} + \frac{420k^2}{13}\sec^4(k\xi)\right)$$
$$= b_0 + b_1\tan(k\xi) + b_2\tan^2(k\xi) + b_3\tan^3(k\xi) + b_4\tan^4(k\xi), \tag{30}$$

where

$$b_0 = a_0 + a_1c + a_2\left(c + \frac{276k^2}{13}\right),\quad b_1 = \frac{270a_1\alpha k}{19},\quad b_2 = \frac{840a_2k^2}{13},\quad b_3 = -120a_1\beta k^3,\quad \text{and}\quad b_4 = \frac{420a_2k^2}{13}.$$

Utilizing the same procedure as before, we have the following solution of the Benney–Kawahara–Lin equation (4)

$$u = c - \frac{36\beta k^2}{7} + \frac{420\alpha k}{13}\tan(k\xi) + \frac{420\alpha k}{13}\tan^3(k\xi) - 1680k^4\delta\tan^4(k\xi). \tag{31}$$

From Ref.[38] it follows that the Kuramoto–Sivashinsky equation (5) admits the following solution

$$u_{\mathrm{KS}} = c - \frac{270\alpha k}{19}\cot(k\xi) + 120\beta k^3\cot^3(k\xi), \tag{32}$$

and that the Kawahara equation (6) admits the following solution

$$u_{\mathrm{K}} = c - \frac{144k^2}{13} + \frac{420k^2}{13}\csc^4(k\xi). \tag{33}$$

According to expressions (32) and (33) and using Eq.(7), we assume that the Benney–Kawahara–Lin equation (4) has the following ansatz solution

$$u = a_0 + a_1\left(c - \frac{270\alpha k}{19}\cot(k\xi) + 120\beta k^3\cot^3(k\xi)\right) + a_2\left(c - \frac{144k^2}{13} + \frac{420k^2}{13}\csc^4(k\xi)\right)$$
$$= b_0 + b_1\cot(k\xi) + b_2\cot^2(k\xi) + b_3\cot^3(k\xi) + b_4\cot^4(k\xi), \tag{34}$$

where

$$b_0 = a_0 + a_1c + a_2\left(c + \frac{276k^2}{13}\right),\quad b_1 = -\frac{270a_1\alpha k}{19},\quad b_2 = \frac{840a_2k^2}{13},\quad b_3 = 120a_1\beta k^3,\quad \text{and}\quad b_4 = \frac{420a_2k^2}{13}.$$

Applying the same procedure as before, we obtain the following solution of the Benney–Kawahara–Lin equation (4)

$$u = c - \frac{36\beta k^2}{7} - \frac{420\alpha k}{13}\cot(k\xi) - \frac{420\alpha k}{13}\cot^3(k\xi) - 1680k^4\delta\cot^4(k\xi). \tag{35}$$

Similarly, based on other solutions of the Kuramoto–Sivashinsky equation (5) and Kawahara equation (6), we may construct other solutions of the Benney–Kawahara–Lin equation (4) by the combination formula (7). Here we omit them because of space limits.

To our knowledge, now no explicit and exact solution of the Benney–Kawahara–Lin equation (4) is available because it is very hard to solve in a usual way. Therefore, the above solutions obtained are all new solutions for the Benney–Kawahara–Lin equation (4), which have been verified by putting them back into Eq.(4) with the aid of Mathematica.

Also, the Benney–Kawahara–Lin equation (4) can be considered to be the combination of the following 5-order KdV equation and KdV–Burgers–Kuramoto equation

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \delta\frac{\partial^5 u}{\partial x^5} = 0, \tag{36}$$

$$\frac{\partial u}{\partial t} + u\frac{\partial u}{\partial x} + \alpha\frac{\partial^2 u}{\partial x^2} + \beta\frac{\partial^3 u}{\partial x^3} + \gamma\frac{\partial^4 u}{\partial x^4} = 0. \tag{37}$$

Likewise, we can find the solutions of the Benney–Kawahara–Lin equation (4) by combination formula (3) with the aid of the solutions of the 5-order KdV equation (36) and KdV–Burgers–Kuramoto equation (37). Here we also omit these details because of space limits.

4. Conclusions and discussion

We present a number of novel explicit and exact solutions of the Benney–Kawahara–Lin equation by our combination method. To study these solutions further so as to investigate the stability and physical significance of these solutions is worthwhile and also important for the explanation of certain physical phenomena related to the Benney–Kawahara–Lin equation. The crucial idea of our approach is to construct solutions of NPDEs which can be regarded as the combination of two or more basic distinct NPDEs by combining solutions of these basic NPDEs. Therefore, the problem for solving NPDEs is significantly simplified. Although it does not seem that our approach is reasonable in theory, it is efficient and feasible in practice for solving the equations of this kind. In fact, our approach is also applicable to some other NPDEs, such as the KdV-Burgers equation which can be regarded as the combination of the Burgers equation and KdV equation, the BBM-Burgers equation which can be regarded as the combination of the Burgers equation and BBM equation, the KdV–Burgers–Kuramoto equation which can be regarded as the combination of the KdV equation Kuramoto–Sivashinsky equation, etc. More importantly, our approach provides a novel line that one may construct solutions of some unknown NPDEs by using those of some known NPDEs, for it is very important and significant to the scientific study. The topic under discussion may be significant for the solution theory of NPDEs, and our obtained results could serve as new examples of solving NPDEs for a supplement to the existing literature. Furthermore, we are convinced that this technique may be used to explore the explicit and exact solutions of other NPDEs with the similar features stated above. We leave this subject to the future study.

References

[1] Malfliet W 1992 *Am. J. Phys.* **60** 650
[2] Ma W X and Zhou D T 1993 *Acta Phys. Sin.* **42** 1731 (in Chinese)
[3] Fan E G 2000 *Phys. Lett.* A **277** 212
[4] Li D S and Zhang H Q 2003 *Acta Phys. Sin.* **52** 1569 (in Chinese)
[5] Zhu J M 2005 *Chin. Phys.* **14** 1290
[6] Wang M L 1995 *Phys. Lett.* A **199** 162
[7] Fan E G and Zhang H Q 1998 *Acta Phys. Sin.* **47** 353 (in Chinese)
[8] Kudryashow N A 1990 *Phys. Lett.* A **147** 287
[9] Xie Y X and Tang J S 2004 *Acta Phys. Sin.* **53** 2828 (in Chinese)
[10] Xie Y X and Tang J S 2006 *Phys. Scr.* **74** 197
[11] Xie Y X 2006 *Nuovo Cimento* B **121** 689
[12] Sirendaoreji and Sun J 2002 *Phys. Lett.* A **298** 133
[13] Xie Y X and Tang J S 2005 *Chin. Phys.* **14** 1303
[14] Xie Y X and Tang J S 2006 *Nuovo Cimento* B **121** 115
[15] Xie Y X and Tang J S 2006 *Int. J. Theor. Phys.* **45** 7
[16] Xie Y X, Su K L and Zhu S H 2007 *Chin. Phys.* **16** 1581
[17] Taogetusang and Sirendaoreji 2006 *Acta Phys. Sin.* **55** 6214 (in Chinese)
[18] Taogetusang and Sirendaoreji 2006 *Chin. Phys.* **15** 2809
[19] Yang X L and Tang J S 2007 *Chin. Phys.* **16** 310
[20] Liu C S 2005 *Acta Phys. Sin.* **54** 2505 (in Chinese)
[21] Liu C S 2005 *Acta Phys. Sin.* **54** 4506 (in Chinese)
[22] Wu Y Y, Wang C and Liao S J 2005 *Chaos, Solitons and Fractals* **23** 1733
[23] Shi Y R, Xu X J, Wu Z X, Wang Y H, Yang H J, Duan W S and Lü K P 2006 *Acta Phys. Sin.* **55** 1555 (in Chinese)

[24] Yang H J, Shi Y R, Duan W S and Lü K P 2007 *Acta Phys. Sin.* **56** 3064 (in Chinese)

[25] Lu B and Zhang H Q 2008 *Chin. Phys.* B **17** 3974

[26] Li W A, Chen H and Zhang G C 2009 *Chin. Phys.* B **18** 400

[27] Hunter J K and Scheurle J 1988 *Phys.* D **32** 253

[28] Kawahara T and Takaoka M 1989 *Phys.* D **39** 43

[29] Kudryashow N 1991 *Phys. Lett.* A **155** 269

[30] Lin S P 1974 *J. Fluid. Mech.* **63** 417

[31] Kuramoto Y and Tsuzuki T 1975 *Prog. Theor. Phys.* **54** 687

[32] Kuramoto Y and Tsuzuki T 1975 *Prog. Theor. Phys.* **55** 356

[33] Nakaya C 1975 *Phys. Fluids* **18** 1407

[34] Sivashinsky G I 1977 *Acta Astronautica* **4** 1177

[35] Topper J and Kawahara T 1978 *J. Phys. Soc. Jpn.* **44** 663

[36] Babchin A J 1983 *Phys. Fluids* **26** 3159

[37] Hooper A P and Grimshaw R 1985 *Phys. Fluids* **28** 37

[38] Xie Y X *Phys. Lett.* A (submitted)