Vol 18 No 1, January 2009
1674-1056/2009/18(01)/0098-08

Chinese Physics B

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Development and transition of spiral wave in the coupled Hindmarsh–Rose neurons in two-dimensional space*

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(Received 12 June 2008; revised manuscript received 30 August 2008)

The dynamics and the transition of spiral waves in the coupled Hindmarsh–Rose (H–R) neurons in two-dimensional space are investigated in the paper. It is found that the spiral wave can be induced and developed in the coupled HR neurons in two-dimensional space, with appropriate initial values and a parameter region given. However, the spiral wave could encounter instability when the intensity of the external current reaches a threshold value of 1.945. The transition of spiral wave is found to be affected by coupling intensity D and bifurcation parameter r. The spiral wave becomes sparse as the coupling intensity increases, while the spiral wave is eliminated and the whole neuronal system becomes homogeneous as the bifurcation parameter increases to a certain threshold value. Then the coupling action of the four sub-adjacent neurons, which is described by coupling coefficient D', is also considered, and it is found that the spiral wave begins to breakup due to the introduced coupling action from the sub-adjacent neurons (or sites) and together with the coupling action of the nearest-neighbour neurons, which is described by the coupling intensity D.

Keywords: spiral wave, Hindmarsh–Rose neurons, instability
PACC: 0545

1. Introduction

From a functional point of view, the sensory neurons play an important role in receiving and transmitting signals in the neuronal system.[1−5] In the previous studies, such problems as the synchronization of coupled neurons,[6−10] the stochastic resonance (SR)[11−14] and the coherence resonance (CR)[15−17] have been investigated extensively. As is known, the appearance of spiral wave in the cardiac tissue is related to heart disease, and the instability of spiral wave in cardiac tissue can induce ventricular fibrillation.[18] The spiral intercellular waves of calcium have also been observed in the slices of hippocampal tissue,[19] and they disinhibited the mammalian neocortex.[20] Usually a spiral wave existing in the cardiac tissue is regarded as a dangerous signal, while the spiral wave in the mammalian neocortex might play an active role in generating periodic activity in a non-oscillatory network without using any individual cellular pacemaker. It is also reported that the spirals can be used to coordinate oscillation phases over a population of neurons, and serve as the functions such as binding sensory information or dynamical temporal storage in working memory. There are many similarities in spiral wave between in the excitable media and in the cardiac tissue. Therefore, many schemes have been proposed to eliminate the harmful spiral wave and turbulence in excitable media.[21−34] As is known, the neuronal system comprises many neurons and complex coupling actions between neurons could be the base to support a normal body. Neurons often respond to the external stimulus sensitively, and thus the firing patterns are changed synchronically. Most of the previous studies of spiral wave were concerned with the formation and the transition of spiral wave in a reaction–diffusion system, but spiral wave in the coupled os-

*Project supported by the National Natural Science Foundation of China (Grant Nos 10747005, 10572056 and 10774130), and the Natural Science Foundation of Lanzhou University, China (Grant No Q200706).
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cillators has received little attention except in some excellent studies.[35–37] In Refs.[35–37], the couplings between oscillators are assumed to be of completely nearest-neighbour type. The famous Hindmarsh–Rose (H–R) neuron model is good enough to describe the electric action of neuron membrane potential in neurons, and it can also be regarded as a special oscillator from the nonlinear dynamics theory. The H–R neurons are also excitable and the coupled H–R neurons in two-dimensional space can also illustrate some properties of the excitable medium. The H–R neuron model can also step into chaotic state in an appropriate parameter region, and thus the coupled H–R neurons can reflect the properties of chaotic medium, such as the spatiotemporal chaos. It is reported that the connection of neurons can be complex, and most of us believe that the connection like small-world type connection could be used to reasonably describe complex actions among neurons. To the best of our knowledge, no spiral wave in the coupled H–R neurons has ever been studied. Under what condition can a stable rotating spiral wave be induced and developed in the coupled H–R neurons? Can the small-world type connection support a spiral wave? How can an instability of spiral wave be induced? Therefore, it is interesting to study these open questions of how the spiral wave is induced and developed in the coupled neurons. Generally, the simplest way to develop a spiral wave numerically in the coupled H–R neurons is to simplify the complex coupling action and assume that the couplings are completely of nearest-neighbour type. In this paper, the coupled HR neurons[3,4] in two-dimensional (2D) space are considered to generate the spiral wave and some controllable parameters are chosen carefully such that the transition of spiral wave is observed. Then other type connections are discussed in brief, and it could throw a light on complex pattern firing in the coupled H–R and other neurons.

2. Coupled Hindmarsh–Rose neurons and numerical results

The excitable dynamics of the H–R model results from the existence of a Hopf bifurcation leading to a limit cycle passing through an activation threshold.[3,4] The H–R model with three variables can be described by

$$\dot{x} = y - ax^3 + bx^2 - z + I_{\text{inject}},$$

$$\dot{y} = c - dx^2 - y,$$

and

$$\dot{z} = r[s(x - x_0) - z], \tag{1}$$

where a, b, c, d, r, s, and x_0 are parameters and their values are often chosen to be a=1, b=3, c=1, d=5, s=4, bifurcation parameter r=0.006 and $x_0 = -1.6$ so that the results can be compared with the previous results obtained by using the same model. In Eq.(1), the variable $x(t)$ represents the difference between electrical potentials across the neuron's membrane and $y(t)$ describes the dynamics of the resetting mechanism that restores the polarity of the membrane after the firing of each potential action. The additional variable $z(t)$ allows the precise tuning of the interspike intervals. The parameter I_{inject}, i.e. the external intensity of a constant (tonic) signal delivered to the neuron, is usually considered as a bifurcation parameter of a dynamical system. Clearly, chaotic state can be observed in the single H–R neuron described by Eq.(1) whose maximum Lyapunov exponent spectrum is calculated with the results as shown in Fig.1. In the following sections, the parameter r=0.006 is used unless otherwise stated.

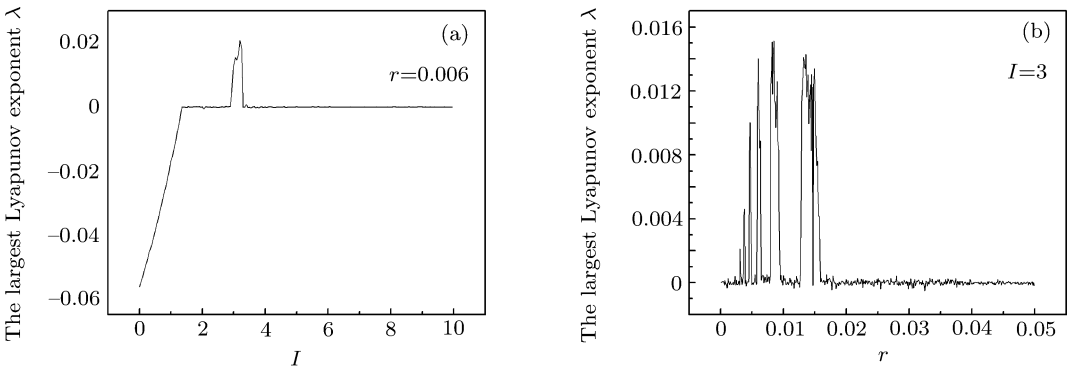


Fig.1. Variation in the maximum Lyapunov exponent of Eq.(1) with parameters I (a) and r (b), separately.

The results in Fig.1 confirm that chaotic state can be detected and observed as an appropriate external forcing current and (or) a bifurcation parameter are (is) used. As is well known, the connection between neurons could be of small-world type or others, and the nearest-neighbour coupling could be of the simplest type. At first, we focus on the case of completely nearest-neighbour connection, and then investigate the influence of other connections on the spiral wave. It is more interesting to study the pattern firing and transition among the coupled H–R neurons. The coupled H–R neurons with completely nearest-neighbour couplings are described by

$$\dot{x}_{i\,j} = y_{i\,j} - a x_{i\,j}^3 + b x_{i\,j}^2 - z_{i\,j} + I_{\text{inject}} + D(x_{i-1\,j} + x_{i+1\,j} + x_{i\,j-1} + x_{i\,j+1} - 4x_{i\,j})$$

and

$$\dot{y}_{i\,j} = c - d x_{i\,j}^2 - y_{i\,j},$$

$$\dot{z}_{i\,j} = r[s(x_{i\,j} - x_0) - z_{i\,j}], \tag{2}$$

where (i, j) describes the position of the neuron and D is the coupling intensity (here D=0.5). The time step h is chosen to be 0.02, i.e. h=0.02, and the no-flux boundary condition is used in the numerical simulation. With the appropriate initial values and the given parameters a stable spiral wave is induced and developed in time sequence, then the external intensity I_{inject} is adjusted in order to observe the transition of the spiral wave. Figure 2 shows the spiral wave and its evolution in time sequence. In our numerical simulation, appropriate initial values are selected in the same scheme as that in Ref.[31] and I_{inject}=1.315 is used to induce and develop a stable rotating spiral wave as shown in Fig.2. Clearly, with the parameters a=1, b=3, c=1, d=5, r=0.006 s=4 and x_0=–1.6, a stable limit circle forms for a single H–R neuron.

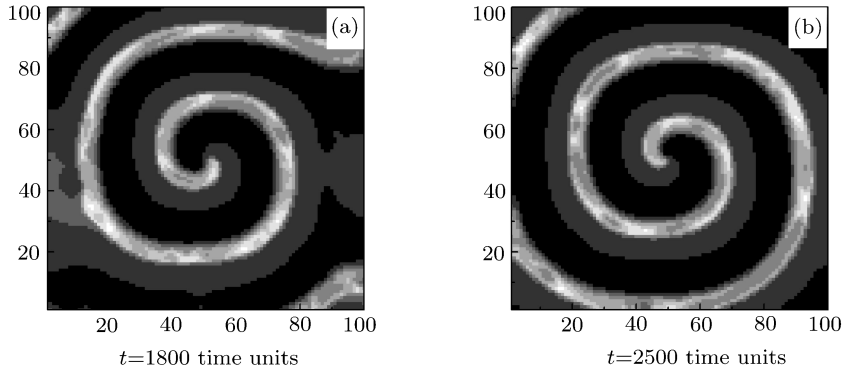


Fig.2. Stable rotating spiral wave in the coupled HR neurons in two-dimensional space, and the final state of activator x is shown in grey with parameters a=1 , b=3, c=1, d=5, r=0.006, s=4, $x_0 = -1.6$, I=1.315 and D=0.5, the time step h=0.02.

In Fig.2, it is found that a stable spiral wave can be induced and developed, and that the developed spiral wave occupies the whole medium within a transient period of about 2500 time units. It is also confirmed that the images of $y(t)$ and $z(t)$ develop into a spiral wave. For simplicity, the images of the activator $x(t)$ are used to investigate the dynamics of the spiral wave in the coupled H–R neurons in 2D space (networks). Furthermore, different values of the coupling coefficient D are also selected to induce the spiral wave. It is found that no regular and stable spiral wave can be generated when the value of coupling coefficient D is less than its threshold value of 0.5. Figure 3 shows the final states as a function of the value of D at t=2000 time units.

The images in Fig.3 confirm that the spiral wave can be induced and it becomes sparse, with the stronger coupling used. Furthermore, the evolution of spiral wave is investigated by changing the values of parameter r after a stable rotating spiral wave has been generated with the same parameters as those in Fig.2 in a transient period of 2000 time units. Specifically, a stable rotating spiral wave is induced with D=0.5 in 2000 time units, where the value of parameter r varies from 0.001 to 0.015. The results are shown in Fig.4.

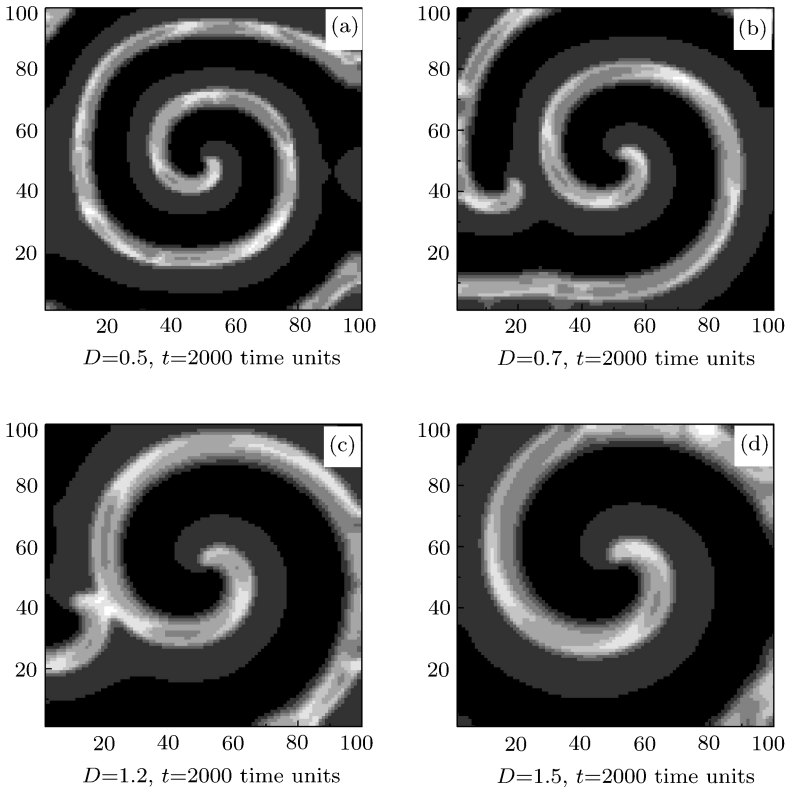
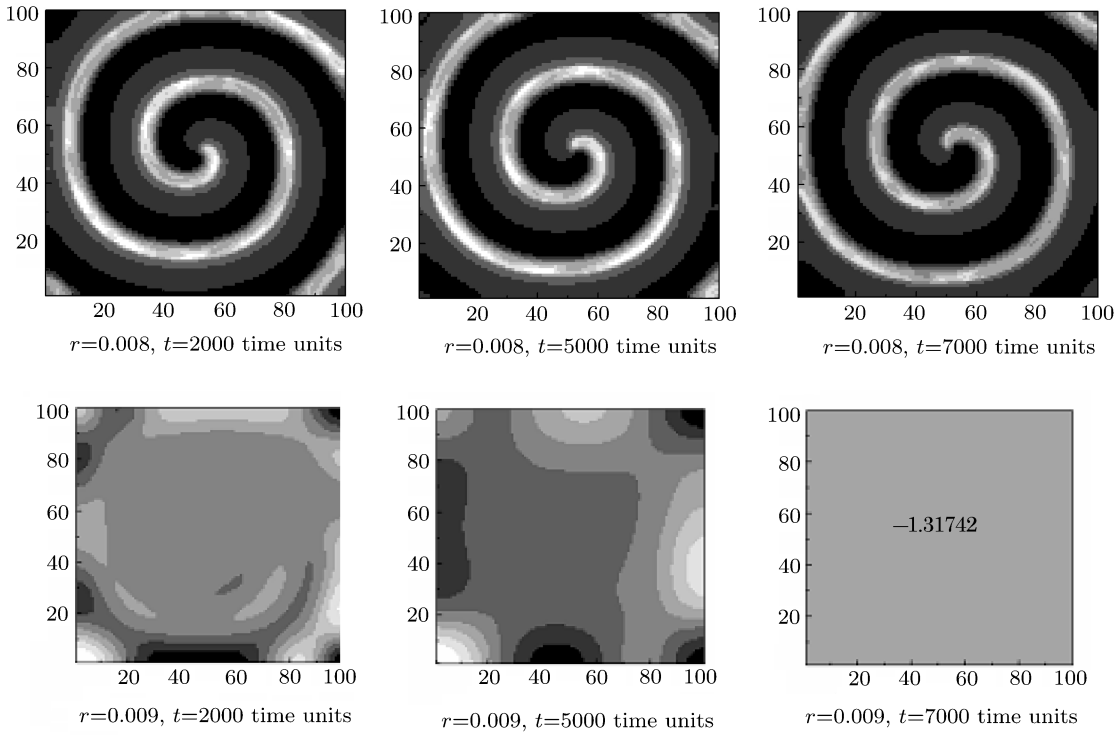


Fig.3. The spiral waves developed, with values of coupling intensity D being 0.5 (a), 0.7 (b), 1.2 (c) and 1.5, and the final state of activator shown in grey.



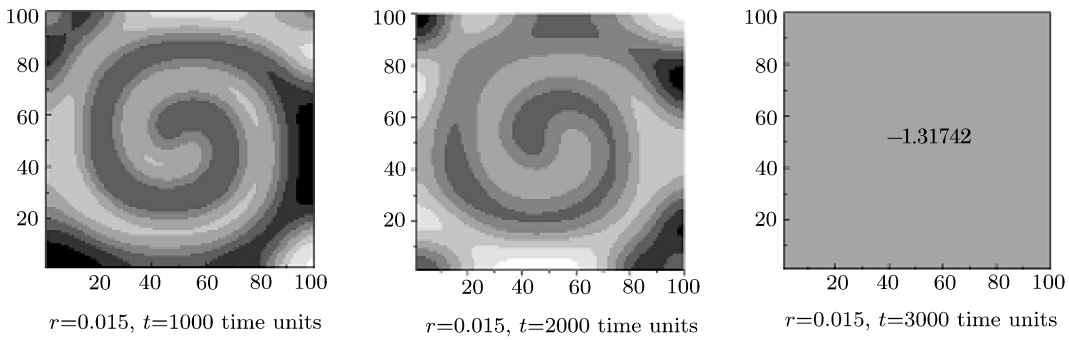


Fig.4. Evolution of spiral wave under different values of parameter r, where the initial spiral wave is induced and developed with coupling coefficient D=0.5 and r=0.006.

The images in Fig.4 confirm that the spiral wave can be destroyed and all the neurons keep their state unchanged till the value of parameter r increase to a threshold value of about 0.009. Next, we begin to investigate the effect of external current I_{inject}, which determines the dynamics of the spiral wave. Compared with the previous results, the external current has been adjusted carefully in order to induce the transition of spiral wave. In the numerical simulations, a stable spiral wave is induced and developed in 2000

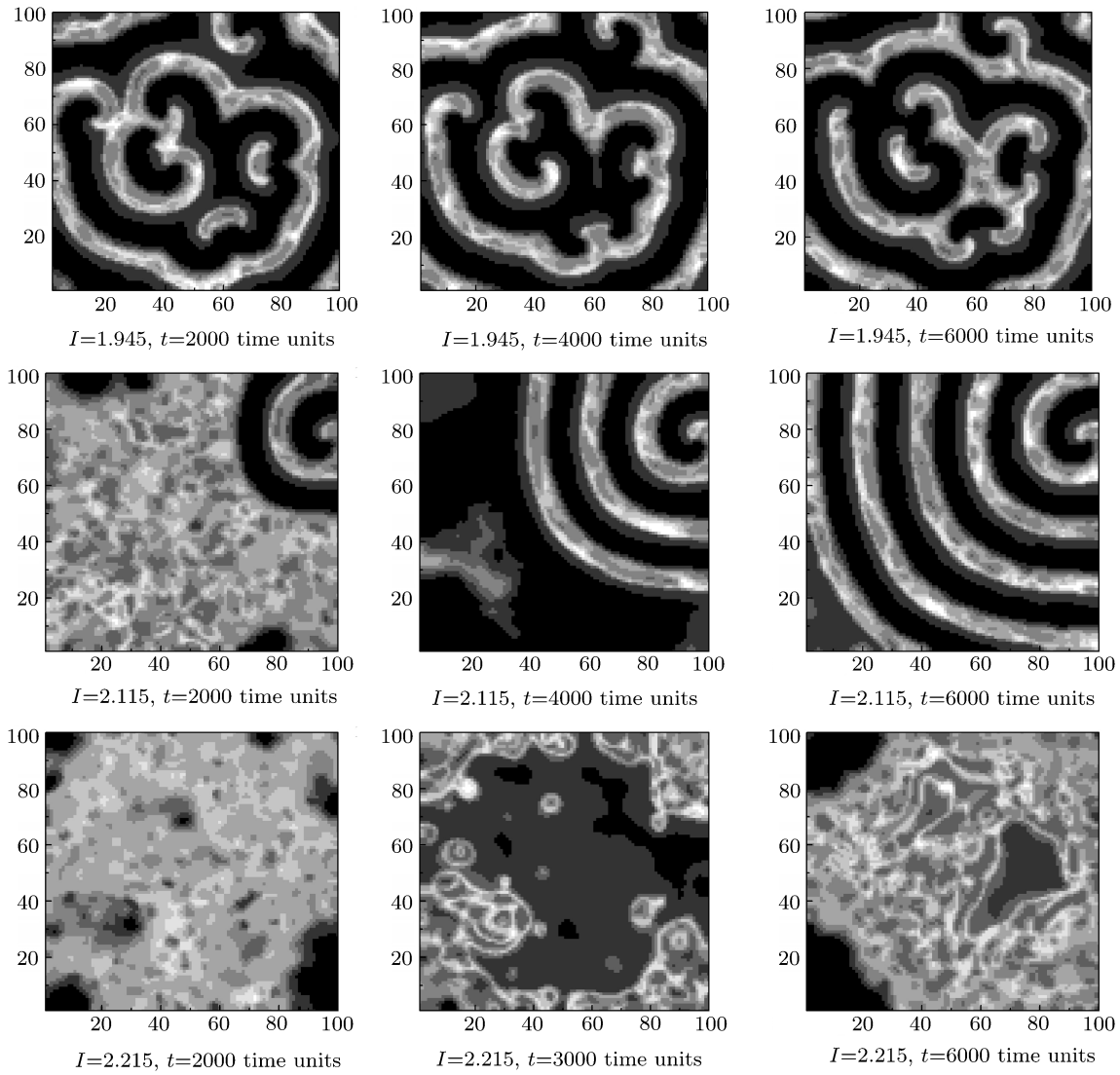


Fig.5. Evolutions of the spiral wave subjected to different intensities of external current, where the initial spiral wave is induced and developed under the same condition as that in Fig.2 and the final state of the activator x is shown in grey.

time units under the given parameters the same as those in Fig.2. Then the intensity of the external currents I is changed in order to study the evolution of spiral wave again, and the simulated results are plotted in Fig.5.

The images in Fig.5 confirm that the spiral wave begins to encounter an instability as the intensity of the external current increases to a threshold value of about 1.945. It is interesting to note that the transition of spiral wave is complex. The spiral wave is destroyed first, then its seed goes up slowly and the spiral wave occupies the whole system finally at I=2.115. The extensive numerical studies show that the breakup of spiral wave is easier to be observed when the intensity of the external current turns strong enough.

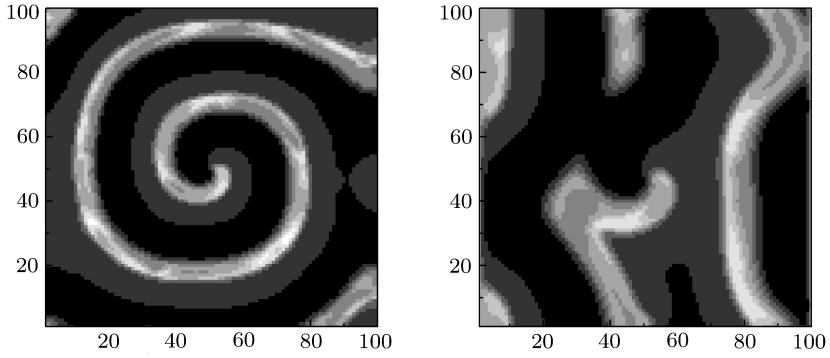
We have studied the formation of spiral wave and the pattern transition induced by the parameter perturbation. As mentioned above, the complex connection between neurons could be of small-world type or other couplings. Unfortunately, our extensive numerical studies cannot reveal the development of spiral wave when the small-world type connection replaces the completely nearest-neighbour couplings, that is to say, the small-world type connection cannot support a standing spiral wave. To simplify this question, we take another way to investigate this problem. Therefore, it is also interesting to investigate the problem about pattern transition induced by the connection of neurons at the sub-adjacent diagonal sites. We will check the influence on spiral wave induced by the sub-adjacent connection. The problem can be described by

$$\begin{aligned}\dot{x}_{i\,j} =& y_{i\,j} - a x_{i\,j}^3 + b x_{i\,j}^2 - z_{i\,j} + I_{\text{inject}} \\ &+ D(x_{i-1\,j} + x_{i+1\,j} + x_{i\,j-1} + x_{i\,j+1} - 4x_{i\,j}) \\ &\times D'(x_{i-1\,j-1} + x_{i+1\,j-1} + x_{i+1\,j+1} \\ &+ x_{i-1\,j+1} - 4x_{i\,j})\end{aligned}$$

and

$$\begin{aligned}\dot{y}_{i\,j} &= c - d x_{i\,j}^2 - y_{i\,j}, \\ \dot{z}_{i\,j} &= r[s(x_{i\,j} - x_0) - z_{i\,j}],\end{aligned} \tag{3}$$

where D' can be the coupling coefficient between neurons at the sub-adjacent diagonal sites and D is the coupling intensity for the nearest neurons. Clearly, the coupling action between the neurons at the sub-adjacent sites has to be considered so that the pattern formation and transition can be understood greatly. In Figs.6 and 7, the pattern transitions are plotted with D=0.5, D'=0.4 and D=0.5, D'=0.5, respectively. In the numerical simulation, appropriate initial values and parameters are needed in order to induce a stable spiral wave in about 2000 time units as shown in Fig.2, but the coupling action between the neurons at sub-adjacent sites is activated for D'=0.5, 0.4.



D=0.5, D'=0, t=2000 time units D=0.5, D'=0.5, t=12000 time units

Fig.6. The evolutions of spiral wave, where the connection of neurons at the adjacent diagonal sites is activated with parameter D=0.5, D'=0.5, and the final state shown in grey.

The images in Fig.6 confirm that the stable spiral wave is destroyed as the connection of neurons at the adjacent diagonal sites begins to come into play. Clearly, the introducing of the connection of the neurons at the diagonal sites can perturb the coupling action. Then the case of coupling coefficient D'=0.4 is investigated and the results are plotted in Fig.7.

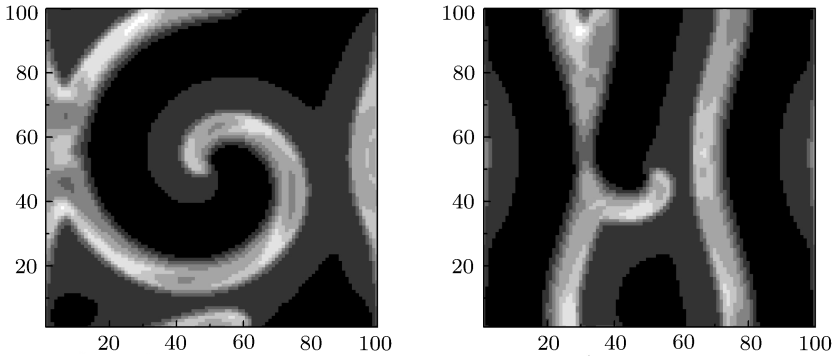


Fig.7. The evolution of spiral wave，where the connection of neurons at the adjacent diagonal sites is activated with parameters D=0.5 and D'=0.4, and the final state shown in grey.

A comparison of the results between in Fig.7 and in Fig.6 confirms further that the spiral wave is destroyed, and other numerical calculations at D=0.5 and D'=0.3, 0.2, 0.6, 0.7 show that the spiral waves still keep alive. In our opinion, both the coupling intensity D and the coupling coefficient D' play an important role in supporting the pattern formation and any perturbation to the coupling intensity and the coupling coefficient can induce the pattern transition, thus the spiral wave may become sparse, dense or destroyed greatly.

Clearly, the dynamics of spiral wave is often changed greatly when other additional connections, even the sub-adjacent connection and couplings, are introduced into the coupled H–R neurons. The instability of spiral wave just indicates the spiral wave will disappear when other complex connections are considered. To our knowledge, the numerical results that complex connection cannot support a standing spiral wave or destroys the spiral wave could account for the fact that the spiral wave cannot be observed in the coupled neuron in two-dimensional space.

Finally, some other questions may arise. How does the spiral wave in the coupled neurons respond to the external periodical forcing? In the previous studies, external periodical forcing was often used to suppress the spiral wave and perturbations if the forcing was applied to the local square sites or all the sites in the medium. In the case of the coupled HR neurons, we choose the periodical forcing to be in the form $I_{\rm inject} = 1.315 + I_0 \cos(\omega t)$ and substitute it into Eq.(2). And the extensive numerical results just show the spiral wave to break up no matter whether the periodical forcing is applied locally or globally. To our knowledge, this character is different from the one in the reaction–diffusion system. The signal and the action between the neurons are activated through coupling by releasing the spike pulses while the action in the reaction–diffusion system is realized through coupling and diffusing. Furthermore, if the external current $I_{\rm inject} = I_0 \cos(\omega t)$ is used, it is found that the spiral wave can be removed as expected. Clearly, the coupling and the activation of the neurons require that the external current should be positive, thereby activates and passes the positive spike pulse, and all the neurons can keep synchronized and the pattern forms. Otherwise, negative external current just introduces negative feedback into the medium, thus all the neurons reach the rest state. Furthermore, stronger and positive external forcing of current just can turn each neuron chaotic and so all the neurons become spatiotemporal chaotic under the action of coupling as concluded from the spectrum of Lyapunov exponents in Fig.1.

3. Discussion and conclusions

It is reported in this paper that numerical study of the spiral wave of the coupled H–R neurons in two-dimensional space is performed, which simplifies the connections between neurons with completely nearest-neighbour couplings. In the simulation, a stable rotating spiral wave is generated and kept developing in time sequence if appropriate initial values and parameters are chosen. To our knowledge, like other excitable medium, the coupled H–R neurons in two-dimensional space can support and develop standing spiral waves only when appropriate initial values and parameters and coupling intensity are used. The spiral wave in the coupled H–R neurons can encounter instability as

the intensity of the external current forcing exceeds a certain threshold value, which may be explained as follows: all the neurons step into a chaotic state as the intensity of current increases to some threshold value and thus the spatiotemporal chaos in the coupled H–R is generated. Then the values of relevant parameters are set again in order to observe the transition of the spiral waves. It is found that the spiral wave turns sparse as the coupling intensity D increases, and it disappears again so that the system of all the coupled H–R neurons becomes homogeneous when the bifurcation parameter r reaches a threshold value of 0.009. Furthermore, the extensive numerical results confirm that the instability of the spiral waves occurs as the intensity of the external current increases to a threshold value of 1.945. Owing to the function of neurons' receiving and passing signals, the complex connections are established during the signal exchange among all the neurons, thus the spatial pattern is induced and developed. We believe that the appearance of spiral waves in the coupled H–R neurons with completely nearest-neighbour coupling results from the self-organization and signal exchange among coupled neurons and it may illustrate a kind of pathway to pass a kind of signal. Further investigations along this direction will be carried out in future.

Acknowledgement

We would like to thank Yu Guang and Professor Jia Ya for useful discussion.

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