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Threshold property of a nematic liquid crystal cell with two grating surface substrates*

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A grating surface can drive the liquid crystal molecules to orientate along the direction parallel or vertical to the projected plane of the grating surface. The nematic liquid crystal (NLC) cell manufactured with two pre-treated grating surface substrates may realize the vertical display, parallel display and twist display. In this paper, the threshold property of this NLC cell is investigated systematically. With the Frank elastic theory and the equivalent anchoring energy formula of grating surface substrate, the analytic expressions of the threshold voltage related to three displays are obtained, which are dependent on their geometrical parameters such as amplitude δ and pitch λ of the grating surface substrate. For a certain anchoring strength, the threshold voltage increases or decreases with the increase of the value δ/λ of the different displays.

Keywords: grating surface, equivalent anchoring energy formula, threshold property, threshold voltage

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1. Introduction

It is well known that the effective utility of liquid crystal in displays relies heavily on the fact that liquid crystal molecules can be controlled by an external influence, including the anchoring action and an electric field or a magnetic one, etc., and the deformation of them can quickly disappear after removing the external influences. The anchoring properties of the liquid crystal in contact with a solid substrate are a hot problem all the time.[1−15] Polymers such as polyamide are found to produce the planar orientation (liquid crystal director parallel to the surface),[1] amphiphilic molecules such as lecithin[2] or silane[3] yield the homeotropic orientation (liquid crystal director perpendicular to the surface), and a linearly polarized ultraviolet radiation can be used to gain an arbitrary oblique angle by the photo-induced method.[4−6]

Additionally, the anchoring properties can still be associated with the geometrical figuration of the substrate. For example, the study on the anchoring properties of grating surface substrate[7−14] has been always developing since Berreman proposed the model of grating surface substrate to discuss the problem on solid surface shape and the alignment of an adjacent nematic liquid crystal in 1972.[15] In our previous work, on the basis of the interaction potential between nematic liquid crystal (NLC) molecules and the molecules of the substrate surface and the increased elastic strain energy, we have derived the equivalent anchoring energy formula as follows:[13,14]

$$g_{\mathrm{s}} = \frac{1}{2}W_1\left(\boldsymbol{n}\cdot\boldsymbol{j}\right)^2 + \frac{1}{2}W_2\left(\boldsymbol{n}\cdot\boldsymbol{i}\right)^2, \qquad (1)$$

where W_1 and W_2 are the equivalent anchoring strengths and are dependent on the geometrical parameters, amplitude δ and pitch λ, of the grating surface substrate. By using this formula, the grating surface can be replaced by its projected plane and its anchoring properties can be described by the equivalent anchoring energy formula. In the following discussion, this formula is used to analyse the anchoring energy of grating surface substrate.

2. Fundamental equation

The structure of a liquid crystal cell with two grating surface substrates is shown in Fig.1. We in-

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troduce a Cartesian coordinate system with its origin at the projected plane of the grating surface of lower substrate, and the z axis is along the direction vertical to the projected plane, the x axis along the direction vertical to the groove, and the y axis along the direction parallel to the groove. The thickness l denotes the spacing between the two projected planes of the grating surface substrate. Suppose that a nematic liquid crystal (NLC) is filled between two grating surface substrates. The director $\boldsymbol{n}$ can be described by the tilt angle θ and the twist angle ϕ, i.e. $\boldsymbol{n} = (\cos\theta\cos\phi, \cos\theta\sin\phi, \sin\theta)$. For simplifying the problem, we suppose that the tilt angle θ and the twist angle ϕ are only function of coordinate z. An external voltage U is applied on the NLC cell and the lower substrate is earthed.

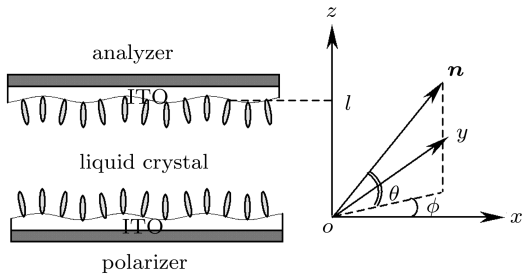


Fig.1. Structure of a liquid crystal cell and the coordinate system.

The NLC cell can be regarded as a sealed system and the Gibbs free energy of this system contains three parts as follows: Frank elastic free energy, dielectric free energy and anchoring energy of the two grating surface substrates. Elastic free energy density proposed by Frank is expressed as

$$g_{\text{elastic}} = \frac{1}{2}k_{11}(\nabla\cdot\boldsymbol{n})^2 + \frac{1}{2}k_{22}(\boldsymbol{n}\cdot\nabla\times\boldsymbol{n})^2 + \frac{1}{2}k_{33}(\boldsymbol{n}\times\nabla\times\boldsymbol{n})^2, \quad (2)$$

where $k_{ii}(i=1,2,3)$ denotes Frank constants. Frank elastic free energy can be written as

$$G_{\text{elastic}} = S\int_0^l \left\{\frac{1}{2}f(\theta)\left(\frac{\mathrm{d}\theta}{\mathrm{d}z}\right)^2 + \frac{1}{2}h(\theta)\left(\frac{\mathrm{d}\phi}{\mathrm{d}z}\right)^2\right\}\mathrm{d}z, \quad (3)$$

where $f(\theta) = k_{11}\cos^2\theta + k_{33}\sin^2\theta$, $h(\theta) = \cos^2\theta\left(k_{22}\cos^2\theta + k_{33}\sin^2\theta\right)$, S denotes the area of the projected plane of the grating surface substrate. Dielectric free energy density induced by an external voltage is

$$g_{\text{dielectric}} = -\frac{1}{2}\boldsymbol{D}\cdot\boldsymbol{E}, \quad (4)$$

where $\boldsymbol{D} = \varepsilon_\perp\boldsymbol{E} + \Delta\varepsilon(\boldsymbol{n}\cdot\boldsymbol{E})\boldsymbol{n}$, $\Delta\varepsilon = \varepsilon_{//} - \varepsilon_\perp$ is the anisotropy of the permittivity, $\varepsilon_{//}$ and $\varepsilon_\perp$ are the permittivities along and across the axis of the liquid crystal, respectively. Dielectric free energy can be expressed as

$$G_{\text{dielectric}} = S\int_0^l \left\{-\frac{1}{2}\frac{D_z^2}{g(\theta)}\right\}\mathrm{d}z, \quad (5)$$

where $g(\theta) = \varepsilon_\perp + \Delta\varepsilon\sin^2\theta$, $D_z = -U/\int_0^l \frac{\mathrm{d}z}{g(\theta)}$. Anchoring energy at the two surfaces of a grating substrate can be calculated by using Eq.(1)

$$G_{\text{surface}} = S\left[\frac{1}{2}\cos^2\theta_0\left(W_{10}\sin^2\phi_0 + W_{20}\cos^2\phi_0\right) + \frac{1}{2}\cos^2\theta_l\left(W_{1l}\sin^2\phi_l + W_{2l}\cos^2\phi_l\right)\right]. \quad (6)$$

Where W_{10} and W_{20} are the equivalent anchoring strengths for the lower substrate, W_{1l} and W_{2l} are the equivalent anchoring strengths for upper substrate, θ_0 and ϕ_0 are the tilt and the twist angle at lower substrate, and θ_l and ϕ_l are the tilt and the twist angle at upper substrate. In the following calculation, we assume that $W_{10} = W_{1l} = W_1$, $W_{20} = W_{2l} = W_2$. Therefore, the total Gibbs free energy can be expressed as

$$G = G_{\text{elastic}} + G_{\text{dielectric}} + G_{\text{surface}}. \quad (7)$$

The equation and boundary condition satisfied by the tilt angle θ and twist angle ϕ can be derived by applying the variational method described in Ref.[13]. Suppose $\theta = \bar{\theta} + \alpha\xi$ and $\phi = \bar{\phi} + \beta\eta$, where $\bar{\theta}$ and $\bar{\phi}$ that make G minimum are functions of z; ξ and η are arbitrary functions of z; α and β are common variables. By substituting these two relationships into Eq.(7), then G is a function of α and β. After solving the two equations $\left.\frac{\partial G}{\partial\alpha}\right|_{\substack{\alpha=0\\\beta=0}} = 0$ and $\left.\frac{\partial G}{\partial\beta}\right|_{\substack{\alpha=0\\\beta=0}} = 0$, we can obtain the equation and boundary condition for θ and ϕ as follows:

$$\frac{1}{2}\frac{\mathrm{d}f(\theta)}{\mathrm{d}\theta}\theta'^2 - \frac{\mathrm{d}}{\mathrm{d}z}\left[f(\theta)\theta'\right] + \frac{1}{2}\frac{\mathrm{d}h(\theta)}{\mathrm{d}\theta}\phi'^2 + \frac{1}{2}D_z^2\frac{\mathrm{d}}{\mathrm{d}\theta}\left(\frac{1}{g(\theta)}\right) = 0, \quad (8)$$

$$\frac{\mathrm{d}}{\mathrm{d}z}\left[h(\theta)\phi'\right] = 0, \quad (9)$$

$$f(\theta_0)\theta_0' = -\frac{1}{2}\sin(2\theta_0)\left(W_1\sin^2\phi_0 + W_2\cos^2\phi_0\right), \quad (10)$$

$$h(\theta_0)\phi_0' = -\frac{1}{2}(W_2 - W_1)\cos^2\theta_0\sin(2\phi_0), \quad (11)$$

$$f(\theta_l)\theta_l' = \frac{1}{2}\sin(2\theta_l)\left(W_1\sin^2\phi_l + W_2\cos^2\phi_l\right), \quad (12)$$

$$h(\theta_l)\phi_l' = \frac{1}{2}(W_2 - W_1)\cos^2\theta_l\sin(2\phi_l). \quad (13)$$

3. Threshold property

The study on threshold property of this kind of the NLC cell is mainly to discuss the threshold voltage. Threshold voltage means the value of voltage on which the director of a liquid crystal starts to generate a deformation.[16] As pointed out in Refs.[13,14], the grating surface substrate can realize the vertical anchoring or planar anchoring according to different geometrical parameters such as amplitude δ and pitch λ. As a result, this kind of NLC cell with two grating surface substrates can realize the vertical display, the parallel display and the twist display. Next, we will discuss the threshold voltage in different display modes.

3.1. Threshold voltage for realizing vertical display

In this mode, we consider a negative material of liquid crystal. In the vicinity of the threshold voltage, the variations of the tilt angle and twist angle of the director are very small and thus can be ignored. Therefore, Eq.(8) can be turned into

$$f(\theta)\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} + \frac{1}{2}D_z^2\frac{\Delta\varepsilon\sin(2\theta)}{g^2(\theta)} = 0. \quad (14)$$

Multiplying the two sides of Eq.(14) with $2\mathrm{d}\theta/\mathrm{d}z$, we obtain

$$\frac{\mathrm{d}}{\mathrm{d}z}\left\{f(\theta)\theta'^2 - \frac{D_z^2}{g(\theta)}\right\} = 0, \quad (15)$$

namely

$$f(\theta)\theta'^2 - \frac{D_z^2}{g(\theta)} = C. \quad (16)$$

With an external voltage, the tilt angle of the director attains a minimum at the middle position of liquid crystal layer, i.e. when $z = l/2$, $\theta = \theta_{\min}$, and the first order derivative of tilt angle of the director equals zero, i.e. $\mathrm{d}\theta/\mathrm{d}z = 0$. By substituting this condition into Eq.(16), the constant C can be re-expressed as

$$C = -\frac{D_z^2}{\varepsilon_\perp + \Delta\varepsilon\sin^2\theta_{\min}}. \quad (17)$$

In the interval between the position of the projected plane of the lower grating surface substrate and the middle position of liquid crystal layer, the first order derivative of the tilt angle of the director is less than zero, i.e. $\mathrm{d}\theta/\mathrm{d}z < 0$. Substituting Eq.(17) into Eq.(16) and considering $D_z < 0$, we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = D_z\sqrt{\frac{\Delta\varepsilon\left(\sin^2\theta_{\min} - \sin^2\theta\right)}{\left(\varepsilon_\perp + \Delta\varepsilon\sin^2\theta\right)\left(\varepsilon_\perp + \Delta\varepsilon\sin^2\theta_{\min}\right)}}\frac{1}{\sqrt{k_{11}\cos^2\theta + k_{33}\sin^2\theta}}. \quad (18)$$

Separating the two variables θ and z in Eq.(18) and taking definite integral for two sides of this equation in the interval of $0 \le z \le l/2$, we have

$$\int_0^{\frac{l}{2}}\mathrm{d}z = \int_{\theta_0}^{\theta_{\min}}\frac{1}{D_z}\sqrt{\frac{\left(\varepsilon_\perp + \Delta\varepsilon\sin^2\theta\right)\left(\varepsilon_\perp + \Delta\varepsilon\sin^2\theta_{\min}\right)}{\Delta\varepsilon\left(\sin^2\theta_{\min} - \sin^2\theta\right)}}\sqrt{k_{11}\cos^2\theta + k_{33}\sin^2\theta}\mathrm{d}\theta. \quad (19)$$

To calculate the threshold voltage, a transformation $\cos\theta = \cos\theta_{\min}\cos\mu$ is necessary. In this case, when $\theta = \theta_0$, then $\mu = \mu_0$; when $\theta = \theta_{\min}$, then $\mu = 0$. After making this transformation, Eq.(19) can be turned into

$$-\frac{l}{2}D_z = \int_0^{\mu_0}\sqrt{\frac{\left(\varepsilon_\perp + \Delta\varepsilon\sin^2\theta_{\min}\right)\left(\varepsilon_{//} - \Delta\varepsilon\cos^2\mu\cos^2\theta_{\min}\right)}{-\Delta\varepsilon}}\sqrt{\frac{k_{33} + (k_{11} - k_{33})\cos^2\mu\cos^2\theta_{\min}}{1 - \cos^2\mu\cos^2\theta_{\min}}}\mathrm{d}\mu. \quad (20)$$

In the vicinity of the threshold voltage, the tilt angle of the director and the dielectric displacement tend to a definite value, i.e. $\theta_{\min} \to \theta_0 \to \pi/2$, $D_z \to D_{zc}$, and then Eq.(20) may be simplified to

$$-\frac{l}{2\varepsilon_{//}}D_{zc}\sqrt{\frac{-\Delta\varepsilon}{k_{33}}} = \arccos\left(\frac{\cos\theta_0}{\cos\theta_{\min}}\right). \quad (21)$$

Subsequently, by setting the cotangent to the two sides of Eq.(21) simultaneously we obtain

$$\cot\left(\frac{l}{2\varepsilon_{//}}D_{zc}\sqrt{\frac{-\Delta\varepsilon}{k_{33}}}\right)=\lim_{\theta_{\min}\to\theta_0\to\pi/2}-\frac{\cos\theta_0}{\sqrt{\sin^2\theta_0-\sin^2\theta_{\min}}}. \tag{22}$$

Under the same condition as that proposed above, Eq.(10) can be simplified to

$$D_{zc}\frac{\sqrt{-k_{33}\Delta\varepsilon}}{\varepsilon_{//}\left(W_1\sin^2\phi_0+W_2\cos^2\phi_0\right)}=\lim_{\theta_{\min}\to\theta_0\to\pi/2}-\frac{\cos\theta_0}{\sqrt{\sin^2\theta_0-\sin^2\theta_{\min}}}. \tag{23}$$

From Eqs.(22) and (23), we obtain the expression of threshold dielectric displacement

$$\cot\left(\frac{l}{2\varepsilon_{//}}D_{zc}\sqrt{\frac{-\Delta\varepsilon}{k_{33}}}\right)=D_{zc}\frac{\sqrt{-k_{33}\Delta\varepsilon}}{\varepsilon_{//}\left(W_1\sin^2\phi_0+W_2\cos^2\phi_0\right)}. \tag{24}$$

To obtain the concrete expression of threshold voltage, we must consider the relationship between threshold dielectric displacement and threshold voltage. Applying this transformation to the formula $D_z=-U/\int_0^l\frac{\mathrm{d}z}{g(\theta)}$, we have

$$U=2\int_0^{\mu_0}\sqrt{\frac{\left(\varepsilon_\perp+\Delta\varepsilon\sin^2\theta_{\min}\right)}{-\Delta\varepsilon}}\sqrt{\frac{k_{33}+(k_{11}-k_{33})\cos^2\mu\cos^2\theta_{\min}}{\left(\varepsilon_{//}-\Delta\varepsilon\cos^2\mu\cos^2\theta_{\min}\right)\left(1-\cos^2\mu\cos^2\theta_{\min}\right)}}\mathrm{d}\mu. \tag{25}$$

At the threshold voltage, Eq.(25) can be simplified to

$$U_{\mathrm{c}}=2\sqrt{\frac{k_{33}}{-\Delta\varepsilon}}\arccos\left(\frac{\cos\theta_0}{\cos\theta_{\min}}\right). \tag{26}$$

By using Eq.(21), the relationship between threshold dielectric displacement and threshold voltage can be written as

$$D_{zc}=-\frac{\varepsilon_{//}}{l}U_{\mathrm{c}}. \tag{27}$$

Substituting Eq.(27) into Eq.(24) and considering $\phi_0=0$, we obtain the practical expression of threshold voltage

$$\cot\left(\frac{1}{2}U_{\mathrm{c}}\sqrt{\frac{-\Delta\varepsilon}{k_{33}}}\right)=\frac{\sqrt{-k_{33}\Delta\varepsilon}}{W_2 l}U_{\mathrm{c}}. \tag{28}$$

3.2. Threshold voltage for realizing parallel display

In this mode, we consider a positive material of liquid crystal. The condition for the threshold voltage is the same as that mentioned in Section 3.1. Under this condition, Eq.(16) can be obtained also. However, at an external voltage, the tilt angle of the director attains a maximum at the middle position of liquid crystal layer, i.e. when $z=l/2$, we have $\theta=\theta_{\max}$. The first order derivative of tilt angle of the director still equals zero, i.e. $\mathrm{d}\theta/\mathrm{d}z=0$. Substituting this condition into Eq.(16), the constant C can be re-written as

$$C=-\frac{D_z^2}{\varepsilon_\perp+\Delta\varepsilon\sin^2\theta_{\max}}. \tag{29}$$

In the interval between the position of the projected plane of the lower grating surface substrate and the middle position of the liquid crystal layer, the first order derivative of the angle of the director is greater than zero, i.e. $\mathrm{d}\theta/\mathrm{d}z>0$. Substituting Eq.(29) into Eq.(16) and considering $D_z<0$, we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}z}=-D_z\sqrt{\frac{\Delta\varepsilon\left(\sin^2\theta_{\max}-\sin^2\theta\right)}{\left(\varepsilon_\perp+\Delta\varepsilon\sin^2\theta\right)\left(\varepsilon_\perp+\Delta\varepsilon\sin^2\theta_{\max}\right)}}\frac{1}{\sqrt{k_{11}\cos^2\theta+k_{33}\sin^2\theta}}. \tag{30}$$

Separating the two variables θ and z in Eq.(30) and carrying out definite integration of the two sides of this equation in the interval of $0\le z\le l/2$, we have

$$-\frac{l}{2}D_z=\int_{\theta_0}^{\theta_{\max}}\sqrt{\frac{\left(\varepsilon_\perp+\Delta\varepsilon\sin^2\theta\right)\left(\varepsilon_\perp+\Delta\varepsilon\sin^2\theta_{\max}\right)}{\Delta\varepsilon\left(\sin^2\theta_{\max}-\sin^2\theta\right)}}\sqrt{k_{11}\cos^2\theta+k_{33}\sin^2\theta}\mathrm{d}\theta. \tag{31}$$

To calculate the threshold voltage, a transformation $\sin\theta = \sin\theta_{\max}\sin\mu$ is necessary. In this case, when $\theta = \theta_0$, we have $\mu = \mu_0$; when $\theta = \theta_{\max}$, we have $\mu = \pi/2$. After making this transformation, Eq.(31) can be turned into

$$-\frac{l}{2}D_z = \int_{\mu_0}^{\pi/2} \sqrt{\frac{\left(\varepsilon_\perp + \Delta\varepsilon\sin^2\theta_{\max}\right)\left(\varepsilon_\perp + \Delta\varepsilon\sin^2\mu\sin^2\theta_{\max}\right)}{\Delta\varepsilon}} \sqrt{\frac{k_{11} + (k_{33} - k_{11})\sin^2\mu\sin^2\theta_{\max}}{1 - \sin^2\mu\sin^2\theta_{\max}}}\mathrm{d}\mu. \quad (32)$$

In the vicinity of threshold voltage, the tilt angle of the director and the dielectric displacement tend to a definite value, i.e. $\theta_{\max} \to \theta_0 \to 0$, $D_z \to D_{zc}$, and then Eq.(32) may be simplified

$$-\frac{l}{2}D_{zc} = \sqrt{\frac{k_{11}}{\Delta\varepsilon}}\varepsilon_\perp\left(\frac{\pi}{2} - \mu_0\right), \quad (33)$$

and the relationship between the threshold dielectric displacement and threshold voltage can be written as

$$D_{zc} = -\frac{\varepsilon_\perp}{l}U_{\mathrm{c}}. \quad (34)$$

At the same threshold voltage, considering $\phi_0 = 0$, we can simplify Eq.(10)

$$\frac{\sqrt{k_{11}\Delta\varepsilon}}{\varepsilon_\perp W_2}D_{zc} = \frac{\sin\theta_0}{\sqrt{\sin^2\theta_{\max} - \sin^2\theta_0}}. \quad (35)$$

Subsequently, we calculate the expression of the threshold voltage. First of all, set the cotangent to the two sides of Eq.(33) simultaneously; then substitute Eqs.(34) and (35) into the expression obtained in the first step and simplify it. The expression of threshold voltage in this case is finally obtained

$$\cot\left(\frac{1}{2}U_{\mathrm{c}}\sqrt{\frac{\Delta\varepsilon}{k_{11}}}\right) = -\frac{\sqrt{k_{11}\Delta\varepsilon}}{W_2 l}U_{\mathrm{c}}. \quad (36)$$

3.3. Threshold voltage for realizing twist display

In this mode, we still consider a positive material of liquid crystal. In the vicinity of the threshold voltage, the variation of the tilt angle is very small and can be ignored. Therefore, Eq.(8) can be turned into

$$f(\theta)\frac{\mathrm{d}^2\theta}{\mathrm{d}z^2} - \frac{1}{2}\frac{\mathrm{d}h(\theta)}{\mathrm{d}\theta}\phi'^2 + \frac{1}{2}\frac{D_z^2\Delta\varepsilon\sin(2\theta)}{g^2(\theta)} = 0. \quad (37)$$

From Eq.(9), we obtain

$$h(\theta)\phi' = C_1. \quad (38)$$

Substituting Eq.(38) into Eq.(37) and multiplying the two sides of Eq.(37) by $2\mathrm{d}\theta/\mathrm{d}z$, we have

$$f(\theta)\theta'^2 + \frac{C_1^2}{h(\theta)} - \frac{D_z^2}{g(\theta)} = C_2. \quad (39)$$

For an external voltage, the condition of variation of tilt angle of the director is the same as that mentioned in Section 3.2. Substituting this condition into Eq.(39), the constant C_2 can be re-written as

$$C_2 = \frac{C_1^2}{h(\theta)} - \frac{D_z^2}{\varepsilon_\perp + \Delta\varepsilon\sin^2\theta_{\max}}. \quad (40)$$

Substituting Eq.(40) into Eq.(39) and considering $D_z < 0$, we obtain

$$\frac{\mathrm{d}\theta}{\mathrm{d}z} = N^{-1/2}(\theta), \quad (41)$$

where

$$N(\theta) = \frac{f(\theta)}{C_1^2\left[\frac{1}{h(\theta_{\max})} - \frac{1}{h(\theta)}\right] - D_z^2\left[\frac{1}{g(\theta_{\max})} - \frac{1}{g(\theta)}\right]}. \quad (42)$$

Separating the two variables θ and z in Eq.(41) and carrying out definite integral of the two sides of this equation in the interval of $0 \le z \le l/2$, we have

$$\frac{l}{2} = \int_{\theta_0}^{\theta_{\max}} N^{1/2}(\theta)\,\mathrm{d}\theta. \quad (43)$$

Using the expression $\mathrm{d}\phi = \frac{\mathrm{d}\phi}{\mathrm{d}z}\frac{\mathrm{d}z}{\mathrm{d}\theta}\mathrm{d}\theta$ and assuming the difference of the twist angles at the upper and lower substrate to be $\Delta\phi$, i.e. $\Delta\phi = \phi_l - \phi_0$, we obtain

$$\frac{\Delta\phi}{2} = \int_{\theta_0}^{\theta_{\max}} \frac{C_1}{h(\theta)}N^{1/2}(\theta)\,\mathrm{d}\theta. \quad (44)$$

To calculate the threshold voltage, we make a similar transformation to that mentioned in Section 3.2. Equation (43) can be turned into

$$\frac{l}{2}=\int_{\mu_0}^{\pi/2}\left\{\frac{f(\theta)}{\dfrac{C_1^2}{h(\theta)\,h(\theta_{\max})}\left[(2k_{22}-k_{33})+(k_{33}-k_{22})\left(\sin^2\theta_{\max}+\sin^2\theta\right)\right]+\dfrac{\Delta\varepsilon D_z^2}{g(\theta_{\max})\,g(\theta)}}\right\}^{1/2}\times\frac{1}{\sqrt{1-\sin^2\theta_{\max}\sin^2\mu}}\mathrm{d}\mu. \tag{45}$$

In the vicinity of threshold voltage, the variation tendencies of the tilt angle of the director and the dielectric displacement are the same as that mentioned in Section 3.2. Then, Eq.(45) may be simplified to

$$\frac{l}{2}=\left[\frac{k_{11}}{\dfrac{C_1^2}{k_{22}^2}(2k_{22}-k_{33})+\dfrac{\Delta\varepsilon D_{zc}^2}{\varepsilon_\perp^2}}\right]^{1/2}\left(\frac{\pi}{2}-\mu_0\right), \tag{46}$$

and Eq.(44) can be simplified to

$$\frac{\Delta\phi}{2}=\frac{C_1}{k_{22}}\left[\frac{k_{11}}{\dfrac{C_1^2}{k_{22}^2}(2k_{22}-k_{33})+\dfrac{\Delta\varepsilon D_{zc}^2}{\varepsilon_\perp^2}}\right]^{1/2}\times\left(\frac{\pi}{2}-\mu_0\right). \tag{47}$$

From Eqs.(46) and (47), the constant C_1 can be determined

$$C_1=\frac{\Delta\phi\cdot k_{22}}{l}. \tag{48}$$

The relationship between threshold dielectric displacement and threshold voltage is given by Eq.(34). Under the same condition of threshold voltage, Eq.(10) can be simplified to

$$\left\{\frac{k_{11}C_1^2}{k_{22}^2}(2k_{22}-k_{33})+\frac{\Delta\varepsilon D_{zc}^2}{\varepsilon_\perp^2}\right\}^{1/2}=-\frac{\sin\theta_0\left(W_1\sin^2\phi_0+W_2\cos^2\phi_0\right)}{\sqrt{\sin^2\theta_{\max}-\sin^2\theta_0}}. \tag{49}$$

Taking cotangent of the two sides of Eq.(46), we obtain

$$\cot\frac{l\left\{\dfrac{C_1^2}{k_{22}^2}(2k_{22}-k_{33})+\dfrac{\Delta\varepsilon D_{zc}^2}{\varepsilon_\perp^2}\right\}^{1/2}}{2\sqrt{k_{11}}}=-\frac{\sin\theta_0}{\sqrt{\sin^2\theta_{\max}-\sin^2\theta_0}}. \tag{50}$$

Substituting Eqs.(34) and (48) into Eqs.(49) and (50) and comparing them, the expression of threshold voltage of this case is finally expressed as

$$\cot\frac{\left[(\Delta\phi)^2(2k_{22}-k_{33})+U_c^2\Delta\varepsilon\right]^{1/2}}{2\sqrt{k_{11}}}=\frac{\left[k_{11}(\Delta\phi)^2(2k_{22}-k_{33})+k_{11}\Delta\varepsilon U_c^2\right]^{1/2}}{l\left(W_1\sin^2\phi_0+W_2\cos^2\phi_0\right)}. \tag{51}$$

From Eq.(51), one can find that the threshold voltage for twist display is only dependent on the property of lower substrate, similar to the case of parallel display. Considering that the twist angles of the upper and lower substrates are $\phi_l=\pi/2$ and $\phi_0=0$, respectively, Eq.(51) can be simplified to

$$\cot\left\{\frac{1}{2}\left[\frac{\pi^2(2k_{22}-k_{33})}{4k_{11}}+\frac{U_c^2\Delta\varepsilon}{k_{11}}\right]^{1/2}\right\}=-\frac{\left[\dfrac{\pi^2k_{11}}{4}(2k_{22}-k_{33})+k_{11}\Delta\varepsilon U_c^2\right]^{1/2}}{W_2 l}. \tag{52}$$

3.4. Results

In this section, we investigate the dependence of the threshold voltage on the geometrical parameters of grating surface substrate. In calculation the parameters of liquid crystals used are: $k_{11}=16.7\,\mathrm{pN}$, $k_{22}=8.0\,\mathrm{pN}$, $k_{33}=15.6\,\mathrm{pN}$, $\Delta\varepsilon=\pm4.2$.[17] The thickness of liquid crystal cell is $l=4.7\,\mu\mathrm{m}$, and the pitch of grating surface substrate is $\lambda=1\,\mu\mathrm{m}$. While the specific expressions of equivalent anchoring strengths W_1 and W_2 are $W_1=A_1-4\pi^3\bar{k}\delta^2/\lambda^3$ and $W_2=A_2$, where A_1 and A_2 are anchoring strengths of grating surface substrates and $\bar{k}$ is the geometric mean value of k_{11} and k_{22}, i.e. $\bar{k}=\sqrt{k_{11}\cdot k_{22}}$. According to the two expressions for equivalent anchoring strengths, we obtain the variation curve of equivalent anchoring strength with the geometrical parameter of grating surface substrate (see Fig.2). As mentioned in Refs.[13, 14], the anchoring easy direction for a grating surface substrate may be along the direction i, j or k according to different geometrical parameters of this grating surface substrate. Hence, for appropriate geometrical parameters of the upper and lower substrates, this kind of NLC cell with two grating surface substrates can realize the vertical display, parallel display and twist display.

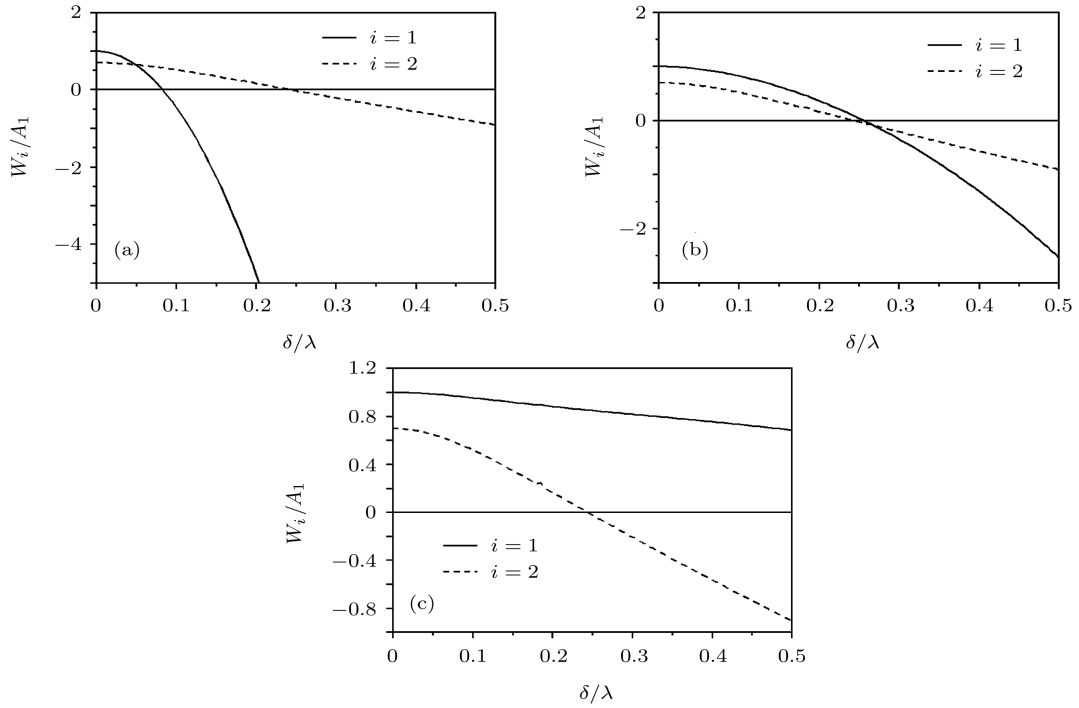
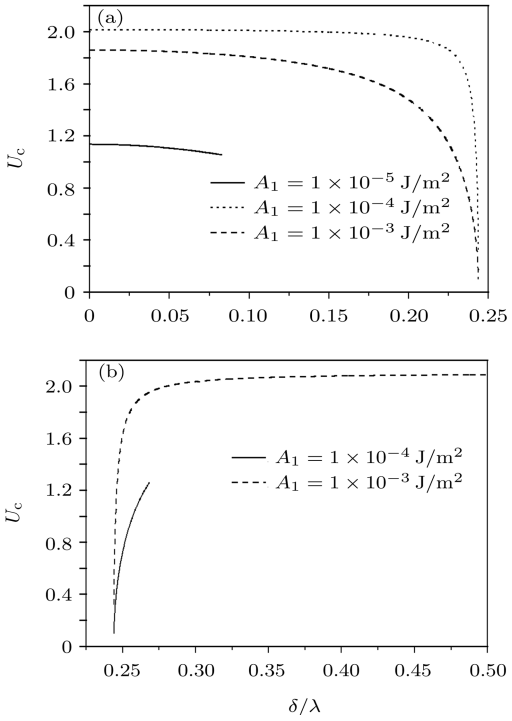


Fig.2. Variations of $W_i/A_1 (i = 1, 2)$ versus δ/λ corresponding to different anchoring strengths, where $A_2/A_1 = 0.7$. (a) $A_1 = 1 \times 10^{-5}\,\mathrm{J/m^2}$; (b) $A_1 = 1 \times 10^{-4}\,\mathrm{J/m^2}$; (c) $A_1 = 1 \times 10^{-3}\,\mathrm{J/m^2}$.

From Fig.2, we can find that the conditions for realizing the vertical display corresponding to different anchoring strengths $A_1 = 1 \times 10^{-5}\,\mathrm{J/m^2}$, $A_1 = 1 \times 10^{-4}\,\mathrm{J/m^2}$ and $A_1 = 1 \times 10^{-3}\,\mathrm{J/m^2}$ are $\delta/\lambda \leq 0.0826$, $\delta/\lambda \leq 0.2439$ and $\delta/\lambda \leq 0.2439$, respectively.

The parallel display corresponding to different anchoring strengths $A_1 = 1 \times 10^{-4}\,\mathrm{J/m^2}$ and $A_1 = 1 \times 10^{-3}\,\mathrm{J/m^2}$ are $0.244 \leq \delta/\lambda \leq 0.2683$ and $0.244 \leq \delta/\lambda \leq 0.5$, respectively. There are four combinations for realizing the twist display as follows: a) if both the anchoring strengths of the upper and lower substrates are $A_1 = 1 \times 10^{-4}\mathrm{J/m^2}$, the conditions for the upper and lower substrates are $0.2684 \leq \delta/\lambda \leq 0.5$ and $0.244 \leq \delta/\lambda \leq 0.2683$, respectively; b) if the anchoring strengths of the upper and lower substrates are $A_1 = 1 \times 10^{-4}\,\mathrm{J/m^2}$ and $A_1 = 1 \times 10^{-3}\,\mathrm{J/m^2}$, the conditions for the upper and lower substrates are $0.2684 \leq \delta/\lambda \leq 0.5$ and $0.244 \leq \delta/\lambda \leq 0.5$, respectively; c) if the anchoring strengths of the upper and lower substrates are $A_1 = 1 \times 10^{-5}\,\mathrm{J/m^2}$ and $A_1 = 1 \times 10^{-4}\,\mathrm{J/m^2}$, the conditions for the upper and lower substrates are $0.0827 \leq \delta/\lambda \leq 0.5$ and $0.244 \leq \delta/\lambda \leq 0.2683$, respectively; d) if the anchoring strengths of the upper and lower substrates are $A_1 = 1 \times 10^{-5}\,\mathrm{J/m^2}$ and $A_1 = 1 \times 10^{-3}\,\mathrm{J/m^2}$, the conditions for the upper and lower substrates are $0.0827 \leq \delta/\lambda \leq 0.5$ and $0.244 \leq \delta/\lambda \leq 0.5$, respectively. From Eqs.(28), (36) and (52) the variation of the threshold voltage U_c with the geometrical parameter of grating surface substrate δ/λ for different anchoring strengths in the three display modes is shown in Fig.3.



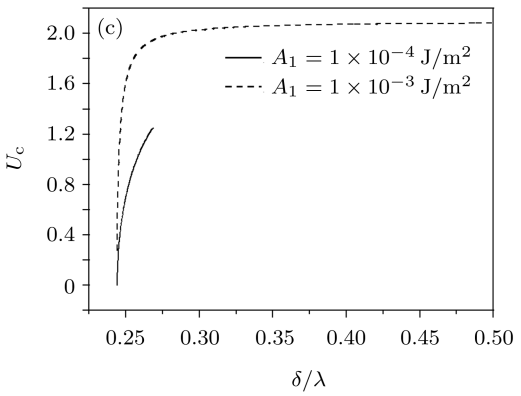


Fig.3. Variation of the threshold voltage U_c with δ/λ for different anchoring strengths in three display modes: (a) vertical display; (b) parallel display; (c) twist display.

For the twist display, we only calculate the two cases of a) and b).

From Fig.3, we see that the threshold voltage decreases with the increase of the value δ/λ for the vertical display; however, the threshold voltage increases with the increase of the value of δ/λ for the parallel and the twist display and their variation is similar.

4. Conclusion and discussion

In this paper, we discuss in detail the threshold voltage for the three displays of the NLC cell with two grating surface substrates. Three expressions of threshold voltage are deduced by using the Frank elastic theory and the equivalent anchoring energy formula of grating surface substrate. By numerical calculation, the variation of threshold voltage with geometrical parameters such as amplitude δ and pitch λ of the grating surface substrate for different cases is obtained. From these curves, one can see that the threshold voltage of this kind of cell is obviously dependent on the geometrical properties of the grating surface substrate.

In addition, in this paper we only discuss the threshold voltage of the NLC cell with two grating surface substrates in which the directions of grooves on the two substrates are the same. However, for a more general case, the direction of groove of the upper and lower substrates should be orthogonal to each other, as pointed in Ref.[10]. We will discuss this problem in another paper.

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