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Application of higher-order KdV–mKdV model with higher-degree nonlinear terms to gravity waves in atmosphere*

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Higher-order Korteweg-de Vries (KdV)-modified KdV (mKdV) equations with a higher-degree of nonlinear terms are derived from a simple incompressible non-hydrostatic Boussinesq equation set in atmosphere and are used to investigate gravity waves in atmosphere. By taking advantage of the auxiliary nonlinear ordinary differential equation, periodic wave and solitary wave solutions of the fifth-order KdV–mKdV models with higher-degree nonlinear terms are obtained under some constraint conditions. The analysis shows that the propagation and the periodic structures of gravity waves depend on the properties of the slope of line of constant phase and atmospheric stability. The Jacobi elliptic function wave and solitary wave solutions with slowly varying amplitude are transformed into triangular waves with the abruptly varying amplitude and breaking gravity waves under the effect of atmospheric instability.

Keywords: gravity waves, higher-order KdV–mKdV equation, propagating, breaking
PACC: 0340, 0220, 0290, 8670G

1. Introduction

Gravity waves are the oscillations of air parcels, induced by the lifting force of bouyancy and the restoring force of gravity.[1,2] These waves propagate vertically as well as horizontally, and actively transport energy and momentum away from the source (for example, thermal forcing, mountains, thunderstorms, jet streams, etc.) to the middle atmosphere and the upper atmosphere.[3−8] Gravity wave and turbulence are often observed to exist simultaneously in the stable atmosphere and in oceans. They are found to affect atmospheric tides in the middle atmosphere and play a significant role in weather and climate in the troposphere, such as the rainstorm of typhoons, orographic precipitation, squall line and front systems.[9−12]

Finding the exact significant wave solutions of nonlinear evolution equations helps one to understand gravity wave excitation, propagation, and breaking properties, and predict hazard weather and climate.[13−16] Many methods to construct exact solutions of nonlinear wave equations have been established and developed, such as the Bäcklund transformation,[17,18] Hirota's bilinear method,[19−21] the tanh-function method,[22−28] the extended tanh-function method,[29,30] the variational iteration methods,[31,32] the collocation method,[33−35] the Adomian Padé approximation,[36] the inverse scattering method,[17,37] the Darboux transformation,[38−42] the F-expansion,[43,44] the auxiliary equation method,[45−51] and so on, which were proposed for constructing periodic wave and solitary wave solutions of some nonlinear evolution equations.

The rest of the present paper is organized as follows. In Section 2, we derive the higher-order mathematical model from the incompressible non-hydrostatic Boussinesq equation, which governs internal gravity waves in atmosphere. In Section 3, we use the auxiliary equation method to find various periodic and solitary wave solutions of the fifth-order KdV–mKdV equation. In Section 4, propagation and transition of gravity waves are investigated under the effect of atmospheric stability. Finally, some conclusions drawn from the present study are given in Section 5.

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2. Derivation of the higher-order KdV–mKdV equation with any order nonlinear terms

The governing system is composed of the nonlinear horizontal momentum equations, mass continuity equation and the nonlinear thermodynamic equation.[2] Considering the two-dimensional nonlinear incompressible non-hydrostatic Boussinesq equations as follows:

$$\frac{\partial u}{\partial t}+u\frac{\partial u}{\partial x}+w\frac{\partial u}{\partial z}=-\frac{1}{\rho}\frac{\partial p'}{\partial x}, \tag{1a}$$

$$\frac{\partial w}{\partial t}+u\frac{\partial w}{\partial x}+w\frac{\partial w}{\partial z}=-\frac{1}{\rho}\frac{\partial p'}{\partial z}+b, \tag{1b}$$

$$\frac{\partial b}{\partial t}+u\frac{\partial b}{\partial x}+N^2\,w=Q, \tag{1c}$$

$$\frac{\partial u}{\partial x}+\frac{\partial w}{\partial z}=0, \tag{1d}$$

where u and w are the speeds, N^2 is the buoyancy frequency per mass unit, p' represents the perturbation pressure, $b=-g\rho'/\rho$ denotes the buoyancy with ρ meaning the density, g denoting gravitational acceleration and ρ' representing the perturbation density.

Considering that diabatic heating consists of two parts, one is latent heating and the other is sensible heating. The form of latent heating is $Q_L=\rho\,\eta\,w$, and the form of sensible heating is

$$Q_H=\rho\,Q_0,\quad Q_0=c_pC_HU(T_o-T_a), \tag{2}$$

where C_H is a non-dimensional coefficient for the transfer of heat and moisture, c_p is the specific heat at constant pressure, U is the absolute wind speed at land or sea surface, T_a is the air temperature and T_o is land or sea surface temperature.

Introducing the transformations

$$\begin{aligned}&u(x,z,t)=u(\xi),\quad w(x,z,t)=w(\xi),\\&\frac{p'(x,z,t)}{\rho(x,z,t)}=p(\xi),\quad b(x,z,t)=b(\xi),\end{aligned} \tag{3}$$

where $\xi=kx+nz-\sigma\,t$ with $\boldsymbol{K}=(k,n)$ meaning the wave vector and σ representing an angular frequency, we can rewrite Eq.(1) as

$$(ku+n\,w-\sigma)\frac{\mathrm{d}u}{\mathrm{d}\xi}=-k\frac{\mathrm{d}p}{\mathrm{d}\xi}, \tag{4a}$$

$$(ku+n\,w-\sigma)\frac{\mathrm{d}w}{\mathrm{d}\xi}=-n\frac{\mathrm{d}p}{\mathrm{d}\xi}+b, \tag{4b}$$

$$(ku-\sigma)\frac{\mathrm{d}b}{\mathrm{d}\xi}+N^2w=\eta\,w+c_pC_HU(T_o-T_a), \tag{4c}$$

$$k\frac{\mathrm{d}u}{\mathrm{d}\xi}+n\frac{\mathrm{d}w}{\mathrm{d}\xi}=0. \tag{4d}$$

Integrating Eq.(4d) with respect to ξ once and taking the integration constant to be zero yield

$$w=-\frac{ku}{n}. \tag{5}$$

Substituting expression (5) into Eqs.(4), we can reduce Eqs.(4) to

$$\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+\frac{(N^2-\eta)k^2\,u-c_pC_HU(T_o-T_a)\,kn}{\sigma\,(k^2+n^2)(\sigma-ku)}=0. \tag{6}$$

If $u\ll\sigma/k$, then $F(u)=1/(\sigma-ku)$ can be expanded as

$$F(u)=\frac{1}{\sigma-ku}=\frac{1}{\sigma}\left(1+\sum_{p=1}^{m}(\frac{ku}{\sigma})^p\right). \tag{7}$$

The substitution of expression (7) into Eq.(6) yields

$$\begin{aligned}&\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2}+\frac{k^2\,(N^2-\eta)\,\sum_{p=1}^{m}(k/\sigma)^p\,u^{p+1}}{\sigma^2\,(k^2+n^2)}\\&-\frac{k\,n\,c_p\,C_H\,U(T_o-T_a)\,\sum_{p=1}^{m}(k/\sigma)^p\,u^p}{\sigma^2\,(k^2+n^2)}\\&+\frac{k^2\,(N^2-\eta)\,u-k\,n\,c_pC_HU(T_o-T_a)}{\sigma^2\,(k^2+n^2)}=0.\end{aligned} \tag{8}$$

Differentiating Eq.(8) once with respect to ξ leads to

$$\begin{aligned}&\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}+\frac{k^2\,(N^2-\eta)\,\sum_{p=1}^{m}(k/\sigma)^p\,(p+1)\,u^p}{\sigma^2\,(k^2+n^2)}\,\frac{\mathrm{d}u}{\mathrm{d}\xi}\\&-\frac{k\,n\,c_p\,C_H\,U(T_o-T_a)\,\sum_{p=1}^{m}(k/\sigma)^p\,p\,u^{p-1}}{\sigma^2\,(k^2+n^2)}\,\frac{\mathrm{d}u}{\mathrm{d}\xi}\\&+\frac{k^2\,(N^2-\eta)}{\sigma^2\,(k^2+n^2)}\,\frac{\mathrm{d}u}{\mathrm{d}\xi}=0,\end{aligned} \tag{9}$$

which is the ordinary differential equation that the generalized KdV–mKdV equation with higher-degree nonlinear terms corresponds to.

For example, choosing $m=1$ in Eq.(9), it is reduced to

$$\begin{aligned}&\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3}+\frac{2\,k^3\,(N^2-\eta)}{\sigma^3\,(k^2+n^2)}\,u\,\frac{\mathrm{d}u}{\mathrm{d}\xi}\\&+\frac{k^2\,\sigma\,(N^2-\eta)-k^2\,n\,c_pC_HU(T_o-T_a)}{\sigma^3\,(k^2+n^2)}\,\frac{\mathrm{d}u}{\mathrm{d}\xi}=0,\end{aligned} \tag{10}$$

which is the ordinary differential equation that KdV equation corresponds to.

Choosing $m = 2$ in Eq.(9), we have

$$\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} + \left[\frac{3\,k^4\,(N^2-\eta)}{\sigma^4\,(k^2+n^2)}\,u^2 + \frac{2\,k^3\,(N^2\,\sigma - n\,c_p\,C_H\,U(\,T_o - T_a) - \eta\,\sigma)}{\sigma^4\,(k^2+n^2)}\,u + \frac{k^2\,(N^2\,\sigma - n\,c_p C_H U(T_o - T_a) - \eta\,\sigma)}{\sigma^3\,(k^2+n^2)}\right]\frac{\mathrm{d}u}{\mathrm{d}\xi} = 0, \tag{11}$$

which is the ordinary differential equation that KdV–mKdV equation corresponds to.

Letting $m = 3$ in Eq.(9) leads to

$$\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} + \left[\frac{4\,k^5\,(N^2-\eta)}{\sigma^5\,(k^2+n^2)}\,u^3 + \frac{3\,k^4\,(N^2\,\sigma - n\,c_p\,C_H\,U(\,T_o - T_a) - \eta\,\sigma)}{\sigma^5\,(k^2+n^2)}\,u^2 + \frac{2\,k^3\,(N^2\,\sigma - n\,c_p C_H U(T_o - T_a) - \eta\,\sigma)}{\sigma^4\,(k^2+n^2)}\,u + \frac{k^2\,(N^2\,\sigma - n\,c_p C_H U(T_o - T_a) - \eta\,\sigma)}{\sigma^3\,(k^2+n^2)}\right]\frac{\mathrm{d}u}{\mathrm{d}\xi} = 0, \tag{12}$$

which is the ordinary differential equation that the generalized KdV–mKdV equation with three-degree nonlinear term corresponds to.

Further, differentiating Eq.(9) twice with respect to ξ leads to

$$\begin{aligned}&\frac{\mathrm{d}^5u}{\mathrm{d}\xi^5} + \left(\frac{3\,k^2\,(N^2-\eta)\,\sum_{p=2}^{m}(k/\sigma)^p\,(p^3-p)\,u^{p-2}}{\sigma^2\,(k^2+n^2)} - \frac{k\,n\,Q_0\,\sum_{p=3}^{m}(k/\sigma)^p\,(p^3-3\,p^2+2p)\,u^{p-3}}{\sigma^2\,(k^2+n^2)}\right)\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)^3 \\ &+ \left(\frac{3\,k^2\,(N^2-\eta)\,\sum_{p=1}^{m}(k/\sigma)^p\,(p^2+p)\,u^{p-1}}{\sigma^2\,(k^2+n^2)} - \frac{3\,k\,n\,Q_0\,\sum_{p=2}^{m}(k/\sigma)^p\,(p^2-p)\,u^{p-2}}{\sigma^2\,(k^2+n^2)}\right)\frac{\mathrm{d}u}{\mathrm{d}\xi}\,\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} \\ &+ \left(\frac{k^2\,(N^2-\eta)\,\sum_{p=1}^{m}(k/\sigma)^p\,(p+1)\,u^p + k^2\,(N^2-\eta)}{\sigma^2\,(k^2+n^2)} - \frac{k\,n\,Q_0\,\sum_{p=1}^{m}(k/\sigma)^p\,p\,u^{p-1}}{\sigma^2\,(k^2+n^2)}\right)\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} = 0,\end{aligned} \tag{13}$$

which is the ordinary differential equation that the fifth-order mKdV equation with higher-degree nonlinear terms corresponds to.

Choosing $m = 1$ and differentiating Eq.(9) twice with respect to ξ, we have

$$\frac{\mathrm{d}^5u}{\mathrm{d}\xi^5} + \left(\frac{2\,k^3\,(N^2-\eta)}{\sigma^3\,(k^2+n^2)}\,u + \frac{k^2\,\sigma\,(N^2-\eta) - k^2\,n\,Q_0}{\sigma^3\,(k^2+n^2)}\right)\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} + \frac{6\,k^3\,(N^2-\eta)}{\sigma^3\,(k^2+n^2)}\,\frac{\mathrm{d}u}{\mathrm{d}\xi}\,\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = 0, \tag{14}$$

which is the ordinary differential equation that fifth-order KdV equation corresponds to.

Letting $m = 2$ and differentiating Eq.(9) twice with respect to ξ arrive at

$$\begin{aligned}&\frac{\mathrm{d}^5u}{\mathrm{d}\xi^5} + \left(\frac{3\,k^4\,(N^2-\eta)}{\sigma^4\,(k^2+n^2)}\,u^2 + \frac{2\,k^3\,(N^2\,\sigma - n\,Q_0 - \eta\,\sigma)}{\sigma^4\,(k^2+n^2)}\,u + \frac{k^2\,(N^2\,\sigma - n\,Q_0 - \eta\,\sigma)}{\sigma^3\,(k^2+n^2)}\right)\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} + \frac{6\,k^4\,(N^2-\eta)}{\sigma^4\,(k^2+n^2)}\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)^3 \\ &+ \left(\frac{18\,k^4\,(N^2-\eta)\,u + 6\,k^3\,\sigma\,(N^2-\eta)}{\sigma^4\,(k^2+n^2)} - \frac{6\,k^3\,n\,Q_0}{\sigma^4\,(k^2+n^2)}\right)\frac{\mathrm{d}u}{\mathrm{d}\xi}\,\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = 0,\end{aligned} \tag{15}$$

which is the ordinary differential equation that the fifth-order KdV–mKdV equation with a two-degree nonlinear term corresponds to. The proposed method can be readily used to derive the higher-order KdV–mKdV equation with higher-degree nonlinear terms.

3. Periodic wave and solitary wave solutions of the fifth-order KdV–mKdV equation

Now we focus our attention on Eq.(15). Introduce the transformations

$$\alpha = \frac{2k^3((N^2-\eta)\,\sigma - n\,c_p\,C_H\,U(T_o - T_a))}{\sigma^4(k^2+n^2)},$$

$$\beta = \frac{3k^4(N^2-\eta)}{\sigma^4(k^2+n^2)},$$
$$\gamma = \frac{k^2((N^2-\eta)\,\sigma - n\,c_p\,C_H\,U(T_o-T_a))}{\sigma^3(k^2+n^2)}, \quad (16)$$

then equation (15) will be changed into

$$\frac{\mathrm{d}^5u}{\mathrm{d}\xi^5} + (\gamma + \alpha\,u + \beta\,u^2)\frac{\mathrm{d}^3u}{\mathrm{d}\xi^3} + 2\beta\left(\frac{\mathrm{d}u}{\mathrm{d}\xi}\right)^3 + 3\,(\alpha + 2\beta\,u)\,\frac{\mathrm{d}u}{\mathrm{d}\xi}\,\frac{\mathrm{d}^2u}{\mathrm{d}\xi^2} = 0. \quad (17)$$

Let $u = \sum_{i=0}^{n} a_i\varphi^i$ with φ satisfying

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \epsilon\sqrt{\sum_{j=0}^{r} c_j\varphi^j}, \quad (18)$$

then we can obtain $r = 2(n+1)$ through balancing terms $u^2\mathrm{d}u/\mathrm{d}\xi$ and $\mathrm{d}^3u/\mathrm{d}\xi^3$ in Eq.(17). For simplicity, we choose $n = 1$ and $r = 4$, and set $u = a_0 + a_1\varphi$ with φ satisfying

$$\frac{\mathrm{d}\varphi}{\mathrm{d}\xi} = \epsilon\sqrt{c_0 + c_1\varphi + c_2\varphi^2 + c_3\varphi^3 + c_4\varphi^4}. \quad (19)$$

Substituting Eq.(19) into Eq.(17) and using the Mathematica, we obtain a system of algebraic equations which gives the following solutions

$$c_2 = -\gamma - \beta{a_0}^2 - \alpha\,a_0,$$
$$c_3 = -\frac{\alpha\,a_1}{3} - \frac{2\,\beta\,a_0\,a_1}{3},$$
$$c_4 = -\frac{\beta\,{a_1}^2}{6}, \quad (20)$$

where a_0, a_1, c_0 and c_1 are arbitrary constants.

Choosing $c_3 = 0$, $c_2 \neq 0$ and $c_4 \neq 0$ in expression (20) gives $a_0 = -\alpha/2\beta$. Solving Eq.(17), we have the following solutions

$$u_1 = -\frac{\alpha}{2\beta} + \frac{\sqrt{6(\alpha^2-4\beta\gamma)}}{2\beta} \times \operatorname{sech}\left(\sqrt{\frac{\alpha^2-4\beta\gamma}{4\beta}}(kx+nz-\sigma\,t)\right), \quad (21)$$

where $\alpha^2 - 4\beta\gamma > 0$, $\beta > 0$.

$$u_2 = -\frac{\alpha}{2\beta} + \frac{1}{2\beta}\sqrt{\frac{4\beta\gamma-\alpha^2}{2}} \times \tanh\left(\sqrt{\frac{4\beta\gamma-\alpha^2}{8\beta}}(kx+nz-\sigma\,t)\right), \quad (22)$$

where $\alpha^2 - 4\beta\gamma < 0$, $\beta > 0$.

$$u_3 = -\frac{\alpha}{2\beta} - \frac{\sqrt{6}}{\sqrt{-\beta}(kx+nz-\sigma\,t)}, \quad (23)$$

where $\alpha^2 - 4\beta\gamma = 0$, $\beta < 0$.

$$u_4 = -\frac{\alpha}{2\beta} + \frac{m}{2\beta}\sqrt{\frac{6(\alpha^2-4\beta\gamma)}{2m^2-1}} \times \operatorname{cn}\left(\sqrt{\frac{\alpha^2-4\beta\gamma}{4\beta(2m^2-1)}}(kx+nz-\sigma\,t)\right), \quad (24)$$

where $\alpha^2 - 4\beta\gamma > 0$, $\beta > 0$.

$$u_5 = -\frac{\alpha}{2\beta} + \frac{m}{2\beta}\sqrt{\frac{4\beta\gamma-\alpha^2}{m^2+1}} \times \operatorname{sn}\left(\sqrt{\frac{4\beta\gamma-\alpha^2}{4\beta(m^2+1)}}(kx+nz-\sigma\,t)\right), \quad (25)$$

where $\alpha^2 - 4\beta\gamma < 0$, $\beta > 0$.

$$u_6 = -\frac{\alpha}{2\beta} + \frac{1}{2\beta}\sqrt{\frac{6(\alpha^2-4\beta\gamma)}{2-m^2}} \times \operatorname{dn}\left(\sqrt{\frac{\alpha^2-4\beta\gamma}{4\beta(2-m^2)}}(kx+nz-\sigma\,t)\right), \quad (26)$$

where $\alpha^2 - 4\beta\gamma > 0$, $\beta > 0$.

$$u_7 = -\frac{\alpha}{2\beta} + \frac{\sqrt{6(\alpha^2-4\beta\gamma)}}{2\beta} \times \sec\left(\sqrt{\frac{4\beta\gamma-\alpha^2}{4\beta}}(kx+nz-\sigma\,t)\right), \quad (27)$$

where $\alpha^2 - 4\beta\gamma > 0$, $\beta < 0$.

$$u_8 = -\frac{\alpha}{2\beta} + \frac{1}{2\beta}\sqrt{\frac{4\beta\gamma-\alpha^2}{2}} \times \tan\left(\sqrt{\frac{\alpha^2-4\beta\gamma}{8\beta}}(kx+nz-\sigma\,t)\right), \quad (28)$$

where $\alpha^2 - 4\beta\gamma < 0$, $\beta < 0$.

When $m \to 1$, u_4 is reduced to u_1, and u_5 is reduced to u_2.

4. Propagation and transition of gravity waves under the influence of atmospheric stability

In the foregoing sections, the fifth-order KdV–mKdV equation depicts four types of solutions, which are Jacobi elliptic function solutions, solitary wave solutions, triangular solutions, and rational solutions.

The two particular solutions of interest to us here are solitary wave solutions u_1 and u_2, and Jacobi elliptic function solutions u_4, u_5 and u_6, which must satisfy the constraint condition $\beta > 0$, that is, $N^2 > \eta$.

As an illustrative sample, the properties of some wave solutions are analysed and shown by some figures.

Firstly, the periodic wave solutions (u_4 and u_6) and solitary wave solution (u_1) must satisfy the constraint condition $\alpha^2 - 4\beta\gamma > 0$, that is,

$$2\sigma^2 (N^2 - \eta)^2 - n\, Q_0 \sigma (N^2 - \eta) - n^2 Q_0^2 < 0, \quad (29)$$

which shows that in stable atmosphere ($N^2 > \eta$), sensible heating ($Q_0 > 0$) administers the generation of upward-propagating Jacobi cnoid periodic waves and solitary waves (the phase speed in the z-direction $c_z = \sigma/n > 0$), but sensible cooling ($Q_0 < 0$) conduces to the production of downward-propagating Jacobi cnoid periodic waves and solitary waves (the phase speed in the z-direction $c_z < 0$).

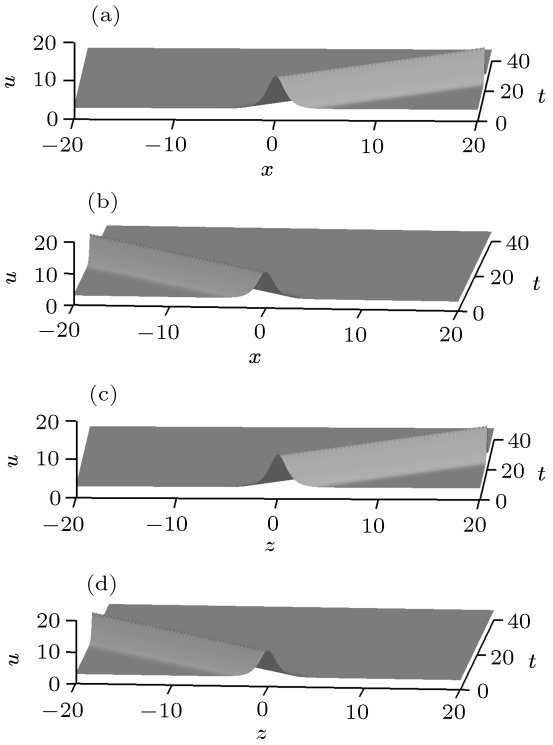


Fig.1. (a) Easterly ($\sigma = 1$), (b) westerly ($\sigma = -1$), (c) upward ($\sigma = 1$) and (d) downward ($\sigma = -1$) propagating solitary wave u_1 with $k = 1,\ n = 1,\ \eta = 0,\ Q = 1$ and $N^2 = 0.1$.

Considering the slope of lines of constant phase in the x–z plane ($\mathrm{d}z/\mathrm{d}x = -k/n$), it is not difficult to see that in the stable atmosphere ($N^2 > \eta$), sensible heating ($Q_0 > 0$) administers the generation of westerly propagating waves (the phase speed in the x-direction $c_x = \sigma/k < 0$) if the slope of lines of constant phase is positive; and sensible heating ($Q_0 > 0$) conduces to the production of easterly propagating waves (the phase speed in the x-direction $c_x > 0$) if the slope of lines of the constant phase is negative. Figures 1 and 2 show the corresponding propagation behaviour in qualitative terms.

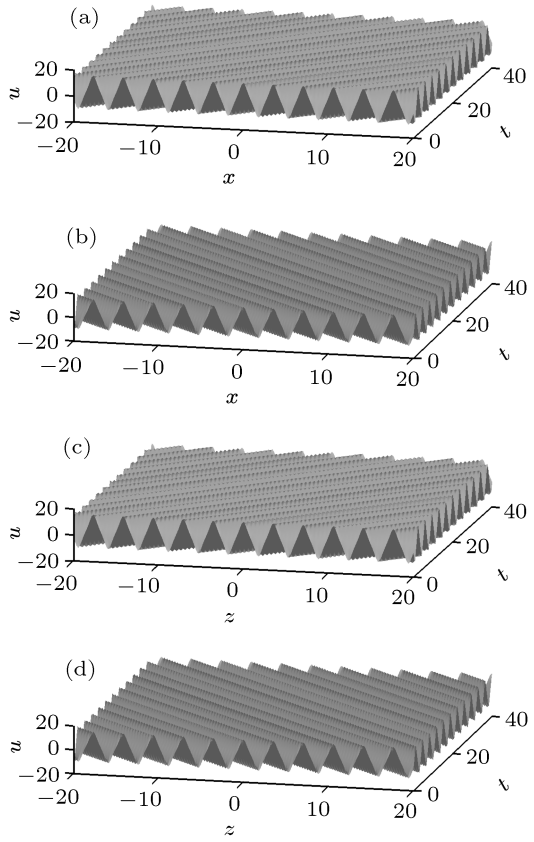


Fig.2. (a) Easterly ($\sigma = 1$), (b) westerly ($\sigma = -1$), (c) upward ($\sigma = 1$) and (d) downward ($\sigma = -1$) propagating elliptic periodic wave u_4 with $k = 1, n = 1,\ \eta = 0,\ Q = 1$ and $N^2 = 0.1$.

However, in the stable atmosphere ($N^2 > \eta$), sensible cooling ($Q_0 < 0$) helps the generation of easterly propagating waves if the slope of lines of the constant phase is positive, and sensible cooling ($Q_0 < 0$) helps the production of westerly propagating waves if the slope of lines of the constant phase is negative.

Similarly, there exist the periodic wave solutions (u_5) and solitary wave solutions (u_2) under the constraint condition $\alpha^2 - 4\beta\gamma < 0$, namely

$$2\sigma^2 (N^2 - \eta)^2 - n\, Q_0 \sigma (N^2 - \eta) - n^2 Q_0^2 > 0,' \quad (30)$$

which indicates that in a stable atmosphere ($N^2 > \eta$), sensible heating ($Q_0 > 0$) administers to the generation of downward-propagating Jacobi snoid periodic waves and a solitary wave ($c_z < 0$), but sensible cooling ($Q_0 < 0$) helps the production of the upward-propagating Jacobi snoid periodic waves and a solitary wave ($c_z > 0$). On the other hand, sensible heating ($Q_0 > 0$) helps the generation of easterly propagating-Jacobi snoid waves and a solitary wave ($c_x > 0$) if the

slope of lines of constant phase is positive, and sensible heating ($Q_0 > 0$) administers to the production of the westerly propagating Jacobi snoid waves and a solitary wave ($c_x < 0$) if the slope of lines of constant phase is negative. The corresponding propagation behaviour is given in Figs.3 and 4.

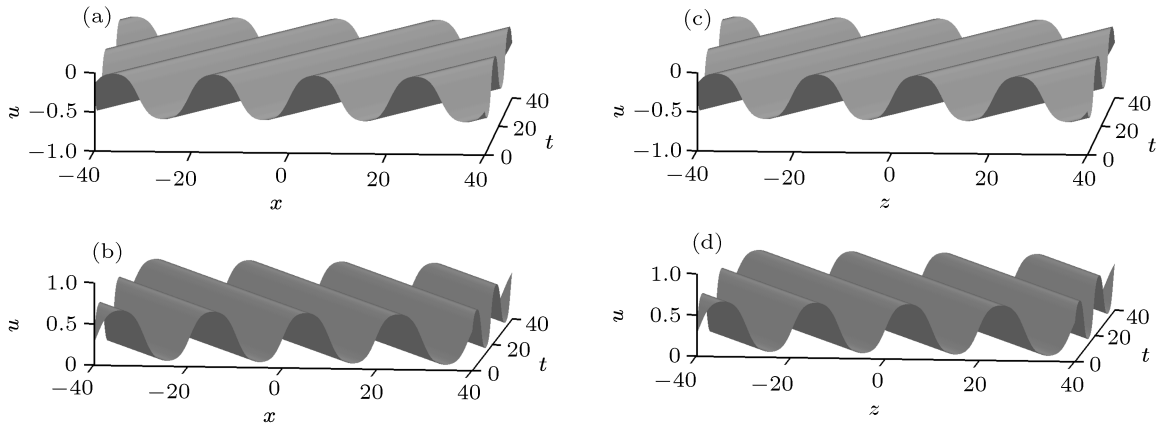


Fig.3. (a) Easterly ($\sigma = 1$), (b) westerly ($\sigma = -1$), (c) upward ($\sigma = 1$) and (d) downward ($\sigma = -1$) propagating elliptic snoid wave u_5 with $k = 1,\ n = 1,\ \eta = 0,\ Q = 0.1$ and $N^2 = 1$.

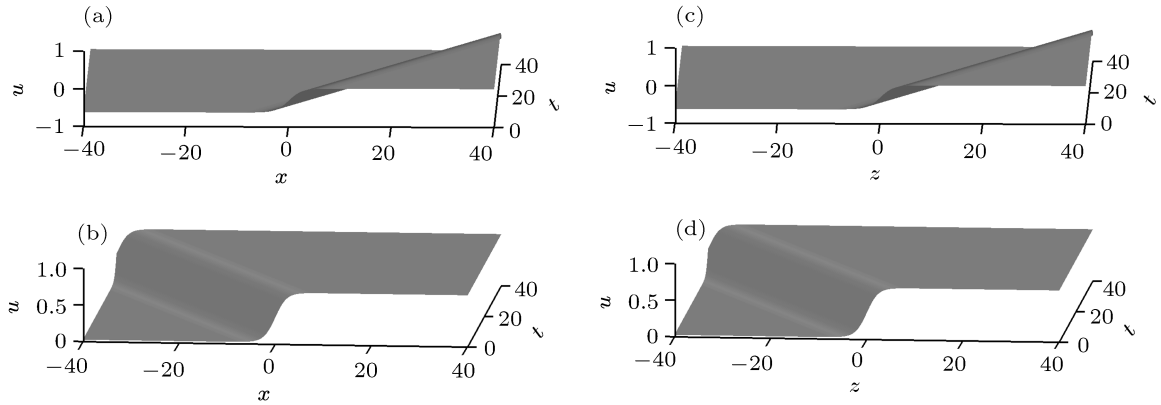


Fig.4. (a) Easterly ($\sigma = 1$), (b) westerly ($\sigma = -1$), (c) upward ($\sigma = 1$) and (d) downward ($\sigma = -1$) propagating kink solitary wave u_2 with $k = 1,\ n = 1,\ \eta = 0,\ Q = 0.1$ and $N^2 = 1$.

Secondly, we restrict our consideration to the more interesting area of triangular solutions u_7 and u_8 with the constraint condition $\beta < 0$, namely, $N^2 < \eta$. The propagation behaviours of triangular waves are given in Figs.5 and 6.

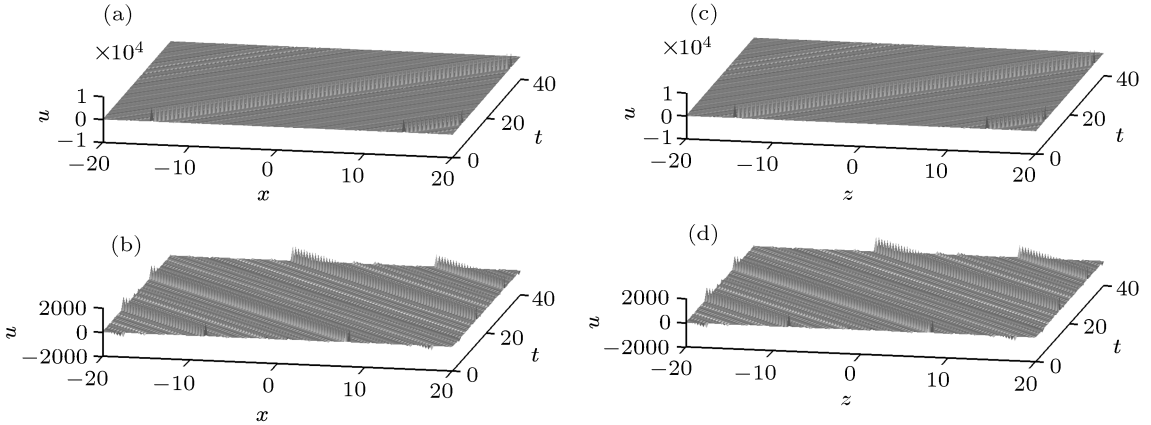


Fig.5. (a) Easterly ($\sigma = 1$), (b) westerly ($\sigma = -1$), (c) upward ($\sigma = 1$) and (d) downward ($\sigma = -1$) propagating triangular wave u_7 with $k = 1,\ n = 1,\ \eta = 0,\ Q = 1$ and $N^2 = -0.1$.

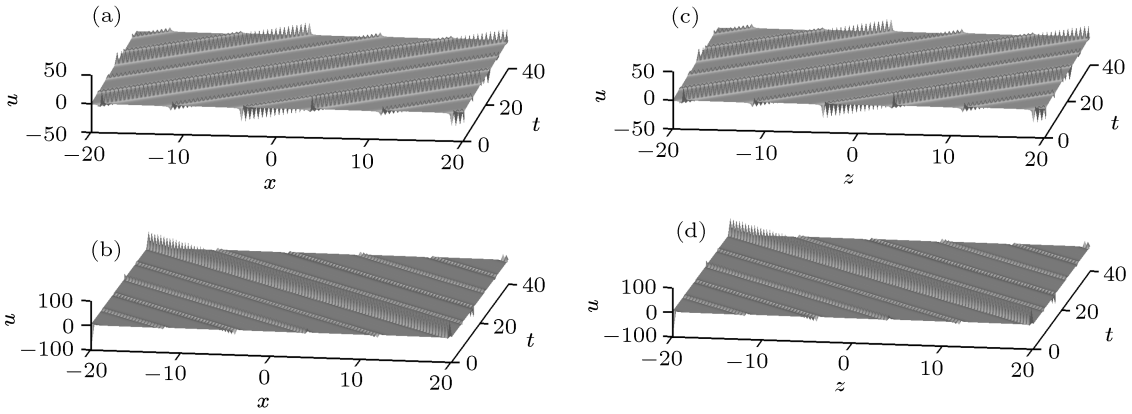


Fig.6. (a) Easterly ($\sigma = 1$), (b) westerly ($\sigma = -1$), (c) upward ($\sigma = 1$) and (d) downward ($\sigma = -1$) propagating triangular wave u_8 with $k = 1,\ n = 1,\ \eta = 0,\ Q = 0.1$ and $N^2 = -1$.

The triangular solution (u_7 with $\alpha^2 - 4\beta\gamma > 0$ or Eq.(29)), shows that, in an unstable atmosphere ($N^2 < \eta$), sensible heating ($Q_0 > 0$) helps the generation of downward-propagating triangular waves ($c_z < 0$), but sensible cooling ($Q_0 < 0$) conduces to the production of upward propagating triangular waves ($c_z > 0$). Considering the slope of the lines of constant phase in the x–z plane, we know that in the unstable atmosphere ($N^2 < \eta$), sensible heating ($Q_0 > 0$) helps the generation of easterly propagating waves ($c_x > 0$) if the slope of lines of the constant phase is positive, and sensible heating ($Q_0 > 0$) conduces to the production of westerly propagating waves ($c_x < 0$) if the slope of lines of the constant phase is negative. However, in the unstable atmosphere ($N^2 < \eta$), cooling ($Q_0 < 0$) helps the generation of westerly propagating waves if the slope of the lines of constant phase is positive, and cooling ($Q_0 < 0$) administers to the production of easterly propagating waves if the slope of lines of constant phase is negative.

On the other hand, there exists the triangular solution (u_8) under $\alpha^2 - 4\beta\gamma < 0$ or Eq.(30), which shows that in an unstable atmosphere ($N^2 < \eta$), heating ($Q_0 > 0$) helps the generation of upward-propagating waves ($c_z > 0$), but cooling ($Q_0 < 0$) conduces to the production of downward-propagating waves ($c_z < 0$). On the other hand, heating ($Q_0 > 0$) helps the generation of westerly propagating waves ($c_x < 0$) if the slope of lines of constant phase is positive, and heating ($Q_0 > 0$) conduces to the generation of easterly propagating waves ($c_x > 0$) if the slope of lines of constant phase is negative.

Thirdly, we take a brief look at rational solutions (u_3), that is, the solutions are rational functions of the independent variables. It is clear that in the unstable atmosphere ($N^2 < \eta$), there is no wave under the constraint condition $\alpha^2 - 4\beta\gamma \leq 0$, namely,

$$2\sigma^2 (N^2 - \eta)^2 - n\, Q_0 \sigma (N^2 - \eta) - n^2 Q_0^2 \geq 0, \quad (31)$$

where gravity wave breaking occurs in an unstable atmosphere ($N^2 < \eta$).

Finally, there are two interesting points mathematically which should not be ignored. We have seen that the Jacobi elliptic function solutions may exist for all the $N^2 > \eta$, when $m \to 1$, u_4 is reduced to u_1 and u_5 is reduced to u_2. On the other hand, all the triangular solutions can be obtained by examining the corresponding solitary wave solutions in an appropriate limit. It is not difficult to see that the solitary wave solutions u_1 and u_2 are valid with $N^2 > \eta$, and the triangular solutions u_7 and u_8 are valid with $N^2 < \eta$. For example, if $N^2 > \eta$, the solitary wave u_1 may exist but the triangular solution u_7 does not exist. However, if $N^2 < \eta$, the triangular solution u_7 may exist but a solitary wave solution u_1 does not exist.

Most importantly, Jacobi elliptic function waves and solitary waves can sometimes be observed in atmosphere and the ocean, which often have slowly varying amplitude and period as in the case of wave train. However, when $N^2 > \eta$ is changed into $N^2 < \eta$, the solitary waves will be transformed into triangular waves with an abrupt amplitude or breaking gravity waves, which may produce damaging wind and rapid changes in aviation weather. Moderate and possibly severe turbulence may also be observed: such a wave

event will continue to pose a serious hazard to local air traffic.

5. Conclusion and discussion

Based on the incompressible non-hydrostatic Boussinesq equation, higher-order KdV–mKdV equations are derived and used to investigate gravity waves in atmosphere. By taking advantage of an auxiliary nonlinear ordinary differential equation, travelling wave solutions of the fifth-order KdV–mKdV equation are obtained, which include the Jacobi elliptic function wave solutions, bell and kink profile solitary wave solutions, triangular periodic wave solutions and singular solutions with various constraint conditions. We have shown in the above section that propagation directions of periodic gravity waves and internal solitary waves in atmosphere depend on atmospheric stability and the slope of the constant phase plane.

Due to atmospheric instability, the Jacobi elliptic function periodic wave and the solitary wave generated in a stable atmosphere can be transformed into triangular waves in an unstable atmosphere, which can cause waves to grow more rapidly and become the breaking gravity waves. Besides, the amplifying gravity waves alter the mean wind distribution in the stratosphere, which can explain several aspects of the results relating to the middle atmospheric circulation. As an example, the westerly acceleration in the stratosphere perhaps can be explained by the changes in gravity wave amplitude and periodic structure in atmosphere.

The current poor understanding of gravity waves and the acknowledged skepticism of their existence, stem from the fact that it is both difficult and tedious to conduct a study of even a single event. Numerical and observational studies of gravity waves will have to be performed to understand both the role of convective instabilities in nonlinear, large amplitude gravity waves and the role of local convective instabilities in producing turbulent modifications and trace constituents in breaking gravity waves, and the propagation and refraction of gravity waves through the observed atmosphere and ocean. The details for these cases will be investigated in our future work.

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