Vol 18 No 1, January 2009
1674-1056/2009/18(01)/0106-13

Chinese Physics B

© 2009 Chin. Phys. Soc.
and IOP Publishing Ltd

Cost and effect of pinning control for network synchronization*

Li Rong(李 嵘)†, Duan Zhi-Sheng(段志生), and Chen Guan-Rong(陈关荣)

State Key Laboratory for Turbulence and Complex Systems, Department of Mechanics and Aerospace Engineering, College of Engineering, Peking University, Beijing 100871, *China*

(Received 5 July 2008; revised manuscript received 4 September 2008)

The problem of pinning control for the synchronization of complex dynamical networks is discussed in this paper. A cost function of the controlled network is defined by the feedback gain and the coupling strength of the network. An interesting result is that a lower cost is achieved by using the control scheme of pinning nodes with smaller degrees. Some strict mathematical analyses are presented for achieving a lower cost in the synchronization of different star-shaped networks. Numerical simulations on some non-regular complex networks generated by the Barabási–Albert model and various star-shaped networks are performed for verification and illustration.

Keywords: complex dynamical network, pinning control, exponential stability
PACC: 0545

1. Introduction and problem formulation

Complex networks are currently being studied crossing many fields of sciences, including physics, chemistry, biology, mathematics, sociology and engineering.[1−10] A complex network is a large set of interconnected nodes, in which a node is a fundamental unit with specific physical contents. Examples of complex networks include the Internet, food webs, cellular neural networks, biological neural networks, electrical power grids, telephone cell graphs, etc. Recently, the synchronization of complex networks of dynamical systems has received a great deal of attention from the nonlinear dynamics community.[11−18] A special control strategy called pinning control is used to achieve the synchronization of complex networks, by which only a fraction of nodes or even a single node is controlled over the whole network.[19−21] This control method has become a common technique for the control, the stabilization and the synchronization of coupled dynamical systems that are networked together. In general, different nodes have different degrees in a network, thus a natural question is how different the effects would be when nodes with different degrees are pinned.

Consider a dynamical network composed of N identical and diffusively coupled nodes, which are an n-dimensional dynamical system each. The state equations of the network are

$$\dot{x}_i(t) = f(x_i(t), t) + c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma} x_j(t), \quad i = 1, 2, \cdots, N, \tag{1}$$

where $f(\cdot)$ is the dynamical function of an isolated node, $\boldsymbol{x}_i = (x_{i1},\ x_{i2}, \ldots,\ x_{in}) \in \boldsymbol{R}^n$ is the state variable of node i, c is the positive constant ($c > 0$) representing the coupling strength, and $\boldsymbol{\Gamma} \in \boldsymbol{R}^{N\times N}$ is the inner linking matrix. Moreover, the coupling matrix $\boldsymbol{A} = (a_{ij}) \in \boldsymbol{R}^{N\times N}$ represents the coupling configuration of the network: if there is a connection between node i and node $j (i \neq j)$, then $a_{ij} = a_{ji} = 1$; otherwise, $a_{ij} = a_{ji} = 0$ $(i \neq j)$; the diagonal entries of $\boldsymbol{A}$ are determined by

$$a_{ii} = -\sum_{\substack{j=1\\ j\neq 1}}^{N} a_{ij}, \quad i = 1, 2, \ldots, N. \tag{2}$$

The network is assumed to be connected if there exists no isolated cluster inside. Then, the coupling matrix $\boldsymbol{A}$ is irreducible. By using Lemma 2 in Ref.[18], it can be proved that zero is an eigenvalue of $\boldsymbol{A}$ with

*Project supported by the National Natural Science Foundation of China (Grant No 60674093) and the Foundation for Key Program of Ministry of Education, China (Grant No 107110).
†Corresponding author. E-mail: lirong@pku.edu.cn

multiplicity one and all the other eigenvalues of $\boldsymbol{A}$ are strictly negative.

Network (1) is said to achieve (asymptotical) synchronization if

$$x_1(t) \to x_2(t) \to \ldots \to x_N(t) \to s(t), \quad \text{as} \quad t \to \infty, \tag{3}$$

where because of the diffusive coupling configuration, $s(t)$ is a solution of an isolated node, which can be an equilibrium, a periodic or a chaotic orbit. As shown in Refs.[19–21], this can be achieved by controlling several nodes (or even only one node) of the network, depending on the network structure. Without loss of generality, suppose that the controllers are added to the last $N-k$ nodes of the network, so the equations of the controlled network can be written as

$$\begin{aligned} \dot{x}_i(t) &= f(x_i(t),t) + c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma} x_j(t), && i = 1,2,\ldots,k, \\ \dot{x}_i(t) &= f(x_i(t),t) + c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma} x_j(t) - c\varepsilon_i\boldsymbol{\Gamma}(x_i(t)-\boldsymbol{s}(t)), && i = k+1,k+2,\ldots,N, \end{aligned} \tag{4}$$

where the feedback gains ε_i are positive constants. It can be seen that synchronizing all states from $x_i(t)$ to $s(t)$ is determined by the dynamics of each isolated node, the coupling strength $c > 0$, the inner linking matrix $\boldsymbol{\Gamma}$, the feedback gains $\varepsilon_i \geq 0$, and the coupling matrix $\boldsymbol{A}$.

As discussed in Refs.[20, 21], to achieve the synchronization of complex dynamical networks, controllers are generally preferred to be added to nodes with larger degrees. However, it is also known that to achieve a certain synchronizability of the network, the feedback gains ε_i usually have to be relatively large. In Ref.[19], when a single controller is used, the coupling strength c has to be quite large in general. From the viewpoint of realistic applications, using large control gains is not expected and sometimes cannot be realized. Practically, a designed control strategy should be effective and also easily implementable. In this paper, for various star-shaped networks and non-regular complex networks, a new concept of cost function is introduced to evaluate the efficiency of the designed controllers. It is surprising to find that the cost can be much lower by controlling nodes with smaller degrees than by controlling nodes with larger degrees. As will be seen, moreover, both the feedback gains ε_i and the coupling strength c can be much smaller than those used in Refs.[19–21].

The rest of this paper is organized as follows. In Section 2, a new definition of cost function and some mathematical preliminaries are given. In Sections 3, 4 and 5, pinning control for synchronization is analysed for three different kinds of star-shaped networks, separately. In Section 6, the pinning control of non-regular complex dynamical networks of chaotic oscillators is studied numerically. In Section 7, the number of controllers required to achieve a certain network synchronizability is discussed. Finally in Section 8 conclusions are drawn from the present study.

2. The cost of pinning control

Denote $e_i(t) = x_i(t) - s(t)$, where $s(t)$ satisfies $\dot{s}(t) = f(s(t))$, then the error equations of network (1) will be written as

$$\begin{aligned} \dot{e}_i(t) =& f(x_i(t),t) - f(s(t)) + c\sum_{j=1}^{N} a_{ij}\boldsymbol{\Gamma} e_j(t), \\ & i = 1,2,\ldots,N, \end{aligned}$$

while the error equations of the controlled network (4) will be given as

$$\begin{aligned} \dot{e}_i(t) =& f(x_i(t),t) - f(s(t)) + c\sum_{j=1}^{N} \tilde{a}_{ij}\boldsymbol{\Gamma} e_j(t), \\ & i = 1,2,\ldots,N, \end{aligned} \tag{5}$$

where $\tilde{a}_{ii} = a_{ii} - \varepsilon_i$, $\varepsilon_i > 0$, $i = k+1,k+2,\ldots,N$, and $\tilde{a}_{ij} = a_{ij}$ otherwise. Letting $\tilde{A} = (\tilde{a}_{ij}) \in R^{N\times N}$, denoting $\boldsymbol{e}(t) = (e_1(t), e_2(t), \ldots, e_N(t))^{\mathrm{T}}$, and differentiating Eq.(5) with respect to $s(t)$ yields

$$\dot{\boldsymbol{e}}(t) = Df(s(t))\boldsymbol{e}(t) + c\boldsymbol{\Gamma}\boldsymbol{e}(t)\tilde{\boldsymbol{A}}^{\mathrm{T}}. \tag{6}$$

By analysing the matrix $\tilde{\boldsymbol{A}}^{\mathrm{T}}$, it is easy to see that all the eigenvalues of $\tilde{\boldsymbol{A}}^{\mathrm{T}}$ are negative, which are denoted by

$$0 > \lambda_1 \geq \ldots \geq \lambda_N.$$

Thus, there exists an orthogonal matrix $\boldsymbol{U}$ such that $\tilde{\boldsymbol{A}}^{\mathrm{T}} = \boldsymbol{UJU}^{-1}$, where $\boldsymbol{J} = \mathrm{diag}\{\lambda_1, \lambda_2, \ldots, \lambda_N\}$. Let $\tilde{\boldsymbol{e}}(t) = \boldsymbol{e}(t)\boldsymbol{U}$, then from Eq.(6) one will have

$$\dot{\tilde{e}}_i(t) = [Df(s(t)) + c\lambda_i \boldsymbol{\Gamma}]\tilde{e}_i(t), \quad i = 1, 2, \ldots, N. \quad (7)$$

Therefore, the local stability problem of network (4) is converted into a stability problem of N independent linear systems (7). When the system function of an isolated node and the inner linking matrix $\boldsymbol{\Gamma}$ are both fixed, the stability problem of systems (7) is dependent on the coupling strength c and the eigenvalues of $\tilde{\boldsymbol{A}}$. Clearly, the smaller the eigenvalues of matrix $\tilde{\boldsymbol{A}}$ ($\lambda_i < 0$, $i = 1, 2, \ldots, N$), the smaller positive coupling strength ($c > 0$) is needed to guarantee the same synchronizability of network (4), if the network has an unbounded synchronization region.[17] In the following, a cost function is introduced to describe the efficiency of the controllers.

Definition 1 (Cost Function) Suppose that the feedback gain matrix is $\boldsymbol{G} = \mathrm{diag}\{\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_N\}$, where $\varepsilon_i \geq 0$, $i = 1, \ldots, N$, are given as those in Eq.(4). A Cost Function is defined as

$$\mathrm{CF} = c\sum_{i=1}^{N} \varepsilon_i.$$

Remark 1 The smaller the CF, the more efficient a control strategy will be to achieve the same goal of control, and the more easily the implementation will be realized.

In order to discuss the effects of pinning control, the following lemmas are needed.

Lemma 2[22] Let $\boldsymbol{A} = (a_{ij}) \in \boldsymbol{C}^{n\times n}$ be Hermitian, rearrange its diagonal entries in increasing order as $a_{nn} \leq \ldots \leq a_{22} \leq a_{11}$, and let the eigenvalues of $\boldsymbol{A}$ be in the order as

$$\lambda_{\min} = \lambda_n \leq \lambda_{n-1} \leq \ldots \leq \lambda_2 \leq \lambda_1 = \lambda_{\max}, \quad (8)$$

then the following inequalities will hold:

(i) $\lambda_n \leq a_{ii} \leq \lambda_1$, for all $i = 1, \ldots, n$,

(ii) $a_{11} + a_{22} \leq \lambda_2$, if $\lambda_1 = 0$.

Remark 2 For the case where a network has an unbounded synchronization region, the smaller the λ_2, the better the synchronizability.[17] However, Lemma 2 shows that λ_2 is related to $a_{11} + a_{22}$, where a_{11} and a_{22} are determined by two smallest nodes. Therefore, in this case, in order to improve the synchronizability, these nodes with small degrees should be pinned.

Lemma 3 Let $\boldsymbol{A} \in \boldsymbol{C}^{n\times n}$ be Hermitian, and

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21}^{\mathrm{T}} & \boldsymbol{A}_2 + D \end{bmatrix},$$

where $\boldsymbol{A}_1 \in \boldsymbol{R}^{k\times k}$, $A_{12} \in \boldsymbol{R}^{k\times(N-k)}$, $A_2 \in \boldsymbol{R}^{(N-k)\times(N-k)}$ and $\boldsymbol{D} = \mathrm{diag}\{\varepsilon_{k+1}, \ldots, \varepsilon_N\}$, then $\tilde{\boldsymbol{A}} < -\alpha I$, $\alpha > 0$ will hold, if and only if $\boldsymbol{A}_1 < -\alpha \boldsymbol{I}_1$ and $\boldsymbol{A}_2 + \boldsymbol{D} - \boldsymbol{A}_{12}^{\mathrm{T}}(\boldsymbol{A}_1 + \alpha \boldsymbol{I}_1)^{-1}\boldsymbol{A}_{12} < -\alpha \boldsymbol{I}_2$, where $\boldsymbol{I}_1$ and $\boldsymbol{I}_2$ are identity matrices with appropriate dimensions.

Remark 3 Matrix $\boldsymbol{D}$, i.e. ε_i can be determined by using the linear matrix inequality (LMI) method.[23] Assume the eigenvalues of $\boldsymbol{A}_1$ to be $\beta_1 \geq \beta_2 \geq \ldots \geq \beta_k$, then $-\alpha > \beta_1$ will hold; and if $\varepsilon_i \to -\infty$, then $-\alpha \to \beta_1$.

3. Simple star-shaped networks

Now, consider a network composed of N identical nodes in a simple star-shaped coupling configuration, where there exists a central node that is connected with all other non-central nodes that are not directly linked up with each other. Such a network is shown as an example in Fig.1.

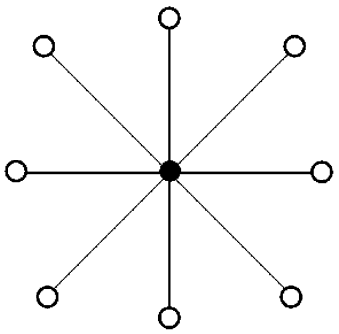


Fig.1. A simple star-shaped network.

The coupling matrix $\boldsymbol{A}$ of network (1) with a simple star-shaped coupling configuration can be written as

$$\boldsymbol{A} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_2 \end{bmatrix}, \quad (9)$$

where $\boldsymbol{A}_1 = -N + 1$, $\boldsymbol{A}_{12} = (1, 1, \ldots, 1) \in \boldsymbol{R}^{(N-1)}$, $\boldsymbol{A}_{21} = \boldsymbol{A}_{12}^{\mathrm{T}}$, and $\boldsymbol{A}_2 = -\boldsymbol{I}_{(N-1)\times(N-1)}$.

Let the feedback gain matrix be $\boldsymbol{G} = \mathrm{diag}\{0, \varepsilon, \ldots, \varepsilon\} \in \boldsymbol{R}^{N\times N}$, that is, the controllers are added to all the non-central nodes for simplicity in this

discussion, then the coupling matrix of the controlled network (4) will be

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_2 - \varepsilon I \end{bmatrix}.$$

Theorem 1 If there exist constants $\varepsilon > 0$ and $\tilde{k} > 0$, such that $N - \tilde{k} - 1 > 0$, $\varepsilon > \tilde{k} - 1$ and $\varepsilon > \dfrac{\tilde{k}(N-\tilde{k})}{N-\tilde{k}-1}$, then $\lambda_i(\tilde{A}) < -\tilde{k}$, $i = 1, \ldots, N$, where $\lambda_i(\tilde{A})$, $i = 1, \ldots, N$, are the eigenvalues of matrix $\tilde{\boldsymbol{A}}$.

Proof Consider the matrix

$$\tilde{\boldsymbol{A}} + \tilde{k}\boldsymbol{I} = \begin{bmatrix} \boldsymbol{A}_1 + \tilde{k} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & (-1-\varepsilon+\tilde{k})\boldsymbol{I} \end{bmatrix}.$$

By the assumptions of the theorem the inequality $(-1-\varepsilon+\tilde{k})\boldsymbol{I} < 0$ is obtained. Let $\bar{\boldsymbol{A}} = (\boldsymbol{A}_1 + \tilde{k}\boldsymbol{I}) - \boldsymbol{A}_{12}((-1-\varepsilon+\tilde{k})\boldsymbol{I})^{-1}\boldsymbol{A}_{21}$, then the following inequality will hold:

$$\begin{aligned} \bar{\boldsymbol{A}} =& -N+1+\tilde{k}+\frac{N-1}{1+\varepsilon-\tilde{k}} \\ <& -N+1+\tilde{k}+\frac{N-\tilde{k}-1}{N-1}(N-1) \\ =& 0. \end{aligned}$$

From the Schur complement lemma, the inequality $\tilde{\boldsymbol{A}} + \tilde{k}\boldsymbol{I} < 0$ is obtained, which leads to the assertion of the theorem.

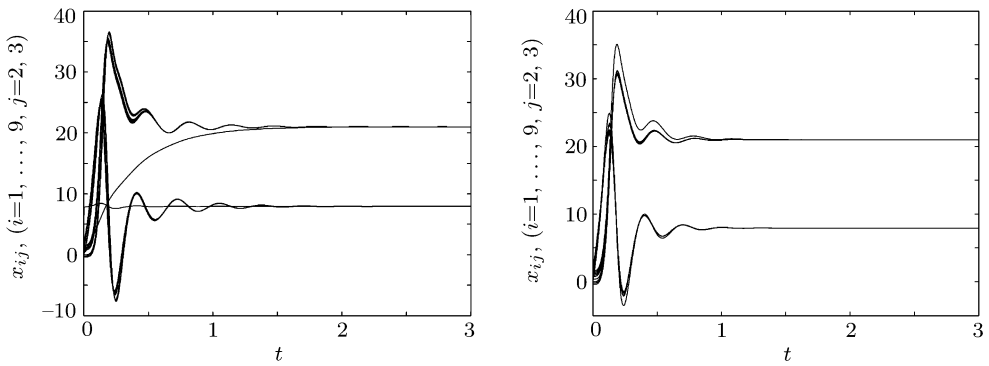
Remark 4 Theorem 1 shows that under the control strategy of adding controllers to all the non-central nodes, as the positive constant ε ($\varepsilon > 0$) increases, the eigenvalues of matrix $\tilde{\boldsymbol{A}}$ will approach $-(N-1)$. Further, suppose that $N > 2$ and take $\tilde{k} = 1$, then based on Theorem 1, to guarantee that $\lambda_i(\tilde{\boldsymbol{A}}) < -1$, $i = 1, \ldots, N$, the feedback gain ε will be required only to satisfy

$$\varepsilon > \frac{N-1}{N-2}. \tag{10}$$

In this case, if the coupling strength of network (9) is c, then the cost function is CF $= c\varepsilon(N-1)$. From inequality (10) and through direct calculations, it is easy to see that under the control strategy of adding controllers to the non-central nodes, CF $> \dfrac{(N-1)^2}{(N-2)}$ suffices to guarantee that $\lambda_i(\tilde{\boldsymbol{A}}) < -1$, $i = 1, \ldots, N$.

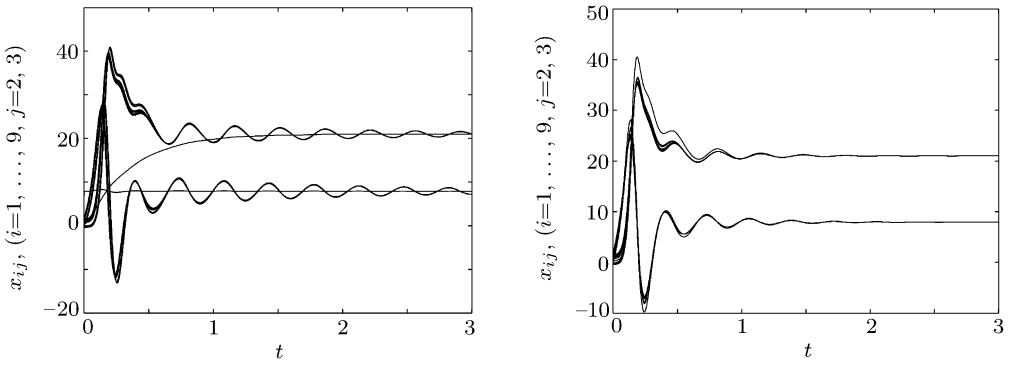
Remark 5 Consider the control strategy that a single controller is added to a central node. In this case, the feedback gain matrix is given by $\boldsymbol{G} = \mathrm{diag}\{\varepsilon, 0, \ldots, 0\} \in \boldsymbol{R}^{N\times N}$ and CF $= c\varepsilon$. Let $\lambda_1(\tilde{\boldsymbol{A}})$ be the largest eigenvalue of matrix $\tilde{\boldsymbol{A}}$, then based on Lemma 2, the inequality $\lambda_1(\tilde{\boldsymbol{A}}) \geq -1$ will hold no matter how large the ε, or equivalently, the cost function CF, is taken to be. This shows that in order to have small values of λ_i, the small degree nodes should be pinned.

Some simulations of network (9) as shown in Fig.1, are presented in Figs.2 and 3, where each node behaves as a chaotic oscillator dynamically,[24] $\boldsymbol{\Gamma} = \mathrm{diag}\{0,\ 1,\ 0\}$, and the synchronized state is set to be $\boldsymbol{s}(t) = [7.9373\ 7.9373\ 21]$, which is an unstable equilibrium point of an isolated node.[25]



(a) CF=$c\varepsilon$=300c=3000 (b) CF=$c\varepsilon(N-1)$=1.5$(N-1)c$=120

Fig.2. Synchronization of the simple star-shaped coupling network (9), where the coupling strength c is 10 (c=10), a single controller is added to the central node (a) and controllers are added to all the non-central nodes (b).



(a) CF=$c\varepsilon$=500c=3500 (b) CF=$c\varepsilon(N-1)$=1.5$(N-1)c$=84

Fig.3. Synchronization of the simple star-shaped coupling network (9), where the coupling strength c is 7 (c=7), the central node is pinned (a) and all the non-central nodes are pinned (b).

From Figs.2 and 3, it can be seen that even with both a small coupling strength c and a low cost function CF, the synchronization of network (9) can be achieved faster when the controllers are added to all the non-central nodes (nodes with smaller degrees) than when a single controller is added to the central node (the node with the largest degree). Although more controllers are needed for pinning nodes with smaller degrees, the total cost is still lower and the control effect is better.

4. Clusters of star-shaped networks with global coupling

Consider a network composed of N identical nodes in clusters of star-shaped coupling configuration, where there are k central nodes that are connected with each other and each may link up with a different number of non-central nodes; all the non-central nodes are not directly connected with each other; and any central node is not linked with any non-central node that is attached to other central node. Figure 4 gives an example of such a network for the case of k=3.

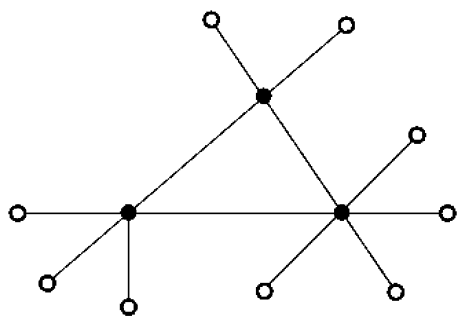


Fig.4. A cluster of star-shaped network with k=3.

Without loss of generality, let the first k nodes be central nodes, and let each of them be connected with n_i non-central nodes, with $n_1 \le n_2 \le \ldots \le n_k$. Clearly, $N = k + \sum_{i=1}^{k} n_i$. The coupling matrix $\boldsymbol{A}$ of network (1) in the clusters of such a kind can be written as

$$A = \begin{bmatrix} \boldsymbol{A}_1 & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_2 \end{bmatrix}, \tag{11}$$

where $\boldsymbol{A}_1 \in \boldsymbol{R}^{k\times k}$, $\boldsymbol{A}_{12} \in \boldsymbol{R}^{k\times(N-k)}$, $\boldsymbol{A}_{21} = \boldsymbol{A}_{12}^{\mathrm{T}} \in \boldsymbol{R}^{(N-k)\times k}$, and $\boldsymbol{A}_2 \in \boldsymbol{R}^{(N-k)\times(N-k)}$. Let $\boldsymbol{\alpha}_i = (1, 1, \ldots, 1) \in \boldsymbol{R}^{n_i}$, $i = 1, \ldots, k$, then the following expressions will hold true:

$$\boldsymbol{A}_1 = \begin{bmatrix} -k+1-n_1 & 1 & \ldots & 1 \\ 1 & -k+1-n_2 & \ldots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \ldots & -k+1-n_k \end{bmatrix},$$

$$\boldsymbol{A}_{12} = \begin{bmatrix} \alpha_1 & & & \\ & \alpha_2 & & \\ & & \ddots & \\ & & & \alpha_k \end{bmatrix},$$

$$\boldsymbol{A}_{21} = \begin{bmatrix} \alpha_1^{\mathrm{T}} & & & \\ & \alpha_2^{\mathrm{T}} & & \\ & & \ddots & \\ & & & \alpha_k^{\mathrm{T}} \end{bmatrix},$$

and

$$\boldsymbol{A}_2 = -\boldsymbol{I}_{(N-k)}.$$

Let the feedback gain matrix be $\boldsymbol{G} = \begin{bmatrix} 0 & 0 \\ 0 & \varepsilon \boldsymbol{I}_{(N-k)} \end{bmatrix}$, then the corresponding coupling matrix $\tilde{\boldsymbol{A}}$ of the con-

trolled network (4) will be

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{G}. \tag{12}$$

Theorem 2 If there exist constants $\varepsilon > 0$ and $\tilde{k} > 0$ such that $n_1 > \tilde{k}$, $\varepsilon > \tilde{k} - 1$, and $\varepsilon > \dfrac{\tilde{k}(n_1 + 1 - \tilde{k})}{n_1 - \tilde{k}}$, then $\lambda_i(\tilde{\boldsymbol{A}}) < -\tilde{k}$, $i = 1, \ldots, N$.

Proof Consider the matrix

$$\tilde{\boldsymbol{A}} + \tilde{k}\boldsymbol{I} = \begin{bmatrix} \boldsymbol{A}_1 + \tilde{k}\boldsymbol{I} & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & (-1 - \varepsilon + \tilde{k})\boldsymbol{I} \end{bmatrix}.$$

By the assumptions of the theorem, the inequality $(-1 - \varepsilon + \tilde{k})I < 0$ holds true. Let $\bar{\boldsymbol{A}} = (\boldsymbol{A}_1 + \tilde{k}\boldsymbol{I}) - \boldsymbol{A}_{12}((-1 - \varepsilon + \tilde{k})\boldsymbol{I})^{-1}\boldsymbol{A}_{21}$, then $\bar{\boldsymbol{A}}$ can be expressed as

$$\bar{\boldsymbol{A}} = \begin{bmatrix} -k + 1 - n_1 + \tilde{k} + \dfrac{n_1}{1 + \varepsilon - \tilde{k}} & 1 & \cdots & 1 \\ 1 & -k + 1 - n_2 + \tilde{k} + \dfrac{n_2}{1 + \varepsilon - \tilde{k}} & \cdots & 1 \\ \vdots & \vdots & \ddots & \vdots \\ 1 & 1 & \cdots & -k + 1 - n_k + \tilde{k} + \dfrac{n_k}{1 + \varepsilon - \tilde{k}} \end{bmatrix}.$$

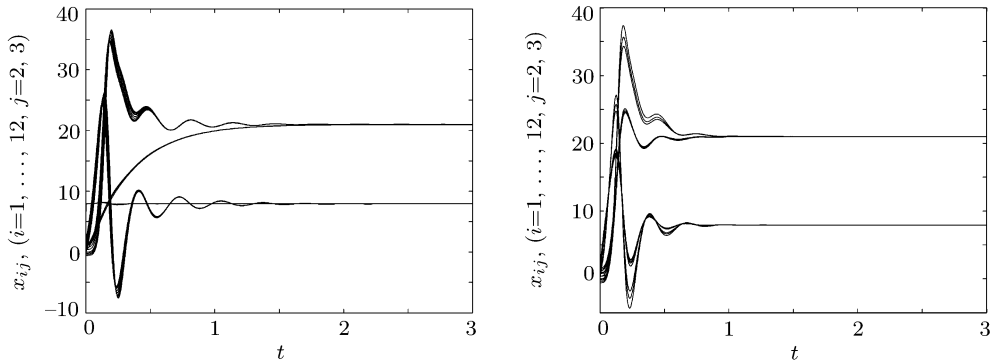
Let $m_i = -k + 1 - n_i + \tilde{k} + \dfrac{n_i}{1 + \varepsilon - \tilde{k}} + (k - 1) \cdot 1$, $i = 1, 2, \ldots, k$, then the following inequality will hold true:

$$\begin{aligned} m_i =& -n_i + \tilde{k} + \frac{n_i}{1 + \varepsilon - \tilde{k}} \\ <& -n_i + \tilde{k} + \frac{n_1 - \tilde{k}}{n_1} n_i \\ =& n_i\left(\frac{-\tilde{k}}{n_1}\right) + \tilde{k} \\ =& \tilde{k}\left(1 - \frac{n_i}{n_1}\right) \le 0. \end{aligned}$$

According to the Gerschgorin disc theorem,[22] $\bar{\boldsymbol{A}} < 0$. By the Schur complement Lemma, the inequality $\tilde{\boldsymbol{A}} + \tilde{k}\boldsymbol{I} < 0$ holds true, which leads to the assertion of the theorem.

Some simulations of network (11) as shown in Fig.4, are presented in Fig.5, where the dynamic behaviours of an isolated node, the inner liking matrix $\boldsymbol{\Gamma}$, and the synchronized state are the same as those given in Section 3.

From Fig.5, it can be seen that although all the non-central nodes are pinned, the cost is lower and the effect is better than the case where the central nodes are pinned.



(a) CF=$c\varepsilon k$=300kc=9000

(b) CF=$c\varepsilon\left(\sum_{i=1}^{k} n_i\right)$=2.5$\left(\sum_{i=1}^{k} n_i\right)c$=225

Fig.5. Synchronization of network (11), where the coupling strength c is 10 (c=10), k central nodes are pinned (a), and all the non-central nodes are pinned (b).

By using the Kronecker product, the controlled network (5) can be rewritten as

$$\dot{\boldsymbol{e}}(t) = \boldsymbol{F}(x,s) + c(\boldsymbol{A}-\boldsymbol{G}) \otimes \boldsymbol{\Gamma}\boldsymbol{e}(t), \qquad (13)$$

where $\boldsymbol{e}(t) = (e_1^{\mathrm{T}}(t), \ldots, e_N^{\mathrm{T}}(t))^{\mathrm{T}}$, $\boldsymbol{F}(x,s) = ((f(x_1(t)) - f(s(t)))^{\mathrm{T}}, \ldots, (f(x_N(t)) - f(s(t)))^{\mathrm{T}})^{\mathrm{T}}$, $\boldsymbol{G} = \mathrm{diag}\{\underbrace{0,\ldots,0}_{k},\underbrace{\varepsilon,\ldots,\varepsilon}_{N-k}\} \in \boldsymbol{R}^{N\times N}$ and $\boldsymbol{\Gamma} = \mathrm{diag}\{r_1, r_2, \ldots, r_n\}$, $r_i = 1$ or 0, $i = 1, \ldots, n$.

Theorem 3 If there exist a positive diagonal matrix $\boldsymbol{P}$ and constants $\mu > 0$, $\varepsilon > 0$, $\tilde{k} > 0$, and $c > 0$, such that $n_1 > \tilde{k}$, $\varepsilon > \tilde{k} - 1$, $\varepsilon > \dfrac{\tilde{k}(n_1 + 1 - \tilde{k})}{n_1 - \tilde{k}}$ and

$$\begin{aligned}&(x-y)^{\mathrm{T}}P(f(x,t) - f(y,t) - c\tilde{k}\boldsymbol{\Gamma}(x-y))\\ \le& -\mu(x-y)^{\mathrm{T}}(x-y),\end{aligned}$$

then the controlled network (4) is globally and exponentially synchronized to $s(t)$.

Proof Choose a Lyapunov function as $V(t) = \dfrac{1}{2}\boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N \otimes \boldsymbol{P})\boldsymbol{e}(t)$. Its time derivative is

$$\begin{aligned}\dot{V}(t) =& \boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N \otimes \boldsymbol{P})\dot{\boldsymbol{e}}(t)\\ =& \boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N \otimes \boldsymbol{P})(\boldsymbol{F}(x,s) + c(\boldsymbol{A}-\boldsymbol{G}) \otimes \boldsymbol{\Gamma}\boldsymbol{e}(t))\\ =& \boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N \otimes \boldsymbol{P})\boldsymbol{F}(x,s)\\ &+ c\boldsymbol{e}^{\mathrm{T}}(t)((\boldsymbol{A}-\boldsymbol{G}) \otimes \boldsymbol{P}\boldsymbol{\Gamma})\boldsymbol{e}(t)\\ =& \boldsymbol{e}^{\mathrm{T}}(t)(\boldsymbol{I}_N \otimes \boldsymbol{P})(\boldsymbol{F}(x,s) - c(\tilde{k}\boldsymbol{I}_n \otimes \boldsymbol{\Gamma})\boldsymbol{e}^{\mathrm{T}}(t))\\ &+ c\boldsymbol{e}^{\mathrm{T}}(t)((\boldsymbol{A}-\boldsymbol{G}) + \tilde{k}\boldsymbol{I}_n) \otimes \boldsymbol{P}\boldsymbol{\Gamma}\boldsymbol{e}(t).\end{aligned}$$

From Theorem 2, the inequality $(\boldsymbol{A}-\boldsymbol{G}) + \tilde{k}\boldsymbol{I}_n < 0$ is obtained. So,

$$\begin{aligned}\dot{V}(t) \le& -\mu\sum\nolimits_{i=1}^{N} e_i^{\mathrm{T}}(t)e_i(t)\\ \le& -\mu\sum\nolimits_{i=1}^{N} \frac{1}{\max\{P_j\}_{1\le j\le n}} e_i^{\mathrm{T}}(t)\boldsymbol{P}e_i(t)\\ =& -\frac{\mu}{\max\{P_j\}_{1\le j\le n}} e_i^{\mathrm{T}}(t)(\boldsymbol{I}_n \otimes \boldsymbol{P})e_i(t)\\ =& -\frac{\mu}{\max\{P_j\}_{1\le j\le n}} V(t),\end{aligned}$$

and the theorem is thus proved.

5. A network with multi-central nodes

Consider a network composed of N identical nodes in a multi-central star-shaped coupling configuration, where there are k central nodes that are not connected with each other, and also there are n non-central nodes that are linked up with each central node, but not directly connected among them. Then, the total of nodes in the network is $N = n + k$. Especially, when k=1, the network is a simple star-shaped network. Figure 6 gives an example of such a network for the case of k=2.

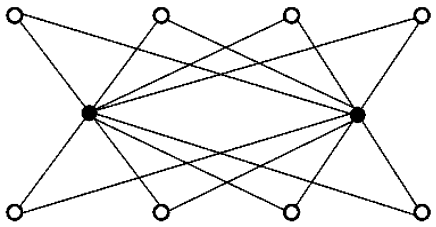


Fig.6. A cluster of star-shaped network with k=2.

Without loss of generality, let the first k nodes be central nodes and $\boldsymbol{A}$ represent the coupling matrix of network (1) in the multi-central star-shaped coupling configuration described above. Then, $\boldsymbol{A}$ can be written as

$$\boldsymbol{A} = \begin{bmatrix}\boldsymbol{A}_1 & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & \boldsymbol{A}_2\end{bmatrix}, \qquad (14)$$

where $\boldsymbol{A}_1 \in \boldsymbol{R}^{k\times k}$, $\boldsymbol{A}_{12} \in \boldsymbol{R}^{k\times(N-k)}$, $\boldsymbol{A}_{21} = \boldsymbol{A}_{12}^{\mathrm{T}} \in \boldsymbol{R}^{(N-k)\times k}$, and $\boldsymbol{A}_2 \in \boldsymbol{R}^{(N-k)\times(N-k)}$. Let $\hat{\alpha} = (1, 1, \ldots, 1) \in \boldsymbol{R}^n$, then the following expressions will be obtained: $\boldsymbol{A}_1 = \begin{bmatrix}-n & & & \\ & -n & & \\ & & \ddots & \\ & & & -n\end{bmatrix}$, $\boldsymbol{A}_{12} = \begin{bmatrix}\hat{\alpha}\\ \hat{\alpha}\\ \vdots\\ \hat{\alpha}\end{bmatrix}$, $\boldsymbol{A}_{21} = \begin{bmatrix}\hat{\alpha}^{\mathrm{T}} & \hat{\alpha}^{\mathrm{T}} & \ldots & \hat{\alpha}^{\mathrm{T}}\end{bmatrix}$, and $\boldsymbol{A}_2 = -k\boldsymbol{I}_{(N-k)}$.

Let the feedback gain matrix be $\boldsymbol{G} = \begin{bmatrix}0 & 0\\ 0 & \varepsilon\boldsymbol{I}_{(N-k)}\end{bmatrix}$, then the corresponding coupling matrix $\tilde{\boldsymbol{A}}$ of the controlled network (4) will be

$$\tilde{\boldsymbol{A}} = \boldsymbol{A} - \boldsymbol{G}. \qquad (15)$$

Theorem 4 If there exist constants $\varepsilon > 0$ and $\tilde{k} > 0$ such that $n > \tilde{k}$, $\varepsilon > \tilde{k} - k$, and $\varepsilon > \dfrac{\tilde{k}(n + k - \tilde{k})}{n - \tilde{k}}$, then $\lambda_i(\tilde{\boldsymbol{A}}) < -\tilde{k}$, $i = 1, \ldots, N$.

Proof Consider the matrix

$$\tilde{\boldsymbol{A}} + \tilde{k}\boldsymbol{I} = \begin{bmatrix}\boldsymbol{A}_1 + \tilde{k}\boldsymbol{I} & \boldsymbol{A}_{12}\\ \boldsymbol{A}_{21} & (-k - \varepsilon + \tilde{k})\boldsymbol{I}\end{bmatrix}.$$

By the assumptions of the theorem, the following inequality holds: $(-k-\varepsilon+\tilde{k})\boldsymbol{I} < 0$. Let $\bar{\boldsymbol{A}} = (\boldsymbol{A}_1+\tilde{k}\boldsymbol{I}) - \boldsymbol{A}_{12}((-k-\varepsilon+\tilde{k})\boldsymbol{I})^{-1}\boldsymbol{A}_{21}$, then $\bar{\boldsymbol{A}}$ can be expressed as

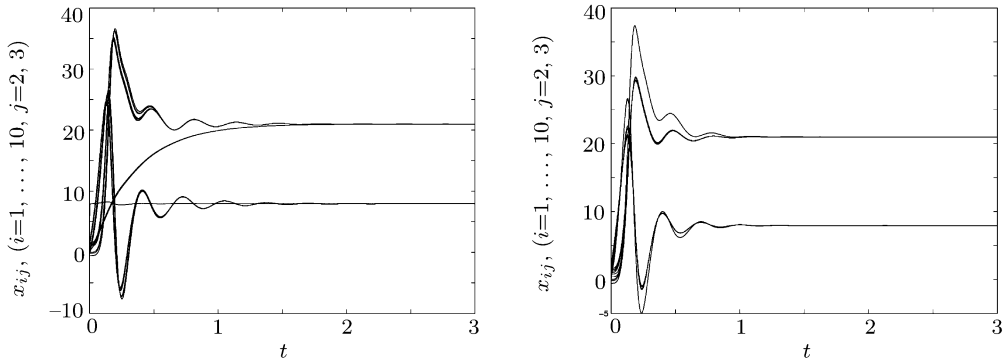
$$\bar{\boldsymbol{A}} = \begin{bmatrix} -n+\tilde{k} & & & \\ & -n+\tilde{k} & & \\ & & \ddots & \\ & & & -n+\tilde{k} \end{bmatrix} + (k+\varepsilon-\tilde{k})^{-1} \begin{bmatrix} n & n & \dots & n \\ n & n & \dots & n \\ \vdots & \vdots & \ddots & \vdots \\ n & n & \dots & n \end{bmatrix}.$$

Since $\bar{\boldsymbol{A}}$ is symmetric, there is an orthogonal matrix $\boldsymbol{P}$ such that

$$\boldsymbol{P}^{\mathrm{T}}\bar{\boldsymbol{A}}\boldsymbol{P} = \begin{bmatrix} -n+\tilde{k} & & & \\ & -n+\tilde{k} & & \\ & & \ddots & \\ & & & -n+\tilde{k} \end{bmatrix} + \frac{n}{k+\varepsilon-\tilde{k}} \begin{bmatrix} k & & & \\ & 0 & & \\ & & \ddots & \\ & & & 0 \end{bmatrix}.$$

Since $\varepsilon > \dfrac{\tilde{k}(n+k-\tilde{k})}{n-\tilde{k}}$ and $-n+\tilde{k}+\dfrac{nk}{\varepsilon+k-\tilde{k}} < 0$, one has $\boldsymbol{P}^{\mathrm{T}}\bar{\boldsymbol{A}}\boldsymbol{P} < 0$, i.e. $\bar{\boldsymbol{A}} < 0$. By the Schur complement Lemma, the inequality $\tilde{\boldsymbol{A}}+\tilde{k}\boldsymbol{I} < 0$ holds, leading to the assertion of the theorem.

Some simulations of network (11) as shown in Fig.6, are presented in Fig.7, where the dynamic behaviours of an isolated node, the inner liking matrix $\boldsymbol{\Gamma}$, and the synchronized state are the same as those given in Section 3.



(a) CF=$c\varepsilon k$=500kc=5000 (b) CF=$c\varepsilon n$=3.5nc=140

Fig.7. Synchronization of the multi-central star-shaped coupling network (14), where the coupling strength c is 5 (c=5), k central nodes are pinned (a), and all the non-central nodes are pinned (b).

Similar to the conclusion given in Section 3, the lower cost and better synchronizability can be achieved by pinning nodes with smaller degrees.

A similar conclusion to that given in Theorem 3 can also be drawn for network (14) by following the same procedure.

6. Some non-regular complex networks

In this section, first, a non-regular coupled network composed of 20 nodes is generated by following the procedure of the well-known Barabási–Albert (BA) model. Pinning controllers are then added to this non-regular network. Figure 8 shows the synchronization of the network with controllers that are added to the three 'biggest' nodes with degrees of 15, 13 and 10, separately. The dynamic behaviours of an isolated node, the inner liking matrix $\boldsymbol{\Gamma}$, and the synchronized state are the same as those given in Section 3. From Figs.8(c) and 8(d), it can be seen that the network does not synchronize any faster, although a larger feedback gain is used. In Fig.9, a different control scheme is applied: eleven nodes with smaller degrees in the network are pinned with coupling strength c=6 and feedback gain ε=5, yielding a cost function value (CF) of 330 (CF=330). Compared with Fig.8, a better control performance is obtained with a much smaller value of the cost function.

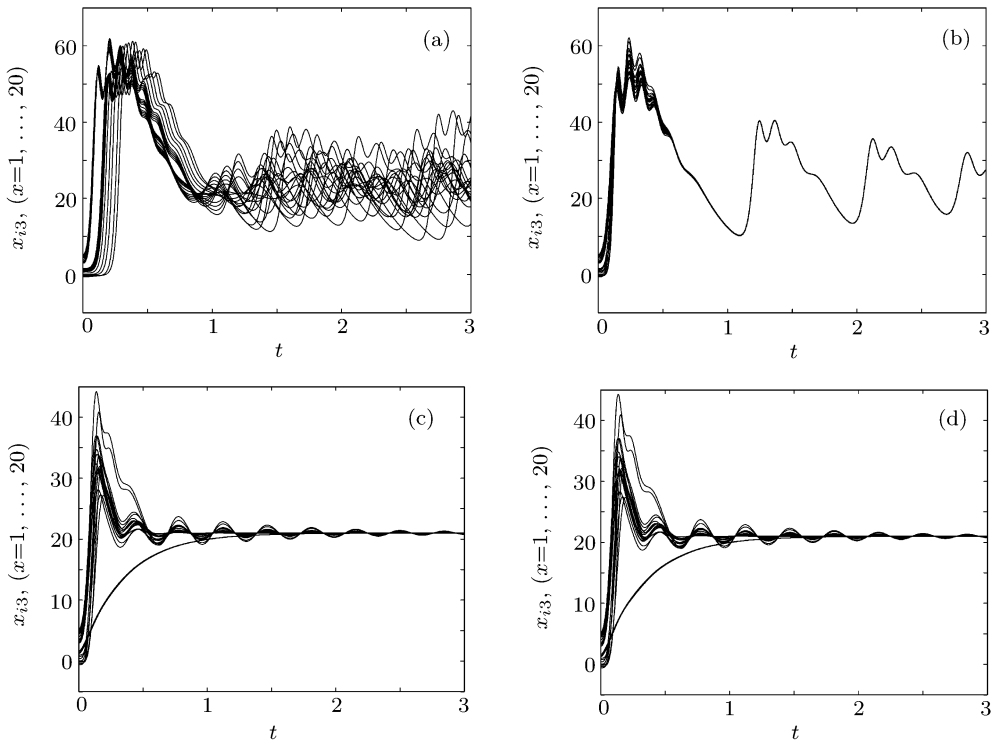


Fig.8. Pinning three 'biggest' nodes with degrees of 15, 13 and 10 in a 20-node non-regular coupled network with (a) $c = 0$, $\varepsilon = 0$, CF $= 0$; (b) $c = 6$, $\varepsilon = 0$, CF $= 0$;(c) $c = 6$, $\varepsilon = 500$, CF $= 9000$; and (d) $c = 6$, $\varepsilon = 1000$, CF $= 18000$.

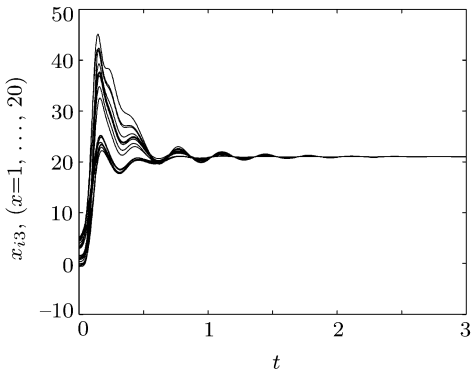
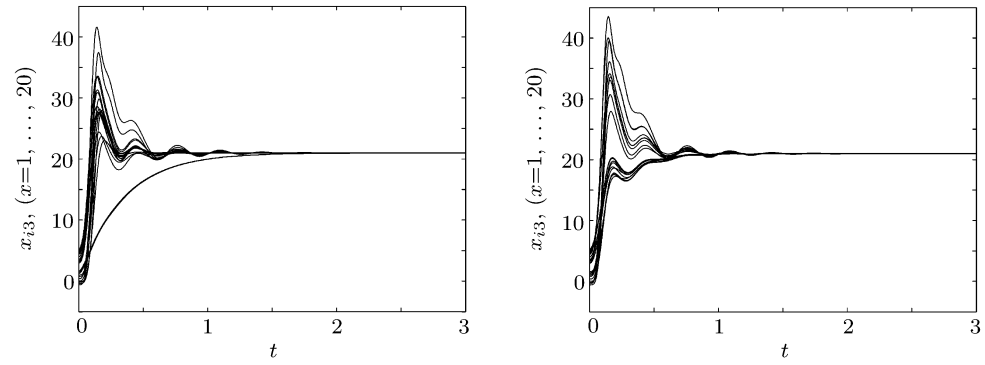


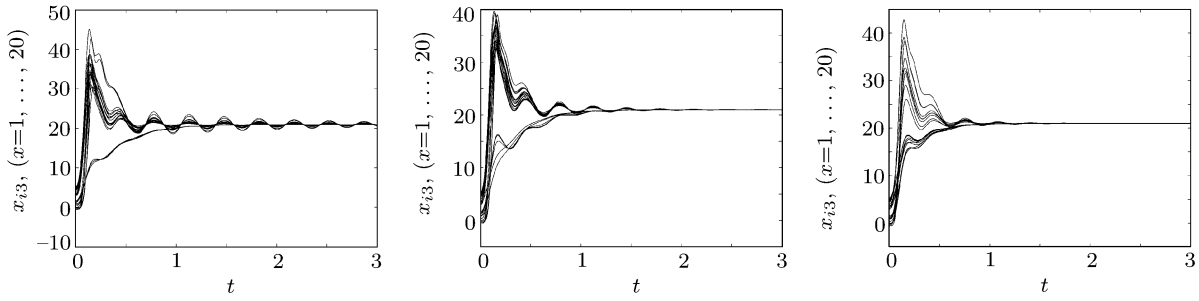
Fig.9. Pinning eleven 'smaller' nodes in a 20-node non-regular coupled network with c=6, ε=5, CF=330.

Figures 10 and 11 also show that it is more efficient by pinning nodes with smaller degrees than by pinning nodes with larger degrees. As shown in Fig. 10, to achieve a similar synchronization effect, a much smaller feedback gain is needed in pinning 'smaller' nodes than in pinning 'larger' ones, thereby the cost function becomes smaller in the former control scheme than in the latter. Figure 11 shows that although the coupling strength, the feedback gain and the cost function are similar, the synchronization effect is better in pinning 'smaller' nodes than in pinning 'larger' ones.



(a) c=8, ε=500, CF=12000 (b) c=6, ε=8, CF=528

Fig.10. (a) Pinning three 'biggest' nodes with degrees of 15, 13 and 10; (b) Pinning eleven 'smaller' nodes.



(a) c=6, ε=55, CF=660 (b) c=6, ε=22, CF=660 (c) c=6, ε=10, CF=660

Fig.11. (a) Pinning two 'biggest' nodes with degrees of 15 and 13; (b) Pinning three 'biggest' nodes with degrees of 15, 13 and 10, and two 'smallest' nodes with the same degrees 3; (c) Pinning eleven 'smaller' nodes.

It seems more efficient to use the control scheme of pinning nodes with smaller degrees than that of pinning nodes with larger degrees for achieving the desired synchronous states in some non-regular coupled dynamical networks. This is an interesting phenomenon, which has not been noticed before.

7. The number of controllers needed in a star-shaped network

As shown above, to achieve a similar synchronization effect on the networks considered, the control scheme of pinning nodes with smaller degrees can be much more efficient than that of pinning nodes with larger degrees. However, because the number of the nodes with smaller degrees in a network is often large, the number of controllers to be applied will also be large if all these nodes are controlled. Thus, another question worthy of consideration in the control scheme of pinning nodes with smaller degrees is the number of controllers required to achieve a certain synchronization effect when the coupling strength c is fixed.

Consider a simple star-shaped network composed of $N = n + 1$ identical nodes, in which there exist one central node and n non-central nodes. Suppose that the number of the controllers used is m, and they are all added to the non-central nodes. Without loss of generality, suppose that the first m non-central nodes are controlled, then the coupling matrix $\tilde{\boldsymbol{A}}$ of the controlled network (4) will be

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} -n & 1 & \ldots & 1 & 1 & \ldots & 1 \\ 1 & -1-\varepsilon & \ldots & 0 & 0 & \ldots & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots & & \vdots \\ 1 & 0 & \ldots & -1-\varepsilon & 0 & \ldots & 0 \\ 1 & 0 & \ldots & 0 & -1 & \ldots & 0 \\ \vdots & \vdots & \ldots & \vdots & \vdots & \ddots & \vdots \\ 1 & 0 & \ldots & 0 & 0 & \ldots & -1 \end{bmatrix}. \tag{16}$$

The characteristic polynomial of matrix (16) is

$$\begin{aligned}\tilde{q}(\lambda) =&(\lambda+1+\varepsilon)^{m-1}(\lambda+1)^{n-m-1}\\ &\times(\lambda^3+(n+\varepsilon+2)\lambda^2\\ &+(n\varepsilon+\varepsilon+n+1)\lambda+m\varepsilon).\end{aligned}\tag{17}$$

Let $a=n+1,\ b=\varepsilon+1$ and $\lambda=y-\dfrac{a+b}{3}$, then a cubic function about y will be obtained from Eq.(17) as

$$f(y)=y^3+py+q,\tag{18}$$

where $p=-\dfrac{(a-b)^2+ab}{3}$ and $q=m\varepsilon+\dfrac{(2(a-b)^2-ab)(a+b)}{27}$. By using the Cardan's formula, roots of the equation $f(y)=0$ can be calculated as

$$\begin{aligned}y_1=&\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}\\ &+\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}},\\ y_2=&\omega\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}\\ &+\omega^2\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}},\end{aligned}$$

and

$$\begin{aligned}y_3=&\omega^2\sqrt[3]{-\frac{q}{2}+\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}}\\ &+\omega\sqrt[3]{-\frac{q}{2}-\sqrt{\left(\frac{q}{2}\right)^2+\left(\frac{p}{3}\right)^3}},\end{aligned}$$

where $\omega=\dfrac{-1+\mathrm{i}\sqrt{3}}{2}$. Thus, the eigenvalues of Eq.(17) can be obtained as $\lambda_i=y_i-\dfrac{a+b}{3}\ (i=1-3),\ \lambda_4=\ldots=\lambda_{m+2}=-1-\varepsilon,\ \lambda_{m+3}=\ldots=\lambda_{n+1}=-1$.

Clearly, the eigenvalues of matrix (16) are dependent on the number m of the applied controllers and the feedback gain ε when n is fixed. Let $\lambda_{\max}(m,\varepsilon)=\max\{\lambda_i(m,\varepsilon)\}_{i=1}^{n+1}$. When the coupling strength c is fixed, an appropriate number of the controllers m and a feedback gain ε can be selected such that $\lambda_{\max}(m,\varepsilon)\le-\dfrac{T}{c}$,where $T>0$ makes $(Df(s(t))-T\boldsymbol{\Gamma})$ Hurwitz stable, which guarantees that the controlled network (16) is exponentially synchronized to $s(t)$.

Consider a simple star-shaped network (see Fig.1), in which each node is a modified Chua circuit with an attraction–repulsion function described as follows:[26]

$$\begin{aligned}\dot{x}_1&=\alpha(x_2-x_1-kf(x_1)),\\ \dot{x}_2&=x_1-x_2+x_3,\\ \dot{x}_3&=\beta(-x_2-\gamma x_3),\end{aligned}\tag{19}$$

where $f(x_1)=ax_1+bx_1\exp(cx_1^2)$. System (19) has a chaotic solution as shown in Fig.12, when parameters of the system are taken as $\alpha=10.5,\ \beta=12,\ \gamma=0.005,\ k=2,\ a=-0.02,\ b=-0.5$, and $c=-0.5$. And the system has three unstable equilibria: $S_0=[0,\ 0,\ 0]$, $S_1=[0.3034,\ 0.0015,\ -0.3019]$ and $S_2=[-0.3034,\ -0.0015,\ 0.3019]$.

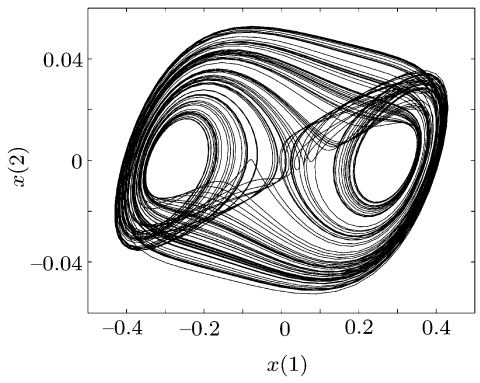


Fig.12. A chaotic solution of system (19).

Choose $s(t)=S_1$, then the value of T required for $(Df(s(t))-T\boldsymbol{\Gamma})$ to be Hurwitz stable will be larger than 8.092, i.e. $T>8.092$. Taking $\boldsymbol{\Gamma}=\mathrm{diag}(1,\ 0,\ 0)$, $\varepsilon=2$ and the coupling strength c=20, direct calculation yields that the required number of controllers (m) is not smaller than 5, i.e. $m\ge5$ $(m_{\max}=8)$, to make the controlled network (14) synchronized to $s(t)$. Figures 13(a), 13(b) and 13(c) display three cases: m=4, 5 and 6 respectively.

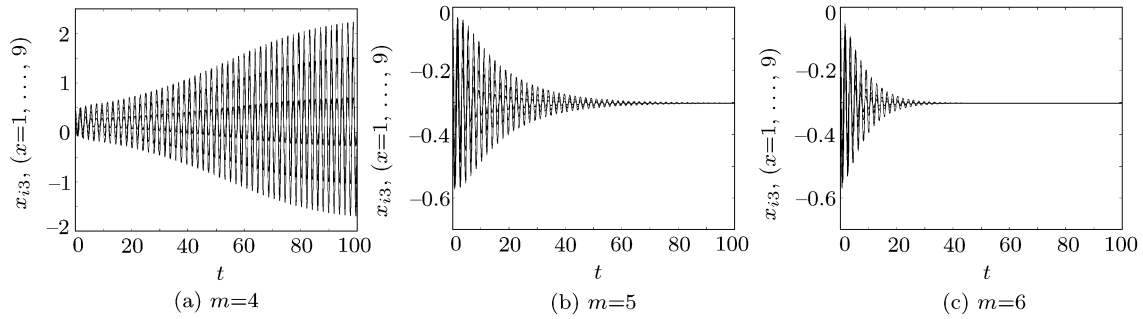


Fig.13. Pinning part of the non-central nodes of system (19).

Remark 7 Suppose that the size of a star-shaped network is very large, and only the central node is controlled, then the coupling matrix $\tilde{\boldsymbol{A}}$ of the controlled network will be

$$\tilde{\boldsymbol{A}} = \begin{bmatrix} -n-d & \boldsymbol{A}_{12} \\ \boldsymbol{A}_{21} & \boldsymbol{A}_2 \end{bmatrix}, \tag{20}$$

where n is the number of non-central nodes, d is the feedback gain, $\boldsymbol{A}_{12} = (1, 1, \ldots, 1) \in \boldsymbol{R}^n$, $\boldsymbol{A}_{21} = \boldsymbol{A}_{12}^{\mathrm{T}}$, and $\boldsymbol{A}_2 = -\boldsymbol{I}_{n\times n}$. The characteristic polynomial of matrix (20) is

$$\hat{q}(\lambda) = (\lambda+1)^{n-1}(\lambda^2 + (n+d+1)\lambda + d). \tag{21}$$

From Eq.(21) and by using Lemma 3, one obtains that (i) if $n \gg d$, then $\lambda_{\max} \to 0$, for which the coupling strength c has to be large in order to achieve the synchronization of the network; (ii) if $d \gg n$, then $\lambda_{\max} > -1$ and $\lambda_{\max} \to -1$. From Remarks 4 and 5, it can be easily seen that the cost is lower and the synchronizability of the network is better if pinning all the non-central nodes. From Eq.(17) it follows that the number m of controllers is dependent on feedback gain ε and coupling strength c. Clearly, the larger the number of m, the better the synchronization effect, when the size of the network is sufficiently large. For example, let $n = 10^6$, i.e. the total of all the nodes in the star-shaped network is $N = 10^6 + 1$. Each node is a modified Chua's circuit described by function (19). Take c=10 and $\varepsilon = 2$. It turns out that the number of controllers required is $m = 9 \times 10^5$ for the controlled network to achieve synchronization. In this case, CF $= 1.8 \times 10^7$ and $\lambda_{\max} = -0.8292$. If only one central node is pinned, then the feedback gain is $d = 4.1 \times 10^6$ to make the controlled network achieve the network synchronization. In this case, CF $= 4.1\times10^7$ and $\lambda_{\max} = -0.8039$. So, once again the lower cost and the better synchronizability are achieved by using the control scheme of pinning nodes with smaller degrees, than by using that of pinning nodes with larger degrees. And, the smaller the coupling strength c, the better the control effect of pinning nodes with smaller degrees. However, it should be noted that when the coupling strength c is very large, the synchronizability of the network in the case of pinning one central node is better than that in the case of pinning only part of the non-central nodes.

8. Conclusions

In this paper, pinning control for the synchronization of complex dynamical networks has been further investigated. In contrast to the general perception, the nodes with smaller degrees play an important role in the synchronizability of some networks. It has been shown by both theoretical analysis and computer simulation that to achieve a similar synchronization effect on some dynamical networks, a smaller coupling strength, a smaller feedback gain and a lower cost function are required by the control scheme of pinning nodes with smaller degrees than those needed in pinning nodes with larger degrees. In other words, it seems more efficient to use the control scheme of pinning nodes with smaller degrees than with larger degrees for some dynamical networks to achieve better synchronization, which is an interesting phenomenon that deserves further investigation in the future.

References

[1] Barabási A L, Albert R and Jeong H 1999 *Phys.* A **272** 173

[2] Boccaletti S, Latora V, Moreno Y, Chavez M and Hwang D U 2006 *Phys. Rep.* **42** 4175

[3] Chen G R, Fan Z P and Li X 2004 *Complex Dynamics in Communication Network* (New York: Springer) pp213–234

[4] Chua L O 1998 *CNN: A Paradigm for Complexity* (Singapore: World Scientific)

[5] Li X, Jin Y Y and Chen G R 2004 *Phys.* A **328** 367

[6] Newman M E J 2003 *SIAM Review* **45** 167

[7] Strogatz S H 2001 *Nature* **410** 268

[8] Watts D J and Strogatz S H 1998 *Nature* **393** 440

[9] Wu Z X, Xu X J and Wang Y H 2005 *Phys. Rev.* E **71** 066124

[10] Feng C F, Xu X J, Wu Z X and Wang Y H 2008 *Chin. Phys.* B **17** 1951

[11] Guo L X, Xu Z Y and Hu M F 2008 *Chin. Phys.* B **17** 836

[12] Li C G and Chen G R 2004 *Chaos, Solitons and Fractals* **20** 353

[13] Li X and Chen G R 2003 *IEEE Trans. on Circuits and Systems I-Fund. Theor. Appl.* **50** 1381

[14] Lou X Y and Cui B T 2008 *Chin. Phys.* B **17** 520

[15] Lü J H and Chen G R 2004 *Phys.* A **334** 281

[16] Posadas C C, Cruz H C and López M D 2005 *Yearly Congress of the Association of Automatic Control Mexico, Cuernavaca Morelos* Oct. 19 2005

[17] Wang X F and Chen G R 2002 *Int. J. Bifur. Chaos* **12** 187

[18] Wu C W and Chua L O 1995 *IEEE Trans. on Circuits and Systems I-Fund. Th. Appl.* **42** 430

[19] Chen T P, Liu X W and Lu W L 2007 *IEEE Trans. on Circuits and Systems I-Fund. Th. Appl.* **54** 1317

[20] Fan Z P and Chen G R 2005 *Proc. IEEE International Symposium on Circuits and Systems* **1** 284

[21] Li X, Wang X F and Chen G R 2004 *IEEE Trans. on Circuits and Systems I-Fund. Th. Appl.* **51** 2074

[22] Horn R and Johnson C 1985 *Matrix Analysis* (Cambridge: Cambridge University Press)

[23] Gahinet P, Nemirovski A and Laub A J 1995 *LMI Control Toolbox* (Natick, Mass: The Math Works, Inc)

[24] Lü J H, Chen G R and Zhang S 2002 *Chaos, Solitons and Fractals* **14** 669

[25] Lü J H, Chen G R and Zhang S 2002 *Int. J. Bifur. Chaos* **12** 1001

[26] Li R, Duan Z S, Chen G R and Wang B 2008 *Int. J. Bifur. Chaos* **18** 1865