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On some classes of two-dimensional local models in discrete two-dimensional monatomic FPU lattice with cubic and quartic potential*

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This paper discusses the two-dimensional discrete monatomic Fermi–Pasta–Ulam lattice, by using the method of multiple-scale and the quasi-discreteness approach. By taking into account the interaction between the atoms in the lattice and their nearest neighbours, it obtains some classes of two-dimensional local models as follows: two-dimensional bright and dark discrete soliton trains, two-dimensional bright and dark line discrete breathers, and two-dimensional bright and dark discrete breather.

Keywords: two-dimensional bright and dark discrete soliton trains, two-dimensional bright and dark discrete breathers, two-dimensional bright and dark line discrete breathers

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1. Introduction

Since the discovery of intrinsic localized modes (ILMs) or discrete breathers (DBs) in Fermi–Pasta–Ulam (FPU) lattices in the late 1980s,[1,2] they have been a topic of increasing focus in view of their significant role in energy localization and transport. Discrete breathers are time-periodic and spatially localized solutions of coupled chain of nonlinear oscillators. In 1994, Mackay and Aubry[3] established the existence of stationary breathers in a broad range of lattice models. In their method breathers are obtained by continuation from the anti-continuum limit at which the network is reduced to an array of uncoupled oscillators. In this limit, breathers are found trivially, since one may take a breather for which only one oscillator is excited, and the others remain at rest.

This method is not applicable to FPU lattices of the type first investigated by Fermi *et al*[4] since these do not possess an uncoupled limit in which trivial breathers exist. Nevertheless, early analytical and numerical work indicated that FPU lattices could indeed support breathers.[5,6] Later, rigorous proofs for existence in particular FPU models were provided, such as the breather or gap breather in diatomic FPU chain,[7−11] the problem of β-FPU,[12,13] asymptotic dynamics of breather in FPU chain,[14] interaction of a kink soliton with a breather in a FPU chain,[15] bright and dark breathers in FPU lattice,[16] q-breathers and the FPU problem,[17] etc. This indicates that the discrete breather can indeed exist in the FPU system.

The papers mentioned above focus on the one-dimensional (1D) nonlinear system. However, the properties of 2D nonlinear models are gradually attracting attention.[18−21] In this paper, we discuss a discrete two-dimensional (2D) monatomic FPU lattice by using the method of multiple-scale and the quasi-discreteness approach.

This paper is organized as follows: in Section 2, we discuss a discrete 2D monatomic FPU lattice only with cubic interactive potential. In Section 3, we discuss a discrete 2D monatomic FPU lattice only with quartic interactive potential. In Section 4, we discuss a discrete 2D monatomic FPU lattice with cubic and quartic interactive potential. The paper concludes with a short summary of the main results in Section 5.

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2. The 2D dark discrete soliton trains and the 2D dark line DBs in discrete 2D monatomic FPU lattice with cubic interactive potential

The Hamiltonian of discrete 2D monatomic FPU lattice with cubic interactive potential is given by

$$\begin{aligned} H =& \sum_{l,m}\left[\frac{1}{2}\dot{u}_{l,m}+\frac{1}{2}(u_{l+1,m}-u_{l,m})^2\right. \\ &\left.+\frac{1}{3}\alpha_l(u_{l+1,m}-u_{l,m})^3\right] \\ &+\sum_{l,m}\left[\frac{1}{2}(u_{l,m+1}-u_{l,m})^2\right. \\ &\left.+\frac{1}{3}\alpha_m(u_{l,m+1}-u_{l,m})^3\right], \end{aligned} \tag{1}$$

where $u_{l,m}$ denotes the displacement of the $(l,\ m)$th atom from its equilibrium position, α_l and α_m are the parameters controlling the strength of the nonlinear coupling. The Hamiltonian (1) yields the equation of motion

$$\begin{aligned} \ddot{u}_{l,m} =& (u_{l+1,m}+u_{l-1,m}-2u_{l,m}) \\ &+\alpha_l[(u_{l+1,m}-u_{l,m})^2-(u_{l-1,m}-u_{l,m})^2] \\ &+(u_{l,m+1}+u_{l,m-1}-2u_{l,m}) \\ &+\alpha_m[(u_{l,m+1}-u_{l,m})^2-(u_{l,m-1}-u_{l,m})^2]. \end{aligned} \tag{2}$$

By using the method of multiple-scale and the quasi-discreteness approach we assume

$$\begin{aligned} u_{l,m}(t) =& \varepsilon u^{(1)}(\xi_l,\zeta_m\tau,\phi_{l,m})+\varepsilon^2u^{(2)}(\xi_l,\zeta_m\tau,\phi_{l,m}) \\ &+\varepsilon^3u^{(3)}(\xi_l,\zeta_m\tau,\phi_{l,m})+\dots \\ =& \sum_{\gamma=1}^{\infty}\varepsilon^\gamma u^{(\gamma)}(\xi_l,\zeta_m\tau,\phi_{l,m}) \\ =& \sum_{\gamma=1}^{\infty}\varepsilon^\gamma u^{(\gamma)}_{lm,lm}, \end{aligned} \tag{3}$$

where ε is a small and ordering parameter denoting the relative amplitude of the excitation and $u^{(\gamma)}_{ir,js}=u^{(\gamma)}(\xi_i,\zeta_r,\tau,\phi_{j,s})$. $\xi_l=\varepsilon(la-\lambda t)$, $\zeta_m=\varepsilon(mb-\eta t)$ and $\tau=\varepsilon^2t$ are multiple-scales slow variables. The fast variables, $\phi_{l,m}=lk_xa+mk_yb-\omega t$, representing the phase of the carrier wave, and k_x, k_y, ω are the numbers and frequency of the carrier wave, respectively. Here a, b are lattice constants and λ, η are parameters to be determined later. We have the Taylor expansion

$$\begin{aligned} u^{(\gamma)}_{l\pm1,m} =& u^{(\gamma)}_{lm,(l\pm1)m}\pm\varepsilon a\frac{\partial u^{(\gamma)}_{lm,(l\pm1)m}}{\partial\xi_n} \\ &+\frac{\varepsilon^2a^2}{2!}\frac{\partial^2u^{(\gamma)}_{lm,(l\pm1)m}}{\partial\xi_n^2}\pm\frac{\varepsilon^3a^3}{3!}\frac{\partial^3u^{(\gamma)}_{lm,(l\pm1)m}}{\partial\xi_n^3} \\ &+\dots, \\ u_{l\pm1,m} =& \sum_{\gamma=1}^{\infty}\varepsilon^\gamma\left[\sum_{\mu=0}^{\infty}\frac{1}{\mu\left(\pm\varepsilon a\dfrac{\partial}{\partial\xi_n}\right)^\mu}u^{(\gamma)}_{lm,(l\pm1)m}\right], \\ u^{(\gamma)}_{l,m\pm1} =& u^{(\gamma)}_{lm,l(m\pm1)}\pm\varepsilon b\frac{\partial u^{(\gamma)}_{lm,l(m\pm1)}}{\partial\zeta_n} \\ &+\frac{\varepsilon^2b^2}{2!}\frac{\partial^2u^{(\gamma)}_{lm,l(m\pm1)}}{\partial\zeta_n^2}\pm\frac{\varepsilon^3b^3}{3!}\frac{\partial^3u^{(\gamma)}_{lm,l(m\pm1)}}{\partial\zeta_n^3} \\ &+\dots, \\ u_{l,m\pm1} =& \sum_{\gamma=1}^{\infty}\varepsilon^\gamma\left[\sum_{\mu=0}^{\infty}\frac{1}{\mu}\left(\pm\varepsilon b\frac{\partial}{\partial\zeta_n}\right)^\mu u^{(\gamma)}_{lm,l(m\pm1)}\right]. \end{aligned} \tag{4}$$

By virtue of the method of the chain, we make a transformation as follows:

$$\frac{\mathrm{d}}{\mathrm{d}t}=\frac{\partial}{\partial t}-\lambda\varepsilon\frac{\partial}{\partial\xi_l}-\eta\varepsilon\frac{\partial}{\partial\zeta_m}+\varepsilon^2\frac{\partial}{\partial\tau}, \tag{5a}$$

$$\begin{aligned} \frac{\mathrm{d}^2}{\mathrm{d}t^2} =& \frac{\partial^2}{\partial t^2}-2\lambda\varepsilon\frac{\partial^2}{\partial t\partial\xi_l}-2\eta\varepsilon\frac{\partial^2}{\partial t\partial\zeta_m}+2\varepsilon^2\frac{\partial^2}{\partial t\partial\tau} \\ &+\lambda^2\varepsilon^2\frac{\partial^2}{\partial\xi_l^2}+2\lambda\eta\varepsilon^2\frac{\partial^2}{\partial\xi_l\partial\zeta_m}-2\lambda\varepsilon^3\frac{\partial^2}{\partial\xi_l\partial\tau} \\ &+\eta^2\varepsilon^2\frac{\partial^2}{\partial\zeta_m^2}-2\eta\varepsilon^3\frac{\partial^2}{\partial\zeta_m\partial\tau}+\varepsilon^4\frac{\partial^2}{\partial\tau^2}. \end{aligned} \tag{5b}$$

Substituting Eqs.(3), (4) and (5) into Eq.(2) we have

$$\begin{aligned} &\varepsilon\frac{\partial^2u^{(1)}_{lm,lm}}{\partial t^2}-2\lambda\varepsilon^2\frac{\partial^2u^{(1)}_{lm,lm}}{\partial t\partial\xi_l}-2\eta\varepsilon^2\frac{\partial^2u^{(1)}_{lm,lm}}{\partial t\partial\zeta_m}+2\varepsilon^3\frac{\partial^2u^{(1)}_{lm,lm}}{\partial t\partial\tau}+\lambda^2\varepsilon^3\frac{\partial^2u^{(1)}_{lm,lm}}{\partial\xi_l^2}+2\lambda\eta\varepsilon^3\frac{\partial^2u^{(1)}_{lm,lm}}{\partial\xi_l\partial\zeta_m} \\ &-2\lambda\varepsilon^4\frac{\partial^2u^{(1)}_{lm,lm}}{\partial\xi_l\partial\tau}+\eta^2\varepsilon^3\frac{\partial^2u^{(1)}_{lm,lm}}{\partial\zeta_m^2}-2\eta\varepsilon^4\frac{\partial^2u^{(1)}_{lm,lm}}{\partial\zeta_m\partial\tau}+\varepsilon^5\frac{\partial^2u^{(1)}_{lm,lm}}{\partial\tau^2}+\varepsilon^2\frac{\partial^2u^{(2)}_{lm,lm}}{\partial t^2}-2\lambda\varepsilon^3\frac{\partial^2u^{(2)}_{lm,lm}}{\partial t\partial\xi_l} \end{aligned}$$

$$
\begin{aligned}
&-2\eta\varepsilon^3\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial t\partial\zeta_m}+2\varepsilon^4\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial t\partial\tau}+\lambda^2\varepsilon^4\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial\xi_l^2}+2\lambda\eta\varepsilon^4\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial\xi_l\partial\zeta_m}-2\lambda\varepsilon^5\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial\xi_l\partial\tau}+\eta^2\varepsilon^4\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial\zeta_m^2}\\
&-2\eta\varepsilon^5\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial\zeta_m\partial\tau}+\varepsilon^6\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial\tau^2}+\varepsilon^3\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial t^2}-2\lambda\varepsilon^4\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial t\partial\xi_l}-2\eta\varepsilon^4\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial t\partial\zeta_m}+2\varepsilon^5\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial t\partial\tau}\\
&+\lambda^2\varepsilon^5\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial\xi_l^2}+2\lambda\eta\varepsilon^5\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial\xi_l\partial\zeta_m}-2\lambda\varepsilon^6\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial\xi_l\partial\tau}+\eta^2\varepsilon^5\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial\zeta_m^2}-2\eta\varepsilon^6\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial\zeta_m\partial\tau}+\varepsilon^6\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial\tau^2}\\
=&\left(\varepsilon u^{(1)}_{lm,(l+1)m}+\varepsilon u^{(1)}_{lm,(l-1)m}-2\varepsilon u^{(1)}_{lm,lm}\pm\varepsilon^2 a\frac{\partial u^{(1)}_{lm,(l\pm1)m}}{\partial\xi_l}+\frac{\varepsilon^3a^2}{2!}\frac{\partial^2 u^{(1)}_{lm,(l\pm1)m}}{\partial\xi_l^2}+\dots\right.\\
&+\varepsilon^2 u^{(2)}_{lm,(l+1)m}+\varepsilon^2 u^{(2)}_{lm,(l-1)m}-2\varepsilon^2 u^{(2)}_{lm,lm}\pm\varepsilon^3 a\frac{\partial u^{(2)}_{lm,(l\pm1)m}}{\partial\xi_l}+\frac{\varepsilon^4a^2}{2!}\frac{\partial^2 u^{(2)}_{lm,(l\pm1)m}}{\partial\xi_l^2}+\dots\\
&\left.+\varepsilon^3 u^{(3)}_{lm,(l+1)m}+\varepsilon^3 u^{(3)}_{lm,(l-1)m}-2\varepsilon^3 u^{(3)}_{lm,lm}\pm\varepsilon^4 a\frac{\partial u^{(3)}_{lm,(l\pm1)m}}{\partial\xi_l}+\frac{\varepsilon^5a^2}{2!}\frac{\partial^2 u^{(3)}_{lm,(l\pm1)m}}{\partial\xi_l^2}+\dots\right)\\
&+\alpha_l[(\varepsilon u^{(1)}_{lm,(l+1)m}-\varepsilon u^{(1)}_{lm,lm})^2-(\varepsilon u^{(1)}_{lm,(l-1)m}-\varepsilon u^{(1)}_{lm,lm})^2+\dots]\\
&+\left(\varepsilon u^{(1)}_{lm,l(m+1)}+\varepsilon u^{(1)}_{lm,l(m-1)}-2\varepsilon u^{(1)}_{lm,lm}\pm\varepsilon^2 a\frac{\partial u^{(1)}_{lm,l(m\pm1)}}{\partial\zeta_m}+\frac{\varepsilon^3a^2}{2!}\frac{\partial^2 u^{(1)}_{lm,l(m\pm1)}}{\partial\zeta_m^2}+\dots\right.\\
&+\varepsilon^2 u^{(2)}_{lm,l(m+1)}+\varepsilon^2 u^{(2)}_{lm,l(m-1)}-2\varepsilon^2 u^{(2)}_{lm,lm}\pm\varepsilon^3 a\frac{\partial u^{(2)}_{lm,l(m\pm1)}}{\partial\zeta_m}+\frac{\varepsilon^4a^2}{2!}\frac{\partial^2 u^{(2)}_{lm,l(m\pm1)}}{\partial\zeta_m^2}+\dots\\
&\left.+\varepsilon^3 u^{(3)}_{lm,l(m+1)}+\varepsilon^3 u^{(3)}_{lm,l(m-1)}-2\varepsilon^3 u^{(3)}_{lm,lm}\pm\varepsilon^4 a\frac{\partial u^{(3)}_{lm,l(m\pm1)}}{\partial\zeta_m}+\frac{\varepsilon^5a^2}{2!}\frac{\partial^2 u^{(3)}_{lm,l(m\pm1)}}{\partial\zeta_m^2}+\dots\right)\\
&+\alpha_m\left[(\varepsilon u^{(1)}_{lm,l(m+1)}-\varepsilon u^{(1)}_{lm,lm})^2-(\varepsilon u^{(1)}_{lm,l(m-1)}-\varepsilon u^{(1)}_{lm,lm})^2+\dots\right].
\end{aligned}
\tag{6}
$$

Comparing the powers of ε in Eq.(6), we can obtain the following equations:

$$
\varepsilon^1:\ \frac{\partial^2 u^{(1)}_{lm,lm}}{\partial t^2}-\left(u^{(1)}_{lm,(l+1)m}+u^{(1)}_{lm,(l-1)m}-2u^{(1)}_{lm,lm}\right)-\left(u^{(1)}_{lm,l(m+1)}+u^{(1)}_{lm,l(m-1)}-2u^{(1)}_{lm,lm}\right)=0, \tag{7a}
$$

$$
\begin{aligned}
&\varepsilon^2:\ \frac{\partial^2 u^{(2)}_{lm,lm}}{\partial t^2}-\left(u^{(2)}_{lm,(l+1)m}+u^{(2)}_{lm,(l-1)m}-2u^{(2)}_{lm,lm}\right)-\left(u^{(2)}_{lm,l(m+1)}+u^{(2)}_{lm,l(m-1)}-2u^{(2)}_{lm,lm}\right)\\
=&\ 2\lambda\frac{\partial^2 u^{(1)}_{lm,lm}}{\partial t\partial\xi_l}+2\eta\frac{\partial^2 u^{(1)}_{lm,lm}}{\partial t\partial\zeta_m}+a\frac{\partial}{\partial\xi_l}\left(u^{(1)}_{lm,(l+1)m}-u^{(1)}_{lm,(l-1)m}\right)+b\frac{\partial}{\partial\zeta_m}\left(u^{(1)}_{lm,l(m+1)}-u^{(1)}_{lm,l(m-1)}\right)\\
&+\alpha_l\left[\left(u^{(1)}_{lm,(l+1)m}-u^{(1)}_{lm,lm}+\dots\right)^2-\left(u^{(1)}_{lm,(l-1)m}-u^{(1)}_{lm,lm}+\dots\right)^2\right]\\
&+\alpha_m\left[\left(u^{(1)}_{lm,l(m+1)}-u^{(1)}_{lm,lm}\dots\right)^2-\left(u^{(1)}_{lm,l(m-1)}-u^{(1)}_{lm,lm}+\dots\right)^2\right],
\end{aligned}
\tag{7b}
$$

$$
\begin{aligned}
&\varepsilon^3:\ \frac{\partial^2 u^{(3)}_{lm,lm}}{\partial t^2}-\left(u^{(3)}_{lm,(l+1)m}+u^{(3)}_{lm,(l-1)m}-2u^{(3)}_{lm,lm}\right)-\left(u^{(3)}_{lm,l(m+1)}+u^{(3)}_{lm,l(m-1)}-2u^{(3)}_{lm,lm}\right)\\
=&\ 2\lambda\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial t\partial\xi_l}+2\eta\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial t\partial\zeta_m}-\left(\lambda^2\frac{\partial^2 u^{(1)}_{lm,lm}}{\partial\xi_l^2}+\eta^2\frac{\partial^2 u^{(1)}_{lm,lm}}{\partial\zeta_m^2}+2\lambda\eta\frac{\partial^2 u^{(1)}_{lm,lm}}{\partial\xi_l\partial\zeta_m}+2\frac{\partial^2 u^{(1)}_{lm,lm}}{\partial t\partial\tau}\right)\\
&+\left[a\frac{\partial}{\partial\xi_l}\left(u^{(2)}_{lm,(l+1)m}-u^{(2)}_{lm,(l-1)m}\right)+\frac{a^2}{2}\frac{\partial^2}{\partial\xi_l^2}\left(u^{(1)}_{lm,(l+1)m}+u^{(1)}_{lm,(l-1)m}\right)\right]\\
&+\left[b\frac{\partial}{\partial\zeta_m}\left(u^{(2)}_{lm,l(m+1)}-u^{(2)}_{lm,l(m-1)}\right)+\frac{b^2}{2}\frac{\partial^2}{\partial\zeta_m^2}\left(u^{(1)}_{lm,l(m+1)}+u^{(1)}_{lm,l(m-1)}\right)\right]\\
&+\alpha_l\left[\left(u^{(1)}_{lm,(l+1)m}-u^{(1)}_{lm,lm}+\dots\right)^2-\left(u^{(1)}_{lm,(l-1)m}-u^{(1)}_{lm,lm}+\dots\right)^2\right]\\
&+\alpha_m\left[\left(u^{(1)}_{lm,l(m+1)}-u^{(1)}_{lm,lm}\dots\right)^2-\left(u^{(1)}_{lm,l(m-1)}-u^{(1)}_{lm,lm}+\dots\right)^2\right].
\end{aligned}
\tag{7c}
$$

Equation (7a) has the same form as the vibration equation of 2D harmonic monatomic lattice. It has solution in the following form:

$$u^{(1)}_{lm,lm} = A_0(\xi_l, \zeta_m, \tau) + A(\xi_l, \zeta_m, \tau)\mathrm{e}^{\mathrm{i}\phi_{l,m}} + \text{c.c.}, \quad (8)$$

and the dispersion relation is

$$\begin{aligned}\omega^2 &= -(\mathrm{e}^{\mathrm{i}k_l a} + \mathrm{e}^{-\mathrm{i}k_l a} - 2) - (\mathrm{e}^{\mathrm{i}k_m b} + \mathrm{e}^{-\mathrm{i}k_m b} - 2)\\ &= 2(1 - \cos(k_l a)) + 2(1 - \cos(k_m b)). \end{aligned} \quad (9)$$

Substituting Eq.(8) into Eq.(7b) we have

$$\begin{aligned}&\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial t^2} - \left(u^{(2)}_{lm,(l+1)m} + u^{(2)}_{lm,(l-1)m} - 2u^{(2)}_{lm,lm}\right) - \left(u^{(2)}_{lm,l(m+1)} + u^{(2)}_{lm,l(m-1)} - 2u^{(2)}_{lm,lm}\right)\\ =& -\mathrm{i}2\omega\left(\lambda\frac{\partial A}{\partial \xi_l} + \eta\frac{\partial A}{\partial \zeta_m}\right)\cdot \mathrm{e}^{\mathrm{i}\phi_{l,m}} + \mathrm{i}2\left(a\sin(k_l a)\frac{\partial A}{\partial \xi_l} + b\sin(k_m b)\frac{\partial A}{\partial \zeta_m}\right)\cdot \mathrm{e}^{\mathrm{i}\phi_{l,m}}\\ &-\mathrm{i}8\left(\alpha_l \sin(k_l a)\sin^2\left(\frac{k_l a}{2}\right) + \alpha_m \sin(k_m a)\sin^2\left(\frac{k_m a}{2}\right)\right)A^2\mathrm{e}^{2\mathrm{i}\phi_{l,m}} + \text{c.c.}. \end{aligned} \quad (10)$$

The terms proportional to $\mathrm{e}^{\mathrm{i}\phi_{l,m}}$ on the right-hand side of Eq.(10) are the secular terms and must be eliminated. Thus we have

$$\begin{aligned}&\frac{\partial^2 u^{(2)}_{lm,lm}}{\partial t^2} - \left(u^{(2)}_{lm,(l+1)m} + u^{(2)}_{lm,(l-1)m} - 2u^{(2)}_{lm,lm}\right) - \left(u^{(2)}_{lm,l(m+1)} + u^{(2)}_{lm,l(m-1)} - 2u^{(2)}_{lm,lm}\right)\\ =& -\mathrm{i}8\left(\alpha_l \sin(k_l a)\sin^2\left(\frac{k_l a}{2}\right) + \alpha_m \sin(k_m a)\sin^2\left(\frac{k_m a}{2}\right)\right)A^2\mathrm{e}^{2\mathrm{i}\phi_{l,m}}, \end{aligned} \quad (11a)$$

$$2\left(\lambda - \frac{a}{\omega}\sin(k_l a)\right)\frac{\partial A}{\partial \xi_l} + 2\left(\eta - \frac{b}{\omega}\sin(k_m b)\right)\frac{\partial A}{\partial \zeta_m} = 0. \quad (11b)$$

From Eq.(11b), we know $\lambda = \dfrac{a}{\omega}\sin(k_l a)$, $\eta = \dfrac{b}{\omega}\sin(k_m b)$, and from Eq.(7c), we obtain

$$\begin{aligned}&\frac{\partial^2 u^{(3)}_{lm,lm}}{\partial t^2} - \left(u^{(3)}_{lm,(l+1)m} + u^{(3)}_{lm,(l-1)m} - 2u^{(3)}_{lm,lm}\right) - \left(u^{(3)}_{lm,l(m+1)} + u^{(3)}_{lm,l(m-1)} - 2u^{(3)}_{lm,lm}\right)\\ =& (a\cos(k_l a) - \lambda^2)\frac{\partial^2 A_0}{\partial \xi_l^2} + 2\lambda\eta\frac{\partial^2 A_0}{\partial \xi_l \partial \zeta_m} + (b\cos(k_m b) - \eta^2)\frac{\partial^2 A_0}{\partial \zeta_m^2} + 8\alpha_l a\sin^2(k_l a)\frac{\partial |A|^2}{\partial \xi_l}\\ &+8\alpha_m b\sin^2(k_m b)\frac{\partial |A|^2}{\partial \zeta_m} + 8\alpha_l a(\cos(k_l a) - 1)\cos k_x a\frac{\partial |A|^2}{\partial \xi_l} + 8\alpha_m b(\cos(k_m b) - 1)\cos(k_m b)\frac{\partial |A|^2}{\partial \zeta_m}\\ &+4\alpha_l \mathrm{ctg}\left(\frac{k_l a}{2}\right)(\cos(2k_l a) - 1)\sin(k_l a)|A|^2 A\mathrm{e}^{\mathrm{i}\phi_{l,m}} - 4\alpha_l \mathrm{ctg}\left(\frac{k_l a}{2}\right)(\cos(k_l a) - 1)\sin(2k_l a)|A|^2 A\mathrm{e}^{\mathrm{i}\phi_{l,m}}\\ &+4\alpha_m \mathrm{ctg}\left(\frac{k_m b}{2}\right)(\cos(2k_m b) - 1)\sin(k_m b)|A|^2 A\mathrm{e}^{\mathrm{i}\phi_{l,m}} - 4\alpha_m \mathrm{ctg}\left(\frac{k_m b}{2}\right)(\cos(k_m b) - 1)\sin(2k_m b)|A|^2 A\mathrm{e}^{\mathrm{i}\phi_{l,m}}\\ &+\left[\mathrm{i}2\omega\frac{\partial A}{\partial \tau} + (a\cos(k_l a) - \lambda^2)\frac{\partial^2 A}{\partial \xi_l^2} + 2\lambda\eta\frac{\partial^2 A}{\partial \xi_l \partial \zeta_m} + (b\cos(k_m b) - \eta^2)\frac{\partial^2 A}{\partial \zeta_m^2}\right]\mathrm{e}^{\mathrm{i}\phi_{l,m}}. \end{aligned} \quad (12)$$

Removing the secular terms on the right-hand side of Eq.(12), we have

$$2\omega\left(\alpha\frac{\partial^2 A_0}{\partial \xi_l^2} + \frac{\lambda\eta}{\omega}\frac{\partial^2 A_0}{\partial \xi_l \partial \zeta_m} + \beta\frac{\partial^2 A_0}{\partial \zeta_m^2}\right) - 8\alpha_l a\cos(k_l a)\frac{\partial |A|^2}{\partial \xi_l} - 8\alpha_m b\cos(k_m b)\frac{\partial |A|^2}{\partial \zeta_m} = 0, \quad (13a)$$

$$\mathrm{i}\frac{\partial A}{\partial \tau} + \alpha\frac{\partial^2 A}{\partial \xi_l^2} + \frac{\lambda\eta}{\omega}\frac{\partial^2 A}{\partial \xi_l \partial \zeta_m} + \beta\frac{\partial^2 A}{\partial \zeta_m^2} + \gamma|A|^2 A = 0, \quad (13b)$$

where $\gamma = \frac{4\alpha_l}{\omega}\mathrm{ctg}\left(\frac{k_l a}{2}\right)(\cos(k_l a - 1))\sin(k_l a) + \frac{4\alpha_m}{\omega}\mathrm{ctg}\left(\frac{k_m b}{2}\right)(\cos(k_m b - 1))\sin(k_m b)$, $\alpha = \frac{1}{2\omega}(a^2\cos(k_l a) - \lambda^2)$ and $\beta = \frac{1}{2\omega}(b^2\cos(k_m b) - \eta^2)$. According to the method of multiple-scale we have $A = \frac{u}{\varepsilon}$, $\xi_l = \varepsilon(la - \lambda t) = \varepsilon X_l$, $\zeta_m = \varepsilon(mb - \eta t) = \varepsilon Y_m$, $\tau = \varepsilon^2 t$. Thus Eqs.(13a) and (13b) can be written as

$$2\omega\left(\alpha\frac{\partial^2 u_0}{\partial X_l^2} + \frac{\lambda\eta}{\omega}\frac{\partial^2 u_0}{\partial X_l \partial Y_m} + \beta\frac{\partial^2 u_0}{\partial Y_m^2}\right) - 8\alpha_l a\cos(k_l a)\frac{\partial|u|^2}{\partial X_l} - 8\alpha_m b\cos(k_m b)\frac{\partial|u|^2}{\partial Y_m} = 0, \tag{14a}$$

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha\frac{\partial^2 u}{\partial X_l^2} + \frac{\lambda\eta}{\omega}\frac{\partial^2 u}{\partial X_l \partial Y_m} + \beta\frac{\partial^2 u}{\partial Y_m^2} + \gamma|u|^2 u = 0. \tag{14b}$$

Here we discuss several types of particular condition. When the FPU lattice is isotropic square lattice, we have $a = b = a_0$, $\alpha_l = \alpha_m = \alpha_0$, $\alpha = \beta = \frac{a_0^2}{8\omega}(3\cos(ka_0) - 1)$, $\frac{\lambda\eta}{\omega} = \frac{a_0^2}{4\omega}(\cos(ka_0) + 1)$ and $k_l = k_m = k$ in the direction of Brillouin zone. Thus Eqs.(14a) and (14b) can be rewritten as

$$\frac{a_0^2}{4}(3\cos(ka_0) - 1)\left(\frac{\partial^2}{\partial X_l^2} + 2\frac{\cos(ka_0) + 1}{3\cos(ka_0) - 1}\frac{\partial^2}{\partial X_l \partial Y_m} + \frac{\partial^2}{\partial Y_m^2}\right) - 8\alpha_0 a_0\cos(ka_0)\left(\frac{\partial}{\partial X_l} + \frac{\partial}{\partial Y_m}\right)|u|^2, \tag{15a}$$

$$\mathrm{i}\frac{\partial u}{\partial t} + \frac{a_0^2}{8\omega}(3\cos(ka_0) - 1)\left(\frac{\partial^2}{\partial X_l^2} + 2\frac{\cos(ka_0) + 1}{3\cos(ka_0) - 1}\frac{\partial^2}{\partial X_l \partial Y_m} + \frac{\partial^2}{\partial Y_m^2}\right)u + \gamma|u|^2 u = 0. \tag{15b}$$

When $k = \pm\frac{\pi}{2a_0}$, Eqs.(15a) and (15b) can be transformed into the following forms:

$$-\frac{a_0^2}{4}\left(\frac{\partial}{\partial X_l} - \frac{\partial}{\partial Y_m}\right)^2 u_0 = 0, \tag{16a}$$

$$\mathrm{i}\frac{\partial u}{\partial t} - \frac{a_0^2}{8\omega}\left(\frac{\partial}{\partial X_l} - \frac{\partial}{\partial Y_m}\right)^2 u + \gamma|u|^2 u = 0, \tag{16b}$$

where $\gamma = -\frac{8\alpha_0}{\omega} > 0$. Under the transformation $X_l = X(r,s) = r + s$, $Y_m = Y(r,s) = -r + s$, we have $\frac{\partial}{\partial r} = \frac{\partial}{\partial X_l}\frac{\partial X_l}{\partial r} + \frac{\partial}{\partial Y_m}\frac{\partial Y_m}{\partial r} = \left(\frac{\partial}{\partial X_l} - \frac{\partial}{\partial Y_m}\right)$, $\frac{\partial}{\partial s} = \frac{\partial}{\partial X_l}\frac{\partial X_l}{\partial s} + \frac{\partial}{\partial Y_m}\frac{\partial Y_m}{\partial s} = \left(\frac{\partial}{\partial X_l} + \frac{\partial}{\partial Y_m}\right)$. Equations (16a) and (16b) can be written as

$$-\frac{a_0^2}{4}\frac{\partial^2 u_0}{\partial r^2} = 0, \tag{17a}$$

$$\mathrm{i}\frac{\partial u}{\partial s} - \frac{a_0^2}{8\omega}\frac{\partial^2 u}{\partial r^2} + \gamma|u|^2 u = 0. \tag{17b}$$

If the initial state of lattice is static, from Eq.(17a) we can obtain $u_0 = 0$. From Eq.(17b) we can assume that its solution has the form of $u = u(r)\mathrm{e}^{-\mathrm{i}s}$. Thus we have

$$u(r,s,t) = \sqrt{c}\tanh\left[\frac{4}{a_0}\sqrt{|\alpha_0|c}(r - vt)\right] \times \mathrm{e}^{-\mathrm{i}\frac{4\omega v}{a_0^2}(r - vt) + \mathrm{i}\omega t - \mathrm{i}s}. \tag{18}$$

Substituting the transformation $r = \frac{X_l - Y_m}{2}$, $s = \frac{X_l + Y_m}{2}$ into Eq.(18) we have

$$u(l,m,t) = \sqrt{c}\tanh\{\delta_0[(l - m)a_0 - vt]\} \times \mathrm{e}^{-\mathrm{i}(\delta_1 l a_0 + \delta_2 m a_0 - \delta_3 t)}, \tag{19}$$

where c and v are integration parameters and $\delta_0 = \frac{2}{a_0}\sqrt{2|\alpha_0|c}$, $\delta_1 = \frac{1}{2} + \frac{2\omega v}{a_0^2}$, $\delta_2 = \frac{1}{2} - \frac{2\omega v}{a_0^2}$, $\delta_3 = \omega + \frac{4\omega v^2}{a_0^2} + \frac{a_0^2}{8\omega}$.

The solution(19) can be shown in Fig.1.

From Fig.1, we know that when $k = \pm\pi/2a_0$, the 2D discrete monatomic FPU lattice with cubic interactive potential has the 2D dark discrete soliton trains and the 2D dark line DBs. They move along the direction of $k_l = k_m = k$ in Brillouin zone.

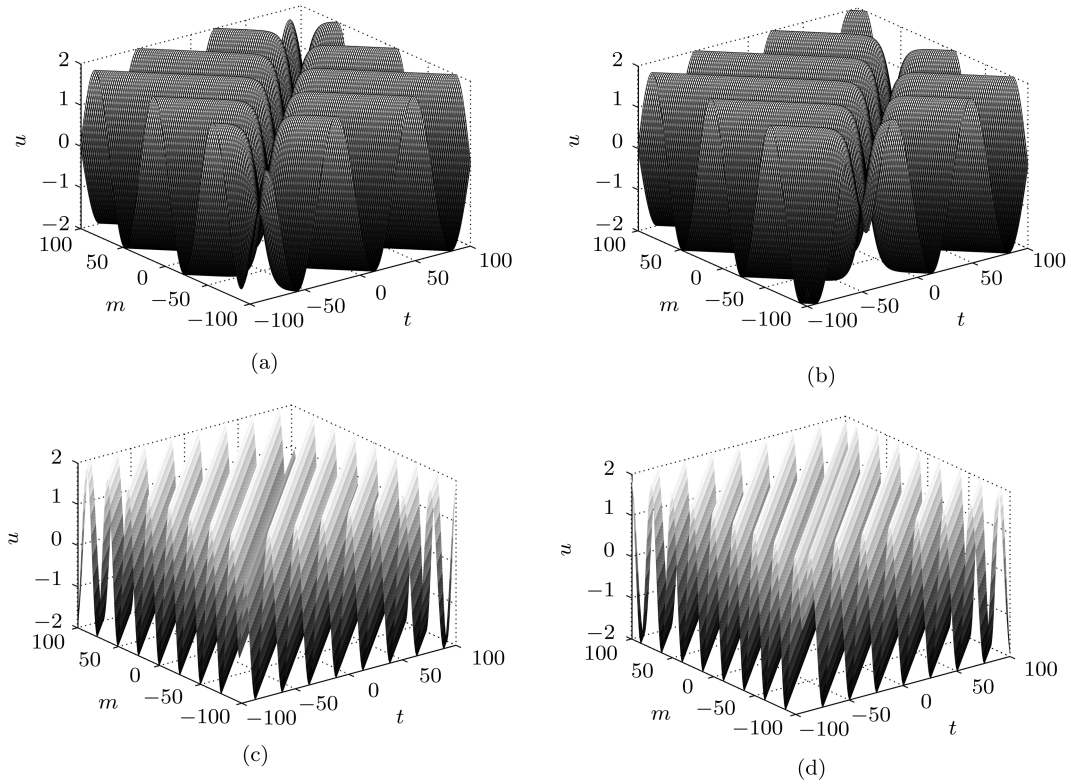


Fig.1. Space–time evolution of the 2D dark discrete soliton trains and 2D dark line DBs, (a) $\delta_2 > 0$, $t = 0$, (b) $\delta_2 > 0$, $t = 4$, (c) $\delta_2 < 0$, $t = 0$, (d) $\delta_2 < 0$, $t = \dfrac{T}{2}$.

3. The 2D bright discrete soliton trains and the 2D bright line DBs in discrete 2D monatomic FPU lattice with quartic interactive potential

The Hamiltonian of discrete 2D monatomic FPU lattice with quartic interactive potential is given by

$$H = \sum_{l,m}\left[\frac{1}{2}\dot{u}_{l,m} + \frac{1}{2}(u_{l+1,m} - u_{l,m})^2 + \frac{1}{4}\beta_l(u_{l+1,m} - u_{l,m})^4\right] + \sum_{l,m}\left[\frac{1}{2}(u_{l,m+1} - u_{l,m})^2 + \frac{1}{4}\beta_m(u_{l,m+1} - u_{l,m})^4\right], \tag{20}$$

where $u_{l,m}$ denotes the displacement of the $(l,\ m)$th atom from its equilibrium position, β_l and β_m are the parameters controlling the strength of the nonlinear coupling. The Hamiltonian (1) yields the equation of motion:

$$\ddot{u}_{l,m} = (u_{l+1,m} + u_{l-1,m} - 2u_{l,m}) + \beta_l[(u_{l+1,m} - u_{l,m})^3 + (u_{l-1,m} - u_{l,m})^3] + (u_{l,m+1} + u_{l,m-1} - 2u_{l,m}) + \beta_m[(u_{l,m+1} - u_{l,m})^3 + (u_{l,m-1} - u_{l,m})^3]. \tag{21}$$

By the same procedure as above, we obtain

$$2\omega\left(\alpha\frac{\partial^2 u_0}{\partial X_l^2} + \frac{\lambda\eta}{\omega}\frac{\partial^2 u_0}{\partial X_l \partial Y_m} + \beta\frac{\partial^2 u_0}{\partial Y_m^2}\right) = 0, \tag{22a}$$

$$\mathrm{i}\frac{\partial u}{\partial t} + \alpha\frac{\partial^2 u}{\partial X_l^2} + \frac{\lambda\eta}{\omega}\frac{\partial^2 u}{\partial X_l \partial Y_m} + \beta\frac{\partial^2 u}{\partial Y_m^2} + \gamma|u|^2 u = 0. \tag{22b}$$

When the FPU lattice is an isotropic square lattice, we have $a = b = a_0$, $\beta_l = \beta_m = \beta_0$, $\alpha = \beta = \dfrac{a_0^2}{8\omega}(3\cos(ka_0) - 1)$, $\dfrac{\lambda\eta}{\omega} = \dfrac{a_0^2}{4\omega}(\cos(ka_0) + 1)$ and $k_x = k_y = k$ in the direction of the Brillouin zone. Thus

Eqs.(22a) and (22b) can be transformed into

$$\frac{a_0^2}{4}(3\cos(ka_0)-1)\left(\frac{\partial^2}{\partial X_l^2}+2\frac{\cos(ka_0)+1}{3\cos(ka_0)-1}\frac{\partial^2}{\partial X_l\partial Y_m}+\frac{\partial^2}{\partial Y_m^2}\right)=0, \tag{23a}$$

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{a_0^2}{8\omega}(3\cos(ka_0)-1)\left(\frac{\partial^2}{\partial X_l^2}+2\frac{\cos(ka_0)+1}{3\cos(ka_0)-1}\frac{\partial^2}{\partial X_l\partial Y_m}+\frac{\partial^2}{\partial Y_m^2}\right)u+\gamma|u|^2u=0. \tag{23b}$$

When $k=\pm\dfrac{\pi}{2a_0}$, Eqs.(23a) and (23b) can be rewritten as

$$-\frac{a_0^2}{4}\left(\frac{\partial}{\partial X_l}-\frac{\partial}{\partial Y_m}\right)^2u_0=0, \tag{24a}$$

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{a_0^2}{8\omega}\left(\frac{\partial}{\partial X_l}-\frac{\partial}{\partial Y_m}\right)^2u+\gamma|u|^2u=0, \tag{24b}$$

here $\gamma=-\dfrac{24K\beta_0}{M\omega}<0$, under the transformation $X_l=X(r,s)=r+s$, $Y_m=Y(r,s)=-r+s$, we have $\dfrac{\partial}{\partial r}=\dfrac{\partial}{\partial X_l}\dfrac{\partial X_l}{\partial r}+\dfrac{\partial}{\partial Y_m}\dfrac{\partial Y_m}{\partial r}=\left(\dfrac{\partial}{\partial X_l}-\dfrac{\partial}{\partial Y_m}\right)$, $\dfrac{\partial}{\partial s}=\dfrac{\partial}{\partial X_l}\dfrac{\partial X_l}{\partial s}+\dfrac{\partial}{\partial Y_m}\dfrac{\partial Y_m}{\partial s}=\left(\dfrac{\partial}{\partial X_l}+\dfrac{\partial}{\partial Y_m}\right)$. Thus Eqs.(24a) and (24b) can be transformed into

$$-\frac{a_0^2}{4}\frac{\partial^2u_0}{\partial r^2}=0, \tag{25a}$$

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{a_0^2}{8\omega}\frac{\partial^2u}{\partial r^2}+\gamma|u|^2u. \tag{25b}$$

If the initial state of lattice is static, then from Eq.(25a) we can obtain $u_0=0$, and from Eq.(25b) we can assume that its solution has the form of $u=u(r)\mathrm{e}^{-\mathrm{i}s}$. Thus we have

$$u(r,s,t)=\sqrt{c}\,\mathrm{sech}\left[\frac{4}{a_0}\sqrt{6\beta_0c}(r-vt)\right]\times\mathrm{e}^{-\mathrm{i}\frac{4\omega v}{a_0^2}(r-vt)+\mathrm{i}\omega t-\mathrm{i}s}. \tag{26}$$

Substituting the transformation $r=\dfrac{X_l-Y_m}{2}$, $s=\dfrac{X_l+Y_m}{2}$ into Eq.(26) we have

$$u(l,m,t)=\sqrt{c}\,\mathrm{sech}\{\delta_0[(l-m)a_0-vt]\}\times\mathrm{e}^{-\mathrm{i}(\delta_1la_0+\delta_2ma_0-\delta_3t)}, \tag{27}$$

where c and v are integration parameters and $\delta_0=\dfrac{2}{a_0}\sqrt{6\beta_0c}$, $\delta_1=\dfrac{1}{2}+\dfrac{2\omega v}{a_0^2}$, $\delta_2=\dfrac{1}{2}-\dfrac{2\omega v}{a_0^2}$, $\delta_3=\omega+\dfrac{4\omega v^2}{a_0^2}+\dfrac{a_0^2}{8\omega}$.

The solution (27) can be shown in Fig.2.

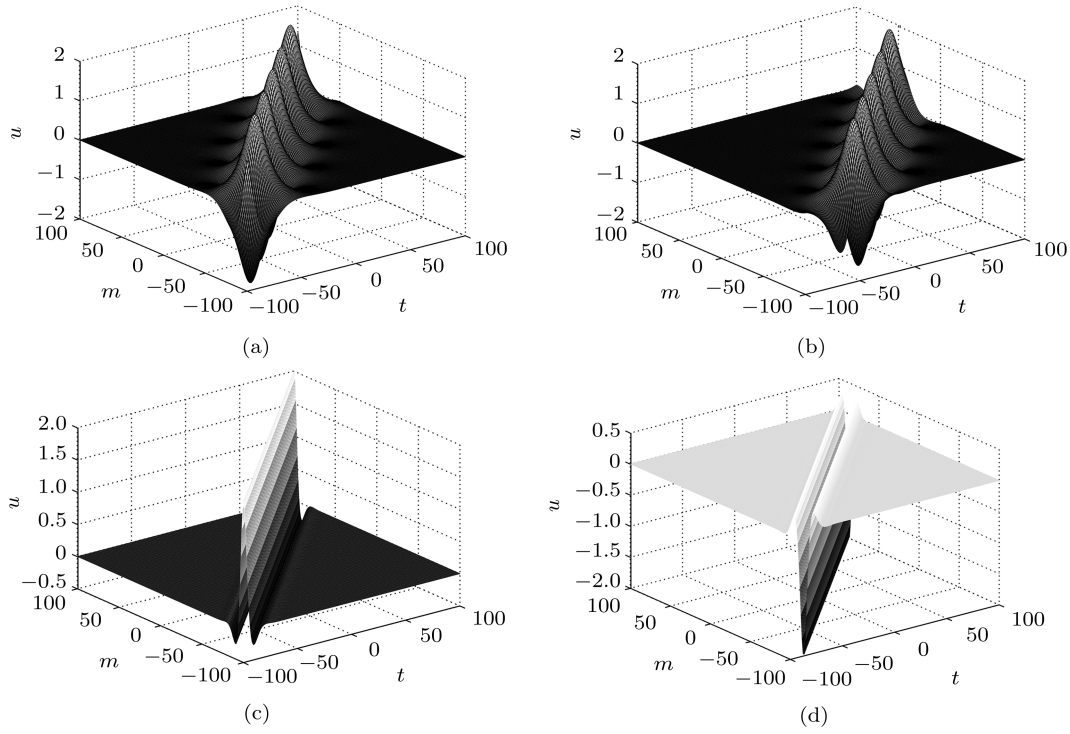


Fig.2. Space–time evolution of the 2D bright discrete soliton trains and 2D bright line DBs, (a) $\delta_2>0$, $t=0$, (b) $\delta_2>0$, $t=4$, (c) $\delta_2<0$, $t=0$, (d) $\delta_2<0$, $t=\dfrac{T}{2}$.

From Fig.2, we know that when $k = \pm\pi/2a_0$, the 2D discrete monatomic FPU lattice with quartic interactive potential has the 2D bright discrete soliton trains and the 2D bright line DBs. They move along the direction of $k_l = k_m = k$ in the Brillouin zone.

4. The 2D bright and dark DBs in discrete 2D monatomic FPU lattice with cubic and quartic interactive potential

The Hamiltonian of discrete 2D monatomic FPU lattice with cubic and quartic interactive potential is given by

$$\begin{aligned} H = & \sum_{l,m}\left[\frac{p_{l,m}^2}{2M}+\frac{1}{2}(u_{l+1,m}-u_{l,m})^2+\frac{1}{2}(u_{l,m+1}-u_{l,m})^2\right] \\ & +\sum_{l,m}\left[\frac{1}{3}\alpha_l(u_{l+1,m}-u_{l,m})^3+\frac{1}{3}\alpha_m(u_{l,m+1}-u_{l,m})^3\right] \\ & +\sum_{l,m}\left[\frac{1}{4}\beta_l(u_{l+1,m}-u_{l,m})^4+\frac{1}{4}\beta_m(u_{l,m+1}-u_{l,m})^4\right], \end{aligned} \tag{28}$$

where $u_{l,m}$ denotes the displacement of the (l, m)th atom from its equilibrium position, α_l, α_m, β_l and β_m are the parameters controlling the strength of the nonlinear coupling. The Hamiltonian (1) yields the equation of motion.

$$\begin{aligned} M\ddot{u}_{l,m} = & (u_{l+1,m}+u_{l-1,m}-2u_{l,m})+(u_{l,m+1}+u_{l,m-1}-2u_{l,m})+\alpha_l[(u_{l+1,m}-u_{l,m})^2-(u_{l-1,m}-u_{l,m})^2] \\ & +\alpha_m[(u_{l,m+1}-u_{l,m})^2-(u_{l,m-1}-u_{l,m})^2]+\beta_l[(u_{l+1,m}-u_{l,m})^3+(u_{l-1,m}-u_{l,m})^3] \\ & +\beta_m[(u_{l,m+1}-u_{l,m})^3+(u_{l,m-1}-u_{l,m})^3]. \end{aligned} \tag{29}$$

By the same procedure as above, we obtain

$$\frac{a_0^2}{4}(3\cos(ka_0)-1)\left(\frac{\partial^2}{\partial X_l^2}+2\frac{\cos(ka_0)+1}{3\cos(ka_0)-1}\frac{\partial^2}{\partial X_l\partial Y_m}+\frac{\partial^2}{\partial Y_m^2}\right)-8\alpha_0 a_0\cos(ka_0)\left(\frac{\partial}{\partial X_l}+\frac{\partial}{\partial Y_m}\right)|u|^2, \tag{30a}$$

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{a_0^2}{8\omega}(3\cos(ka_0)-1)\left(\frac{\partial^2}{\partial X_l^2}+2\frac{\cos(ka_0)+1}{3\cos(ka_0)-1}\frac{\partial^2}{\partial X_l\partial Y_m}+\frac{\partial^2}{\partial Y_m^2}\right)u+\gamma|u|^2u=0, \tag{30b}$$

where

$$\begin{aligned} \gamma = & \frac{4\alpha_l}{\omega}\mathrm{ctg}\left(\frac{k_la}{2}\right)(\cos(k_la)-1)\sin(k_la)+\frac{4\alpha_m}{\omega}\mathrm{ctg}\left(\frac{k_mb}{2}\right)(\cos(k_mb)-1)\sin(k_mb) \\ & +\frac{4\beta_l}{\omega}[\sin^2(k_la)(\cos(k_la)-1)+2(\cos(k_la)-1)^2(2\cos(k_la)-1)] \\ & +\frac{4\beta_m}{\omega}[\sin^2(k_mb)(\cos(k_mb)-1)+2(\cos(k_mb)-1)^2(2\cos(k_mb)-1)]. \end{aligned}$$

When $k = \pm\dfrac{\pi}{2a_0}$, Eqs.(30a) and (30b) can be transformed into

$$-\frac{a_0^2}{4}\left(\frac{\partial}{\partial X_l}-\frac{\partial}{\partial Y_m}\right)^2u_0=0, \tag{31a}$$

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{a_0^2}{8\omega}\left(\frac{\partial}{\partial X_l}-\frac{\partial}{\partial Y_m}\right)^2u+\gamma|u|^2u=0, \tag{31b}$$

where $\gamma = -\dfrac{8(\alpha_0-3\beta_0)}{\omega}$, under the transformation $X_l = X(r,s) = r+s$, $Y_m = Y(r,s) = -r+s$, we have $\dfrac{\partial}{\partial r} = \dfrac{\partial}{\partial X_l}\dfrac{\partial X_l}{\partial r}+\dfrac{\partial}{\partial Y_m}\dfrac{\partial Y_m}{\partial r} = \left(\dfrac{\partial}{\partial X_l}-\dfrac{\partial}{\partial Y_m}\right)$, $\dfrac{\partial}{\partial s} = \dfrac{\partial}{\partial X_l}\dfrac{\partial X_l}{\partial s}+\dfrac{\partial}{\partial Y_m}\dfrac{\partial Y_m}{\partial s} = \left(\dfrac{\partial}{\partial X_l}+\dfrac{\partial}{\partial Y_m}\right)$. Thus Eqs.(33a) and (31b) can be rewritten as

$$-\frac{a_0^2}{4}\frac{\partial^2 u_0}{\partial r^2}=0, \tag{32a}$$

$$\mathrm{i}\frac{\partial u}{\partial t}+\frac{a_0^2}{8\omega}\frac{\partial^2 u}{\partial r^2}+\frac{8(\alpha_0-3\beta_0)}{\omega}|u|^2u=0. \tag{32b}$$

If $\alpha_0<3\beta_0$, Eq.(32b) has the 2D bright discrete soliton trains and the 2D bright line DBs. If $\alpha_0>3\beta_0$ Eq.(32b) has the 2D dark discrete soliton trains and the 2D dark line DBs.

When $k=\pm\dfrac{\pi}{a_0}$, Eq.(30b) can be rewritten as

$$\mathrm{i}\frac{\partial u}{\partial t}+\alpha\frac{\partial^2 u}{\partial X_l^2}+\beta\frac{\partial^2 u}{\partial Y_m^2}+\gamma|u|^2u=0, \tag{33}$$

where $\alpha=\beta=-\dfrac{a_0^2}{2\omega}$, $\gamma=-\dfrac{32(\alpha_0-6\beta_0)}{\omega}$, and Eq.(33) is a standard cubic nonlinear Schrödinger equation.

When $k=\pm\pi/a_0$, $\alpha_0>6\beta_0$, we have $\alpha\beta>0$ and $\alpha\gamma>0$, Eq.(33) has the following solutions

$$\begin{aligned}&u(X_l,Y_m,t)\\&=\pm A_0\mathrm{sech}(B_0\sqrt{(la-\lambda t)^2+(mb-\eta t)^2})\\&\quad\times\cos[la+mb+(\Omega-2\omega)t].\end{aligned} \tag{34}$$

The solution (34) can be shown in Fig.3.

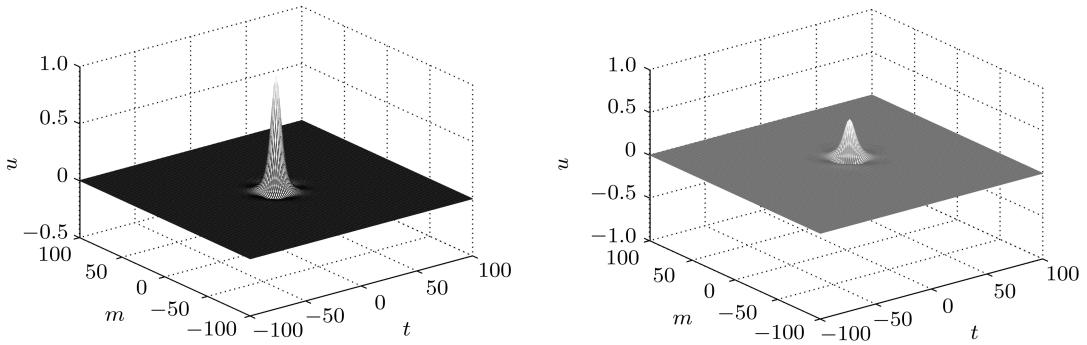


Fig.3. Space–time evolution of the 2D bright DBs, (a) $t=0$, (b) $t=\dfrac{T}{4}$.

From Fig.3, we know that when $k=\pm\pi/a_0$, $\alpha_0>6\beta_0$, the 2D discrete monatomic FPU lattice with cubic and quartic interactive potential has the 2D bright DBs, and they move along the direction of $k_l=k_m=k$ in the Brillouin zone.

When $k=\pm\pi/a_0$, $\alpha_0<6\beta_0$, we have $\alpha\beta>0$ and $\alpha\gamma<0$, Eq.(33) has the following solutions

$$\begin{aligned}&u(X_l,Y_m,t)\\&=\pm A_0\tanh(B_0\sqrt{(la-\lambda t)^2+(mb-\eta t)^2})\\&\quad\times\cos[la+mb+(\Omega-2\omega)t]\end{aligned} \tag{35}$$

The solution (35) can be shown in Fig.4.

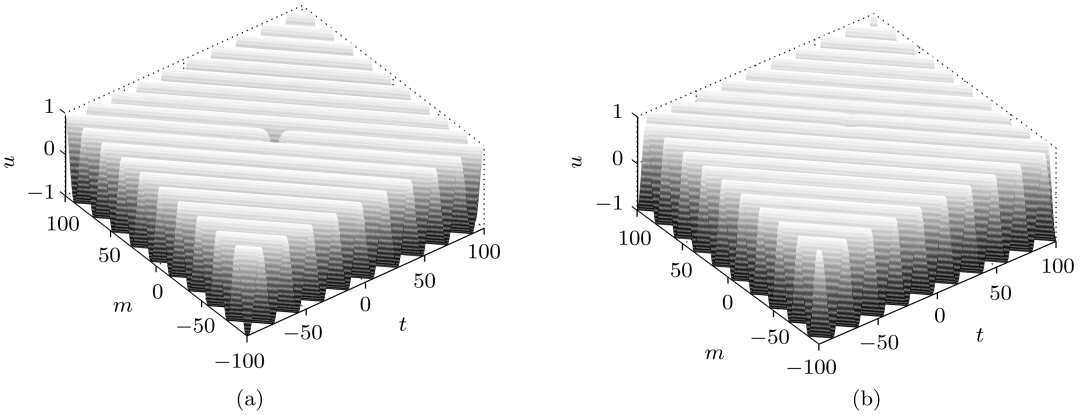


Fig.4. Space–time evolution of the 2D dark DBs, (a) $t=0$, (b) $t=\dfrac{T}{2}$.

From Fig.4, we know that when $k = \pm\pi/a_0$, $\alpha_0 < 6\beta_0$, the 2D discrete monatomic FPU lattice with cubic and quartic interactive potential has the 2D dark DBs, and they move along the direction of $k_l = k_m = k$ in the Brillouin zone.

5. Conclusions

In this paper we have investigated the 2D discrete monatomic FPU lattice only with cubic interactive potential, only with quartic interactive potential, and with cubic and quartic interactive potential, separately. We obtain some classes of 2D local models. The 2D dark discrete soliton trains and the 2D dark line DBs can exist in the 2D discrete monatomic FPU lattice only with cubic interactive potential, and with cubic and quartic interactive potential. The 2D bright discrete soliton trains and the 2D bright line DBs can exist in the 2D discrete monatomic FPU lattice only with quartic interactive potential, and with cubic and quartic interactive potential. The 2D bright and dark DBs can exist in the 2D discrete monatomic FPU lattice with cubic and quartic interactive potential. The 2D bright and dark discrete soliton trains and the 2D bright and dark line DBs have not appeared in Refs.[1–21], they are new kinds of discrete models, and only exist in the 2D discrete system. This is a difference between the 1D discrete system and the 2D discrete system, because the bright and dark DBs can also exist in 1D discrete system.[16,22]

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